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# INTEGER SOLUTIONS TO INTERVAL LINEAR EQUATIONS AND UNIQUE MEASUREMENT

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Abstract. Every system of $n$ linearly independent homogeneous linear equations in $n+1$ unknowns with coefficients in $\{1, 0, -1\}$ has a unique (up to multiplication by $-1$) non-zero solution vector $d = (d_1, d_2, \dots, d_{n+1})$ in which the $d_j$'s are integers with no common divisor greater than 1. It is known that, for large $n$, $|\sum d_j|$ can be arbitrarily greater than $2^n$. We prove that if every equation, written as $\sum_A x_i - \sum_B x_i = 0$, is such that $A$ and $B$ are intervals of contiguous indices, then $|\sum d_j| \leq 2^n$. This confirms conjectures of the author and Fred Roberts that arose in the theory of unique finite measurement.

## 1. Introduction

Let $S_n$ be a set of $n$ linearly independent homogeneous linear equations in $n+1$ unknowns $x_1, x_2, \dots, x_{n+1}$ with coefficients in $\{1, 0, -1\}$. Then $S_n$ has a non-zero solution $d = (d_1, d_2, \dots, d_{n+1})$ that is unique up to multiplication by a non-zero constant [7, pp. 250–251] and, because all coefficients are rational, there are solutions in which every $d_i$ is an integer. We refer to a non-zero integer solution $d$ in which the $d_i$ have no common integer divisor greater than 1 as a *minimal integer solution*. $S_n$ has exactly two minimal integer solutions: they are the negatives of each other.

Our aim is to prove that when $S_n$ is a so-called interval system, it has a non-zero integer solution in which $|\sum d_j| \leq 2^n$. This result was motivated by research in the theory of finite measurement [2, 3, 4] and confirms conjectures that arose in that context. We will illustrate our main result, Theorem 1, with examples from the measurement-theory context.

Each equation in $S_n$ has the form

$$\sum_{j \in A} x_j - \sum_{j \in B} x_j = 0 \ , \tag{1}$$

where $A, B \subseteq \{1, 2, \dots, n+1\}$, $A \cap B = \emptyset$ and $A \cup B \neq \emptyset$. We say that $S_n$ is an *interval system* if the indices $1, 2, \dots, n+1$ can be arranged singly around a circle so that every non-empty $A$ and $B$ of (1) is an interval or arc of contiguous indices on the circle. Figure 1 illustrates equations of an interval system for $n = 8$ under the natural clockwise order of $1, 2, \dots, 9$.

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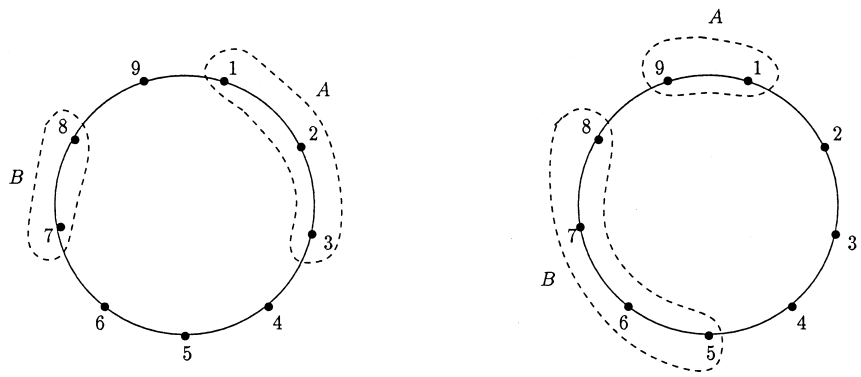


$x_1 + x_2 + x_3 = x_7 + x_8$ $\qquad$ $x_9 + x_1 = x_5 + x_6 + x_7 + x_8$

FIGURE 1. Interval system equations for $n = 8$.

**Theorem 1.** *Every interval system $S_n$ has a non-zero integer solution $d = (d_1, d_2, \ldots, d_{n+1})$ with*

$$\left| \sum_{j=1}^{n+1} d_j \right| \leq 2^n \ . \tag{2}$$

The theorem is proved in the next section.

Equations like (1) arise in the theory of measurement [1, 6, 8] from qualitative equality comparisons in assessments of subjective probability, subset evaluation, and comparable preference differences. Examples include equally likely events $A$ and $B$, equally valuable subsets $A$ and $B$, and, for a finite set $\{1, 2, \ldots\}$ of items that increase in preference in the order $1, 2, \ldots$, qualitatively equal differences in preference between $i$ and $j > i$, and between $k$ and $l > k$. The finite-measurement references [2, 3, 4] focus on qualitative equality comparisons that translate into $S_n$ sets that yield additive numerical subjective probabilities or item utilities that are unique up to multiplication by a non-zero constant. Most of this work is concerned with minimal integer solutions in which all $d_j$ are positive, so we will illustrate Theorem 1 in the positive-solution mode.

The simplest interval system is $\{x_1 = x_2, x_2 = x_3, \ldots, x_n = x_{n+1}\}$ with solution $d = (1, 1, \ldots, 1)$. The first two of the following three examples of interval systems attain the upper bound of $2^n$ for $\sum d_j$.

**1.** $\{x_2 = x_1,\ x_3 = x_1 + x_2,\ x_4 = x_1 + x_2 + x_3, \ldots, x_{n+1} = x_1 + x_2 + \cdots + x_n\}$. The positive minimal integer solution is $d = (1, 1, 2, 4, \ldots, 2^{n-1})$ with $\sum d_j = 2^n$.

**2.** $d = (5, 5, 6, 4, 4, 8)$ with $\sum d_j = 2^5$ is a solution to

$$\begin{aligned} &x_1 = x_2 \\ &x_1 + x_2 = x_3 + x_4 \\ &x_1 + x_2 + x_3 = x_4 + x_5 + x_6 \\ &x_4 = x_5 \\ &x_4 + x_5 = x_6 \ . \end{aligned}$$

These equations and those of example **1** have the feature that $A$ and $B$ of (1) are adjacent intervals, or that $A \cup B$ itself is an interval under the natural order of $1, 2, \ldots, n + 1$.

**3.** $d = (7, 6, 1, 2, 3, 9)$ with $\sum d_j = 28 < 2^5$ is the positive minimal integer solution of

$$\begin{aligned}
&x_1 = x_2 + x_3\\
&x_1 + x_2 + x_3 = x_4 + x_5 + x_6\\
&x_2 = x_3 + x_4 + x_5\\
&x_2 + x_3 + x_4 = x_6\\
&x_3 + x_4 = x_5 \ .
\end{aligned}$$

$A$ and $B$ ({2,3,4} and {6}) are not adjacent in the fourth equation, and there is no way to reorder the indices around a circle to yield an equivalent $S_5$ that is an interval system in which every $A \cup B$ is an interval.

Violations of (2) occur with $S_n$'s that are **not** interval systems.

**4.** The simplest case is $d = (4, 1, 2, 3)$ for

$$\begin{aligned}
&x_1 = x_2 + x_4\\
&x_1 + x_2 = x_3 + x_4\\
&x_2 + x_3 = x_4 \ .
\end{aligned}$$

The first equation departs from the interval format under the natural order 1, 2, 3, 4, and there is no way to reorder 1, 2, 3 and 4 around a circle so that the three equations form an interval system.

**5.** The largest $\sum d_j$ presently known [3] for a positive minimal integer solution at $n = 6$ in which one $d_j$ equals 1 is $d = (1, 5, 14, 18, 36, 44, 74)$ for

$$\begin{aligned}
&x_1 + x_4 = x_2 + x_3\\
&x_1 + x_5 = x_2 + x_3 + x_4\\
&x_1 + x_3 + x_6 = x_2 + x_4 + x_5\\
&x_1 + x_2 + x_6 = x_3 + x_5\\
&x_1 + x_2 + x_7 = x_5 + x_6\\
&x_1 + x_2 + x_3 + x_4 + x_5 = x_7 \ .
\end{aligned}$$

This has $\sum d_j = 192$ in contrast to the upper bound of 64 for an interval system.

It is proved in [3] that

$$2^n(\min d_j)/\sum d_j \to 0 \quad \text{as} \quad n \to \infty$$

for some sequence of $S_n$'s with strictly positive $d$ solutions. However, we could find no *interval system* with a strictly positive $d$ solution in which $2^n(\min d_j)/\sum d_j < 1$, and conjectured [2, 4] that all such cases have

$$\sum d_j/(\min d_j) \le 2^n \ .$$

A stronger conjecture asserts that, regardless of the magnitude of $\min d_j$, every interval system with a strictly positive minimal integer solution $d$ satisfies $\sum d_j \le 2^n$. This is now confirmed by Theorem 1.

## 2. Proof of Theorem 1

We assume that $S_n$ is an interval system under the natural clockwise order of $1, 2, \ldots, n+1$. Let $C$ be the $n \times (n+1)$ coefficient matrix of $S_n$ with equations as in (1), let $C_j$ be the $j^{\text{th}}$ column of $C$, so $C = [C_1\ C_2 \cdots C_{n+1}]$, and let $C^{-j}$ be the $n \times n$ matrix obtained by removing $C_j$ from $C$. It follows from [5, pp. 22–23] that

one non-zero integer solution of $S_n$ is

$$d_j = (-1)^{j+1}|C^{-j}|, \quad j = 1, 2, \ldots, n+1 \ .$$

We show that this implies (2).

Elementary operations on determinants yield

$$\sum_{j=1}^{n+1} d_j = |C_2 - C_1 \ C_3 - C_2 \ \cdots \ C_{n+1} - C_n| \ .$$

For example,

$$\begin{aligned}
\sum d_j &= |C_2C_3\cdots C_{n+1}| - |C_1C_3\cdots C_{n+1}| + |C_1C_2C_4\cdots C_{n+1}| \\
&\quad -|C_1C_2C_3C_5\cdots C_{n+1}| + \cdots \\
&= |C_2 - C_1 \ C_3\cdots C_{n+1}| - |C_2 - C_1 \ C_2C_4\cdots C_{n+1}| \\
&\quad +|C_2 - C_1 \ C_2C_3C_5\cdots C_{n+1}| - \cdots \\
&= |C_2 - C_1 \ C_3 - C_2 \ C_4\cdots C_{n+1}| - |C_2 - C_1 \ C_3 - C_2 \ C_3\cdots C_{n+1}| + \cdots \\
&= |C_2 - C_1 \ C_3 - C_2 \ C_4 - C_3 \ C_5\cdots C_{n+1}| - \cdots \ .
\end{aligned}$$

Let $\Delta C = [C_2 - C_1 \ C_3 - C_2 \ \cdots \ C_{n+1} - C_n]$.

Because each row of $C$ has a block of 1's and/or a block of $-1$'s and 0's elsewhere, with possible circular wrap-around, the left-to-right non-zero entries in every row of $\Delta C$ is a member of $R \cup (-R)$, where, with $c^* = -c$,

$$R = \{22^*, 21^*1^*, 12^*1, 112^*, 21^*, 12^*, 2, 111^*1^*, 11^*1^*1, 111^*, 11^*1^*, 11, 11^*, 1\} \ .$$

We remark that $11^*11^*$ and $11^*1$ and their negatives can be added to $R \cup (-R)$ without affecting ensuing conclusions.

It follows that every row of $\Delta C$ can be written as the sum of one or two vectors in $\{1, 0, -1\}^n$ whose non-zero entries are $11^*$, $1^*1$, 1 or $1^*$. For example,

$$\begin{aligned}
00202^*0 &= 00101^*0 + 00101^*0 \\
201^*01^*0 &= 101^*000 + 10001^*0 \\
01^*0110 &= 01^*0100 + 000010 \ ,
\end{aligned}$$

and so forth. Then, by sequential row splits [5, p. 10], $|\Delta C|$ equals the sum of $2^n$ or fewer determinants in which every row's non-zero entries are $11^*$, $1^*1$, 1 or $1^*$. A straightforward induction on $k$ for $k \times k$ matrices with such rows shows that each of the latter determinants is in $\{1, 0, -1\}$. Because $\sum d_j = |\Delta C|$, it follows that $|\sum d_j| \leq 2^n$.

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