GEODESICS ON THE SPACE OF LAGRANGIAN SUBMANIFOLDS IN COTANGENT BUNDLES

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Abstract. We prove that the space of Hamiltonian deformations of zero section in a cotangent bundle of a compact manifold is locally flat in the Hofer metric and we describe its geodesics.

1. Introduction

Let $M$ be a compact smooth manifold. Denote by $\mathcal{L}(M)$ the space of Hamiltonian deformations of zero section $O_M$ in its cotangent bundle $T^*M$. For $L_0 \in \mathcal{L}(M)$ and for a path $L_t := \phi^H_t(L_0)$, where $\phi^H_t$ is a Hamiltonian isotopy generated by a smooth compactly supported Hamiltonian function $H : [0, 1] \times T^*M \to \mathbb{R}$, we define

\[
\text{length}(\{L_t\}) := \inf_H \int_0^1 (\max_x H(t, x) - \min_x H(t, x)) \, dt,
\]

where the infimum is taken over all $H$ such that $\phi^H_t(L_0) = L_t$. For $L_1 \in \mathcal{L}(M)$, define distance between $L_0$ and $L_1$ as the infimum of lengths over connecting paths. More precisely,

\[
d(L_0, L_1) := \inf_{\phi^H_t} \int_0^1 (\max_x H(t, x) - \min_x H(t, x)) \, dt \mid \phi^H_t(L_0) = L_1.
\]

Note that $\{L_t\}$ depends only on values of $H(t, x)$ for $x$ near $\cup L_t$. Since the distance defined by (2) is the infimum of lengths over all connecting paths, $d$ does not depend on whether the maximum and the minimum are taken over $x \in \cup L_t$ or over $T^*M$. In this paper $\max_x$ and $\min_x$ will denote the maximum and minimum over $x \in \cup L_t$.

A group of Hamiltonian diffeomorphisms of $T^*M$ acts on $\mathcal{L}(M)$ via $(\psi, L) \mapsto \psi(L)$. It is known that $d$ is an invariant distance on $\mathcal{L}(M)$ with respect to this action. The most delicate fact in the proof of this fact is the non-degeneracy of $d$ (see [2], [8]). A proof given by Oh [8] is based on a study of invariants defined by

\[
\rho(H) = \inf \{\lambda \mid HF^*(-\infty, \lambda)_+ (H) \to HF_+(H) \text{ is surjective}\},
\]

where $HF_+$ is Floer homology. After a certain normalization, $\rho(H)$ depends only on $L = \phi^H_t(O_M)$ and it is denoted by $\rho(L)$ (see [8] or Section 2 below for more details).
In this paper we prove that every \( L \in \mathcal{L}(M) \) has a flat \( C^1 \)-neighborhood. More precisely, let \( \mathcal{G} \) be the \( C^1 \)-neighborhood (in the space of Lagrangian embeddings) of \( O_M \) such that if \( L \in \mathcal{G} \), then \( L = \text{Graph}(dS) \) for some smooth function \( S \) on \( M \). We prove that for \( L_i = \text{Graph}(dS_i) \in \mathcal{G} \), \( i \in \{0,1\} \),
\[
d(L_0, L_1) = \|S_1 - S_0\| := \max(S_1 - S_0) - \min(S_1 - S_0).
\]
As a corollary, we obtain a description of geodesics on \( \mathcal{L}(M) \). This generalizes an analogous result by Bialy and Polterovich for Hamiltonian diffeomorphisms in \( \mathbb{R}^{2n} \). A description of geodesics on the group of Hamiltonian diffeomorphisms in general symplectic manifolds is obtained by Lalonde and McDuff [5].

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2. Preliminaries

Let \( \{L_t\}_{0 \leq t \leq 1} \) be a smooth regular path in \( \mathcal{L} \), i.e. \( \frac{d}{dt} L_t \neq 0 \) for every \( t \in [0,1] \). \( L_t \) is called a minimal geodesic if \( \text{length}(\{L_t\}) = d(L_0, L_1) \). It is called geodesic if it is a minimal geodesic locally on \([0,1]\) (compare [1]).

A Hamiltonian \( H(t,x) \) is called quasi-autonomous if there exist \( x_+, x_- \in \bigcup_t L_t \) such that \( \max_x H(t,x) = H(t,x+) \) and \( \min_x H(t,x) = H(t,x-) \) for every \( t \). This is equivalent to
\[
\int_0^1 (\max_x H(t,x) - \min_x H(t,x)) dt = \max_x \int_0^1 H(t,x) dt - \min_x \int_0^1 H(t,x) dt
\]
(see [1]).

Let us recall a construction of symplectic invariants by Oh [8]. For \( H \in C_0^\infty([0,1] \times T^*M) \) consider the classical action functional
\[
A_H(\gamma) := \int_0^1 \gamma^* pdq - Hdt,
\]
for \( \gamma : [0,1] \to T^*M \), \( \gamma(0) \in O_M \), where \( pdq \) is a canonical Liouville 1-form on \( T^*M \). Floer chain complexes \( CF_*(H) \) are defined as free \( \mathbb{Z} \)-modules over
\[
\text{Crit}_*(A_H) := \{ \gamma : [0,1] \to T^*M \mid \frac{d\gamma}{dt} = X_H(\gamma), \ \gamma(0), \gamma(1) \in O_M \}
\]
where \( X_H \) is a Hamiltonian vector field corresponding to \( H \). They are graded by the Maslov index, and filtered by level sets of \( A_H : CF_*^\lambda(H) \) denotes a free \( \mathbb{Z} \)-module over
\[
\text{Crit}_*(A_H) := \{ \gamma \in CF_*(H) \mid A_H(\gamma) \leq \lambda \}.
\]
Recall that \( HF_*(H) \) is defined as a homology group of \( CF_*(H) \) with respect to the boundary homomorphism
\[
\partial : CF_*(H) \to CF_*(H), \ \partial(x) := \sum_{y \in CF(H)} n(x,y)y,
\]
where \( n(x,y) \) is the number of solutions of
\[
\begin{align*}
\frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial \tau} - X_H(u)) &= 0, \\
u(\tau,0), u(\tau,1) &\in O_M, \\
u(-\infty, t) &= x(t), u(+\infty, t) = y(t)
\end{align*}
\]
(“negative gradient flow of \( A_H \)). Here \( J \) is some almost complex structure compatible with the symplectic form. Since \( A_H \) decreases along its “negative gradient
lines" [2], \( \partial \) restricts to \( CF^\lambda_\ast(H) \); the corresponding homology group is denoted by \( HF^\lambda_\ast(H) \). An obvious inclusion \( CF^\lambda_\ast(H) \to CF_\ast(H) \) induces a homomorphism \( j^\lambda_\ast : HF^\lambda_\ast(H) \to HF_\ast(H) \).

Oh [8] defined for a generic \( H \in C^\infty_0([0,1] \times T^*M) \)

\[ \rho(H) = \inf \{ \lambda \mid j^\lambda_\ast : HF^\lambda_\ast(H) \to HF_\ast(H) \text{ is surjective} \}, \]

and proved that it depends only on \( L := \phi^H_1(O_M) \), but not on a particular choice of normalized \( H \in C^\infty_0([0,1] \times T^*M) \) that generates \( L \). More precisely, denote by \( W_H \) the wave front of \( H \), i.e.

\[ W_H := \{(q,s) \in M \times \mathbb{R} \mid q = \pi(x), \ s = A_H(\phi^H_t \circ (\phi^H_s)^{-1}(x)), \ x \in L\}, \]

where \( \pi : T^*M \to M \) is the canonical projection. Then, if \( \phi^H_1(O_M) = \phi^K_1(O_M) \) and \( W_H = W_K, \ \rho(H) = \rho(K) \) (Theorem 8.1 in [5]).

Let \( \mathcal{H}(M) \) be a set of Hamiltonians normalized so that their wave fronts depend only on \( L := \phi^H_1(O_M) \):

\[ \mathcal{H}(M) := \{ H \in C^\infty_0([0,1] \times T^*M) \mid \max_{(x,s) \in W_H} s + \min_{(x,s) \in W_H} s = 0 \}. \]

Note that definitions [1] and [2] remain the same if we take the infimum over \( H \in \mathcal{H}(M) \) only. Indeed, it is clear that

\[
\inf_{H \in C^\infty_0([0,1] \times T^*M)} \left\{ \int_0^1 \left( \max_x H(t,x) - \min_x H(t,x) \right) dt \mid \phi^H_t(0_M) = L_t \right\}
\leq \inf_{H \in \mathcal{H}(M)} \left\{ \int_0^1 \left( \max_x H(t,x) - \min_x H(t,x) \right) dt \mid \phi^H_t(0_M) = L_t \right\}.
\]

For every \( H \in C^\infty_0([0,1] \times T^*M) \) there exist \( c_0 \in \mathbb{R} \) and \( \chi \in C^\infty_0(T^*M) \) such that \( \chi = 1 \) in a neighborhood of \( \bigcup_t \text{supp} H(t, \cdot) \), \( \chi \leq 1 \), and \( H^{c_0,\chi} := (H + c_0)\chi \in \mathcal{H}(M) \). Since \( H^{c_0,\chi} \) generates \( L_t \) and

\[ \max_x H^{c_0,\chi}(t,x) - \min_x H^{c_0,\chi}(t,x) \leq \max_x H(t,x) - \min_x H(t,x), \]

it follows that [4] is an equality.

Recall that for generic \( H, K \in C^\infty_0([0,1] \times T^*M) \) the homomorphism \( h_\ast : HF_\ast(H) \to HF_\ast(K) \) is induced by the homomorphism defined on the chain level as

\[ h : CF_\ast(H) \to CF_\ast(K), \ h(x) := \sum_{y \in CF(K)} n(x,y)y, \]

where \( n(x,y) \) is the number of solutions of

\[
\begin{cases}
\frac{\partial u}{\partial t} + J\left(\frac{\partial u}{\partial x} - X_H(u)\right) = 0, \\
u(\tau, 0), u(\tau, 1) \in O_M, \\
u(-\infty, t) = x(t), u(+\infty, t) = y(t).
\end{cases}
\]

Here \( \tilde{H}(\tau, t, x) \) is a generic path in \( C^\infty_0([0,1] \times T^*M) \), such that \( \tilde{H}(\tau, t, x) = H(t, x) \) for \( \tau \leq -1 \) and \( \tilde{H}(\tau, t, x) = K(t, x) \) for \( \tau \geq 1 \). Since \( A_H \) decreases along its "negative gradient lines" [2], it follows that \( j^\lambda_\ast \) commutes with \( h_\ast \). This fact, together with careful analysis of change of \( A_H \) along the trajectories [2] makes it possible to express effects of the homomorphism \( h_\ast \) on the level sets of the action functional and to prove that \( \rho \) is \( C^0 \) continuous. This extends the definition of \( \rho \)
to all (not necessarily generic) $H \in \mathcal{H}(M)$ (we refer the reader to [S] for details). Main properties of $\rho$ are summarized in the following

**Proposition 1.** The function $\rho : \mathcal{H}(M) \to \mathbb{R}$ satisfies

1. If $L := \phi^H_t(O_M) = \phi^K_t(O_M)$, then $\rho(H) = \rho(K)$; hence we can denote $\rho(H)$ by $\rho(L)$.
2. $\rho(L) \in \text{Spec}(L) := \{A_h(\phi^H_t \circ (\phi^H_t)^{-1}(x)) \mid x \in O_M \cap L\}$
3. 

\[-\int_0^1 \max_{x \in T^*M} (H(t,x) - K(t,x))dt \leq \rho(H) - \rho(K) \]

\[\leq -\int_0^1 \min_{x \in T^*M} (H(t,x) - K(t,x))dt.\]

In particular, $\rho$ is $C^0$-continuous and monotone, i.e. if $K \leq H$, then $\rho(H) \leq \rho(K)$.
4. $\rho(0) = 0$.
5. $\rho(\phi^H_1(O_M)) + \rho((\phi^H_t)^{-1}(O_M)) \leq d(O_M, \phi^H_1(O_M))$.
6. If $S : M \to \mathbb{R}$ is a smooth function, then $\rho(-\pi^*S) = \max S$, where $\pi : T^*M \to M$ is a canonical projection.

**Proof.** 1. is the contents of Theorem 8.1 in [S]. 2.-4. are contained in Theorem II in [S]. From 3. and 4. follows

\[(5) \quad \rho(\phi^H_1(O_M)) \leq -\int_0^1 \min_{x \in T^*M} H(t,x)dt.\]

Since $(\phi^H_t)^{-1} = \phi^\mathcal{T}_t$, where $\mathcal{T}(t,x) = -H(t, \phi^H_t(x))$, (5) gives

\[\rho((\phi^H_t)^{-1}(O_M)) \leq -\int_0^1 \min_{x \in T^*M} \mathcal{T}(t,x)dt \]

\[= -\int_0^1 \min_{x \in T^*M} \{-H(t,x)\}dt \]

\[= \int_0^1 \max_{x \in T^*M} H(t,x)dt.\]

Adding (5) and (6) we get

\[\rho(\phi^H_1(O_M)) + \rho((\phi^H_t)^{-1}(O_M)) \leq \text{length}(\{\phi^H_1(O_M)\}).\]

Taking $\inf H$ we get 5.

To prove 6., note that $CF_*(\pi^*S)$ consists of constant paths (critical points of $S$), and thus for $x \in CF_*(-\pi^*S)$

\[A_{-\pi^*S}(x) = S(x).\]

Let $x_+ \in M$ be such that $S(x_+) = \max S$. By the non-triviality of the cap action (see [3], or [9], [6] for similar arguments) for a generic $x \in M$ there exists a trajectory $\mathcal{J}$ connecting the generators of $CF_{\dim M}(-\pi^*S)$ and $CF_0(-\pi^*S)$ that contribute to the generators of $HF_{\dim M}(-\pi^*S)$ and $HF_0(-\pi^*S)$, where the grading comes from Maslov (or Morse) index (see [3]). Since $A_{-\pi^*S}$ (and thus $S$) increases along the gradient lines $\mathcal{J}$, by choosing $x \in S^{-1}(\max S - \epsilon, \max S)$ for $\epsilon$ small
enough, we see that if $\mu$ is a generator of $HF_{\dim M}(-\pi^*S)$, then
\[
\mu = k[x_+] + \sum_{x_+ \neq x, i \in CF_{\dim M}} k_i[x_i]
\]
for some integers $k, k_i$, where $k \neq 0$. Assume that $\lambda < S(x_+)$. Then $x_+ \notin CF_{\dim M}(-\pi^*S)$, so that for every $a \in HF_{\dim M}(-\pi^*S)$,
\[
j_{\lambda}^{-1}(a) = \sum_{x_+ \neq x, i \in CF_{\dim M}} l_i[x_i].
\]
Since $HF_{\dim M}(-\pi^*S)$ is a free module with one generator $\mu$, it must be $j_{\lambda}^{-1}(a) = c\mu$ for some integer $c$, i.e.
\[
ck[x_+] + \sum_{x_+ \neq x, i \in CF_{\dim M}} (ck_i - l_i)[x_i] = 0.
\]
Since
\[
CF_{\dim M}(-\pi^*S) \ni ck x_+ + \sum_{x_+ \neq x, i \in CF_{\dim M}} (ck_i - l_i)x_i \notin \text{Image}(\theta)
\]
and since $k \neq 0$, (7) is possible only if $c = 0$. Hence, $j_{\lambda}^{-1}$ is not surjective. \(\text{q.e.d.}\)

3. Flatness and geodesics

Denote by $F(M)$ the space of normalized smooth functions on $M$:
\[
F(M) := \{ S \in C^\infty(M) \mid \int_M S(q) dq = 0 \},
\]
where $dq$ is the Lebesgue measure induced by the Riemannian metric on $M$. Note that for every $C^1$-small Hamiltonian deformation $L$ of a zero section $O_M$ there is a unique $S \in F(M)$ such that $L = \text{Graph}(dS)$. Define a norm on $F(M)$ by
\[
\|S\| := \max_{q \in M} S - \min_{q \in M} S.
\]

**Theorem 2.** There exist $C^1$-neighborhoods $G$ of $O_M \in \mathcal{L}$ and $U$ of $0 \in F(M)$ such that the mapping
\[
\Phi : U \rightarrow G, \; L \mapsto \text{Graph}(dS)
\]
is an isometry.

**Proof.** $L := \text{Graph}(dS) \in G$ is generated by the time-independent Hamiltonian $-\pi^*S$ (to simplify notation we keep the same notation for $-\pi^*S$ multiplied by a cut-off function). Hence $\rho(L) = \rho(\phi_1^{-\pi^*S}(O_M)) = \rho(-\pi^*S)$. It is easy to see that $\rho((\phi_1^{-\pi^*S})^{-1}(O_M)) = \rho(\pi^*S)$. Therefore, by Proposition [4]
\[
\begin{align*}
\max S - \min S &= \max S + \max(-S) \\
&= \rho(\pi^*S) + \rho(-\pi^*S) \\
&\leq d(O_M, L) \\
&\leq \int_0^1 (\max S - \min S) dt \\
&= \max S - \min S.
\end{align*}
\]
Hence, \( d(O_M, L) = \|S\| \). The statement of the theorem follows from the invariance of \( d \):

\[
d(Graph(dS_0), Graph(dS_1)) = d(O_M, (\phi^{-1}_{0} \circ \phi^{-1}_{1})^* S_1 (O_M)) = d(O_M, \phi^*_{1} S_0 - \phi^*_{1} S_1) = d(O_M, Graph(d(S_1 - S_0))) = \|S_1 - S_0\|.
\]

q.e.d.

**Remark 3.** Theorem 2 complements the conclusion made at the end of Section 7 of [7] that, if \( c(\mu, L), c(1, L) \) are Viterbo’s invariants of \( L \) (see [10]), then \( c(\mu, L) - c(1, L) \leq d(O_M, L) \). From Theorem 2 it follows that the strict equality holds at least for \( L \) close to \( O_M \).

Now we prove several statements analogous to the ones in [1] (or Section 5.7 in [4]).

**Lemma 4 (Hamilton-Jacobi equation).** Let \( H(t, x) \) be a Hamiltonian generating \( L_t = Graph(dS_t) \). Then

\[
\frac{\partial S}{\partial t}(q) + H(t, dS_t(q)) = \text{const}.
\]

**Proof.** Let \( q = \pi(\phi^H_t(y)) \) for \( y \in M, s \in [0, 1] \). Let \((q, p)\) be the canonical coordinates around \( \phi^H_t(y) \). Denote \((q_t, p_t) = \phi^H_t(y)\) for \( t \) near \( s \). Then

\[
(q_t, p_t) = \left( q, \frac{\partial S}{\partial q}(q) \right).
\]

Differentiating with respect to \( t \) we get

\[
\left( \frac{dq}{dt}, \frac{dp}{dt} \right) = \left( \frac{dq}{dt}, \frac{\partial S}{\partial q} \frac{\partial^2 S}{\partial q^2} \right).
\]

On the other hand, differentiating \( H(t, dS(q)) := H(t, (q, \frac{\partial S}{\partial q})) \) with respect to \( q \) we obtain

\[
\frac{\partial}{\partial q} \left( H(t, q, \frac{\partial S}{\partial q}) \right) = \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p} \frac{\partial^2 S}{\partial q^2}.
\]

Applying Hamiltonian equations

\[
\begin{align*}
\frac{dq}{dt} &= \frac{\partial H}{\partial p}, \\
\frac{dp}{dt} &= -\frac{\partial H}{\partial q}
\end{align*}
\]

we obtain

\[
\frac{\partial}{\partial q} \left( H(t, dS(q)) + \frac{\partial S}{\partial t}(q) \right) = 0.
\]

q.e.d.

**Corollary 5.** If \( H(t, x) \) is a Hamiltonian generating \( L_t = Graph(dS_t) \), then

\[
\left\| \frac{\partial S}{\partial t} \right\| \leq \max_{x \in T^* M} H(t, x) - \min_{x \in T^* M} H(t, x)
\]

for every \( t \).
Proof. By Lemma 4

\[
\begin{align*}
\max_{x \in T^*M} H(t, x) & \geq \max_{q \in M} H(t, dS_t(q)) \\
& = c + \max_{q \in M} \left( -\frac{\partial S}{\partial t}(q) \right) \\
& = c - \min_{q \in M} \left( \frac{\partial S}{\partial t}(q) \right)
\end{align*}
\]

and

\[
\begin{align*}
\min_{x \in T^*M} H(t, x) & \leq \min_{q \in M} H(t, dS_t(q)) \\
& = c + \min_{q \in M} \left( -\frac{\partial S}{\partial t}(q) \right) \\
& = c - \max_{q \in M} \left( \frac{\partial S}{\partial t}(q) \right),
\end{align*}
\]

i.e.

\[
\begin{align*}
- \min_{x \in T^*M} H(t, x) & \geq -c + \max_{q \in M} \left( \frac{\partial S}{\partial t}(q) \right).
\end{align*}
\]

Adding (10) and (11) finishes the proof. \(q.e.d.\)

Corollary 6 (Compare [1], Proposition 3.3.A). If \(L_t = \text{Graph}(dS_t)\), then \(\frac{\partial S}{\partial t}\) is quasi-autonomous if and only if \(\{L_t\}\) is generated by a quasi-autonomous Hamiltonian.

Proof. Note that \(\{L_t\}\) is completely determined by values of \(H(t, x)\) for \(x\) near \(\cup_t L_t\). Therefore, according to the comment after (2), we can assume that

\[
\max_{x} H(t, x) = \max_{x \in \cup_t L_t} H(t, x), \quad \min_{x} H(t, x) = \min_{x \in \cup_t L_t} H(t, x).
\]

Assume first that \(\frac{\partial S}{\partial t}\) is quasi-autonomous, so that \(\max_{q \in M} \frac{\partial S}{\partial t} = \frac{\partial S}{\partial t}(q_+)\) and \(\min_{q \in M} \frac{\partial S}{\partial t} = \frac{\partial S}{\partial t}(q_-)\). Then

\[
0 = \frac{\partial}{\partial t} \frac{\partial S}{\partial q}(q_+) = \frac{\partial}{\partial t} \frac{\partial S}{\partial q}(q_-)
\]

and thus \(x_\pm := dS_t(q_\pm)\) does not depend on \(t\). Let \(H_t\) be a Hamiltonian generating \(\{L_t\}\). By Lemma 4, for \(x \in \cup_t L_t\)

\[
H(t, x) = c - \frac{\partial S}{\partial t}(\pi(x)) \leq c - \frac{\partial S}{\partial t}(q_-)
\]

Similarly, \(H(t, x) \geq H(t, x_-)\), i.e. \(H\) is quasi-autonomous.

Assume now that \(H(t, x)\) is a quasi-autonomous Hamiltonian, such that \(\phi_t^H(O_M) = L_t\). Let \(\max_x H(t, x) = H(t, x_+)\) and \(\min_x H(t, x) = H(t, x_-)\). Again, we can
assume that $x_\pm \in \bigcup L_t$. Let $q_\pm := \pi(x_\mp)$. Then by Lemma 3

$$\frac{\partial S}{\partial t}(q) = c - H(t, dS_t(q))$$

$$\leq c - H(t, x_-)$$

$$= \frac{\partial S}{\partial t}(q_+).$$

Similarly, $\frac{\partial S}{\partial t}(q) \geq \frac{\partial S}{\partial t}(q_-)$, hence $S$ is quasi-autonomous.

Remark 7. From the proof of Corollary 6 it follows that, if $H_t$ is a quasi-autonomous Hamiltonian generating $Graph(dS_t)$, then (after modifying $H$ away from $\bigcup L_t$ if necessary)

$$\max_{x \in T^* M} H(t, x) - \min_{x \in T^* M} H(t, x) = \max_{q \in M} \frac{\partial S}{\partial t}(q) - \min_{q \in M} \frac{\partial S}{\partial t}(q).$$

Theorem 8. A regular path $\{L_t\} \in \mathcal{L}$ is a geodesic if and only if it is generated by a locally quasi-autonomous Hamiltonian function.

Proof. Assume, without loss of generality, that $L_0 = O_M$. Choose $\epsilon > 0$ such that $L_t \in \mathcal{G}$ for $t \in (0, \epsilon)$. Let $S_t = \Phi^{-1}(L_t)$, where $\Phi$ is as in Theorem 2. $\{L_t\}_{0 \leq t \leq \epsilon}$ is a minimizing geodesic if and only if for every $\delta > 0$ there exists a Hamiltonian $H$ such that $\phi_t^H(O_M) = L_t$ and

$$\int_0^\epsilon (\max_x H(t, x) - \min_x H(t, x)) dt - \delta \leq d(O_M, L_\epsilon). \quad (12)$$

Since $\Phi$ is an isometry (Theorem 2)

$$d(O_M, L_\epsilon) = \|S_\epsilon\|$$

$$= \left\| \int_0^\epsilon \frac{\partial S}{\partial t} dt \right\|$$

$$\leq \int_0^\epsilon \left\| \frac{\partial S}{\partial t} \right\| dt$$

$$\leq \int_0^\epsilon (\max_x H(t, x) - \min_x H(t, x)) dt \quad (13)$$

(the last inequality follows from Corollary 5). Since $\delta$ in (12) is arbitrary, it follows from (12) and (13) that

$$\int_0^\epsilon \left\| \frac{\partial S}{\partial t} \right\| dt = \left\| \int_0^\epsilon \frac{\partial S}{\partial t} dt \right\|,$$

which according to (13) means that $\frac{\partial S}{\partial t}$ is quasi-autonomous. It follows from Corollary 6 that this is equivalent to $H(t, x)$ being quasi-autonomous. Vice versa, if $H$ (and thus $S$ as well) is quasi-autonomous, by (13) and Remark 7 both inequalities in (13) are equalities, and this gives

$$d(O_M, L_\epsilon) = \int_0^\epsilon (\max_x H(t, x) - \min_x H(t, x)) dt.$$

$q.e.d.$

Theorem 8 extends Theorem 1.3.D in [1] which states that a regular path in a group $Ham(\mathbb{R}^{2n})$ of compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ is a geodesic if and only if it is generated by a locally quasi-autonomous Hamiltonian function.
function. Indeed, a graph of every $\psi \in Ham(\mathbb{R}^{2n})$ is a Hamiltonian deformation of a diagonal $\Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. We can identify $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, dq \wedge dp - dQ \wedge dP)$ with $(T^*\Delta, -d(pdq))$ through a symplectic identification

$$(q, p, Q, P) \mapsto \left(\frac{q + Q}{2}, \frac{p + P}{2}, P - p, q - Q\right).$$

Since $\psi$ is compactly supported, the image of $\text{Graph}(\psi)$ coincides with the zero section of $T^*M$ outside a compact set. Thus, after adding the fiber at infinity, $\text{Graph}(\psi)$ can be considered as a Hamiltonian deformation of the zero section in $T^*S^{2n}$. Hence, Theorem 2 and Theorem 8 extend the analogous results in [1].

REFERENCES


