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INTEGER SOLUTIONS TO INTERVAL LINEAR EQUATIONS AND UNIQUE MEASUREMENT

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Abstract. Every system of n linearly independent homogeneous linear equations in $n+1$ unknowns with coefficients in $\{1,0,-1\}$ has a unique (up to multiplication by -1) non-zero solution vector $d=(d_1,d_2,\dots,d_{n+1})$ in which the d_j's are integers with no common divisor greater than 1. It is known that, for large n, $|\sum d_j|$ can be arbitrarily greater than 2^n. We prove that if every equation, written as $\sum_A x_i - \sum_B x_i = 0$, is such that A and B are intervals of contiguous indices, then $|\sum d_j| \le 2^n$. This confirms conjectures of the author and Fred Roberts that arose in the theory of unique finite measurement.

1. Introduction

Let S_n be a set of n linearly independent homogeneous linear equations in $n+1$ unknowns $x_1,\ x_2,\dots,x_{n+1}$ with coefficients in $\{1,0,-1\}$. Then S_n has a non-zero solution $d=(d_1,d_2,\dots,d_{n+1})$ that is unique up to multiplication by a non-zero constant [7, pp. 250–251] and, because all coefficients are rational, there are solutions in which every d_i is an integer. We refer to a non-zero integer solution d in which the d_i have no common integer divisor greater than 1 as a *minimal integer solution*. S_n has exactly two minimal integer solutions: they are the negatives of each other.

Our aim is to prove that when S_n is a so-called interval system, it has a non-zero integer solution in which $|\sum d_j| \le 2^n$. This result was motivated by research in the theory of finite measurement [2, 3, 4] and confirms conjectures that arose in that context. We will illustrate our main result, Theorem 1, with examples from the measurement-theory context.

Each equation in S_n has the form

$$\sum_{j\in A} x_j - \sum_{j\in B} x_j = 0\ , \tag{1}$$

where $A,B \subseteq \{1,2,\dots,n+1\}$, $A\cap B=\emptyset$ and $A\cup B\neq\emptyset$. We say that S_n is an *interval system* if the indices $1,2,\dots,n+1$ can be arranged singly around a circle so that every non-empty A and B of (1) is an interval or arc of contiguous indices on the circle. Figure 1 illustrates equations of an interval system for $n=8$ under the natural clockwise order of $1,2,\dots,9$.

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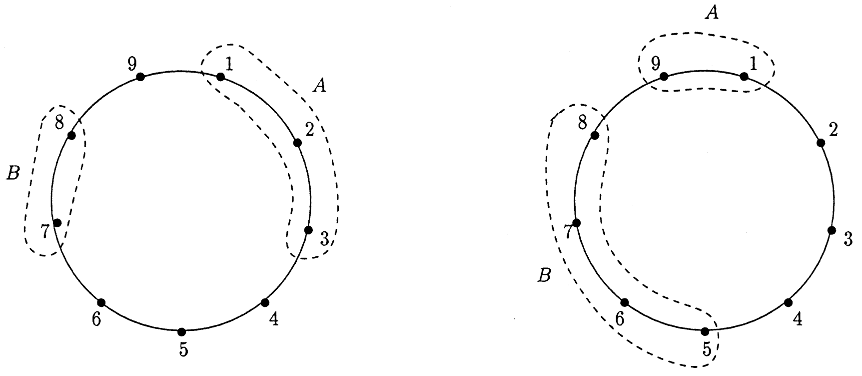

$x_1 + x_2 + x_3 = x_7 + x_8$ $\qquad$ $x_9 + x_1 = x_5 + x_6 + x_7 + x_8$

FIGURE 1. Interval system equations for $n = 8$.

Theorem 1. *Every interval system* S_n *has a non-zero integer solution* $d = (d_1, d_2, \ldots, d_{n+1})$ *with*

$$\left| \sum_{j=1}^{n+1} d_j \right| \leq 2^n \ . \tag{2}$$

The theorem is proved in the next section.

Equations like (1) arise in the theory of measurement [1, 6, 8] from qualitative equality comparisons in assessments of subjective probability, subset evaluation, and comparable preference differences. Examples include equally likely events A and B, equally valuable subsets A and B, and, for a finite set $\{1, 2, \ldots\}$ of items that increase in preference in the order $1, 2, \ldots$, qualitatively equal differences in preference between i and $j > i$, and between k and $l > k$. The finite-measurement references [2, 3, 4] focus on qualitative equality comparisons that translate into S_n sets that yield additive numerical subjective probabilities or item utilities that are unique up to multiplication by a non-zero constant. Most of this work is concerned with minimal integer solutions in which all d_j are positive, so we will illustrate Theorem 1 in the positive-solution mode.

The simplest interval system is $\{x_1 = x_2, x_2 = x_3, \ldots, x_n = x_{n+1}\}$ with solution $d = (1, 1, \ldots, 1)$. The first two of the following three examples of interval systems attain the upper bound of 2^n for $\sum d_j$.

1. $\{x_2 = x_1,\ x_3 = x_1 + x_2,\ x_4 = x_1 + x_2 + x_3, \ldots, x_{n+1} = x_1 + x_2 + \cdots + x_n\}$. The positive minimal integer solution is $d = (1, 1, 2, 4, \ldots, 2^{n-1})$ with $\sum d_j = 2^n$.

2. $d = (5, 5, 6, 4, 4, 8)$ with $\sum d_j = 2^5$ is a solution to

$$\begin{aligned} &x_1 = x_2 \\ &x_1 + x_2 = x_3 + x_4 \\ &x_1 + x_2 + x_3 = x_4 + x_5 + x_6 \\ &x_4 = x_5 \\ &x_4 + x_5 = x_6 \ . \end{aligned}$$

These equations and those of example **1** have the feature that A and B of (1) are adjacent intervals, or that $A \cup B$ itself is an interval under the natural order of $1, 2, \ldots, n + 1$.

3. $d = (7,6,1,2,3,9)$ with $\sum d_j = 28 < 2^5$ is the positive minimal integer solution of

$$\begin{aligned}
&x_1 = x_2 + x_3\\
&x_1 + x_2 + x_3 = x_4 + x_5 + x_6\\
&x_2 = x_3 + x_4 + x_5\\
&x_2 + x_3 + x_4 = x_6\\
&x_3 + x_4 = x_5\ .
\end{aligned}$$

A and B ({2,3,4} and {6}) are not adjacent in the fourth equation, and there is no way to reorder the indices around a circle to yield an equivalent S_5 that is an interval system in which every $A \cup B$ is an interval.

Violations of (2) occur with S_n's that are **not** interval systems.

4. The simplest case is $d = (4,1,2,3)$ for

$$\begin{aligned}
&x_1 = x_2 + x_4\\
&x_1 + x_2 = x_3 + x_4\\
&x_2 + x_3 = x_4\ .
\end{aligned}$$

The first equation departs from the interval format under the natural order 1, 2, 3, 4, and there is no way to reorder 1, 2, 3 and 4 around a circle so that the three equations form an interval system.

5. The largest $\sum d_j$ presently known [3] for a positive minimal integer solution at $n = 6$ in which one d_j equals 1 is $d = (1,5,14,18,36,44,74)$ for

$$\begin{aligned}
&x_1 + x_4 = x_2 + x_3\\
&x_1 + x_5 = x_2 + x_3 + x_4\\
&x_1 + x_3 + x_6 = x_2 + x_4 + x_5\\
&x_1 + x_2 + x_6 = x_3 + x_5\\
&x_1 + x_2 + x_7 = x_5 + x_6\\
&x_1 + x_2 + x_3 + x_4 + x_5 = x_7\ .
\end{aligned}$$

This has $\sum d_j = 192$ in contrast to the upper bound of 64 for an interval system.

It is proved in [3] that

$$2^n(\min d_j)/\sum d_j \to 0 \quad \text{as} \quad n \to \infty$$

for some sequence of S_n's with strictly positive d solutions. However, we could find no *interval system* with a strictly positive d solution in which $2^n(\min d_j)/\sum d_j < 1$, and conjectured [2, 4] that all such cases have

$$\sum d_j/(\min d_j) \leq 2^n\ .$$

A stronger conjecture asserts that, regardless of the magnitude of $\min d_j$, every interval system with a strictly positive minimal integer solution d satisfies $\sum d_j \leq 2^n$. This is now confirmed by Theorem 1.

2. PROOF OF THEOREM 1

We assume that S_n is an interval system under the natural clockwise order of $1, 2, \ldots, n+1$. Let C be the $n \times (n+1)$ coefficient matrix of S_n with equations as in (1), let C_j be the j^{th} column of C, so $C = [C_1\ C_2 \cdots C_{n+1}]$, and let C^{-j} be the $n \times n$ matrix obtained by removing C_j from C. It follows from [5, pp. 22–23] that one non-zero integer solution of S_n is

$$d_j = (-1)^{j+1}|C^{-j}|, \quad j = 1, 2, \ldots, n+1 .$$

We show that this implies (2).

Elementary operations on determinants yield

$$\sum_{j=1}^{n+1} d_j = |C_2 - C_1 \; C_3 - C_2 \; \cdots \; C_{n+1} - C_n| .$$

For example,

$$\begin{aligned} \sum d_j &= |C_2C_3 \cdots C_{n+1}| - |C_1C_3 \cdots C_{n+1}| + |C_1C_2C_4 \cdots C_{n+1}| \\ &\quad -|C_1C_2C_3C_5 \cdots C_{n+1}| + \cdots \\ &= |C_2 - C_1 \; C_3 \cdots C_{n+1}| - |C_2 - C_1 \; C_2C_4 \cdots C_{n+1}| \\ &\quad +|C_2 - C_1 \; C_2C_3C_5 \cdots C_{n+1}| - \cdots \\ &= |C_2 - C_1 \; C_3 - C_2 \; C_4 \cdots C_{n+1}| - |C_2 - C_1 \; C_3 - C_2 \; C_3 \cdots C_{n+1}| + \cdots \\ &= |C_2 - C_1 \; C_3 - C_2 \; C_4 - C_3 \; C_5 \cdots C_{n+1}| - \cdots . \end{aligned}$$

Let $\Delta C = [C_2 - C_1 \; C_3 - C_2 \; \cdots \; C_{n+1} - C_n]$.

Because each row of C has a block of 1's and/or a block of -1's and 0's elsewhere, with possible circular wrap-around, the left-to-right non-zero entries in every row of ΔC is a member of $R \cup (-R)$, where, with $c^* = -c$,

$$R = \{22^*, 21^*1^*, 12^*1, 112^*, 21^*, 12^*, 2, 111^*1^*, 11^*1^*1, 111^*, 11^*1^*, 11, 11^*, 1\} .$$

We remark that 11^*11^* and 11^*1 and their negatives can be added to $R \cup (-R)$ without affecting ensuing conclusions.

It follows that every row of ΔC can be written as the sum of one or two vectors in $\{1, 0, -1\}^n$ whose non-zero entries are 11^*, 1^*1, 1 or 1^*. For example,

$$\begin{aligned} 00202^*0 &= 00101^*0 + 00101^*0 \\ 201^*01^*0 &= 101^*000 + 10001^*0 \\ 01^*0110 &= 01^*0100 + 000010 , \end{aligned}$$

and so forth. Then, by sequential row splits [5, p. 10], $|\Delta C|$ equals the sum of 2^n or fewer determinants in which every row's non-zero entries are 11^*, 1^*1, 1 or 1^*. A straightforward induction on k for $k \times k$ matrices with such rows shows that each of the latter determinants is in $\{1, 0, -1\}$. Because $\sum d_j = |\Delta C|$, it follows that $|\sum d_j| \leq 2^n$.

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