INTEGER SOLUTIONS TO INTERVAL LINEAR EQUATIONS AND UNIQUE MEASUREMENT

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Abstract. Every system of \( n \) linearly independent homogeneous linear equations in \( n + 1 \) unknowns with coefficients in \( \{1, 0, -1\} \) has a unique (up to multiplication by \(-1\)) non-zero solution vector \( d = (d_1, d_2, \ldots, d_{n+1}) \) in which the \( d_j \)'s are integers with no common divisor greater than 1. It is known that, for large \( n \), \( |\sum d_j| \) can be arbitrarily greater than \( 2^n \). We prove that if every equation, written as \( \sum_A x_i - \sum_B x_i = 0 \), is such that \( A \) and \( B \) are intervals of contiguous indices, then \( |\sum d_j| \leq 2^n \). This confirms conjectures of the author and Fred Roberts that arose in the theory of unique finite measurement.

1. Introduction

Let \( S_n \) be a set of \( n \) linearly independent homogeneous linear equations in \( n + 1 \) unknowns \( x_1, x_2, \ldots, x_{n+1} \) with coefficients in \( \{1, 0, -1\} \). Then \( S_n \) has a non-zero solution \( d = (d_1, d_2, \ldots, d_{n+1}) \) that is unique up to multiplication by a non-zero constant [7, pp. 250–251] and, because all coefficients are rational, there are solutions in which every \( d_i \) is an integer. We refer to a non-zero integer solution \( d \) in which the \( d_i \) have no common integer divisor greater than 1 as a minimal integer solution. \( S_n \) has exactly two minimal integer solutions: they are the negatives of each other.

Our aim is to prove that when \( S_n \) is a so-called interval system, it has a non-zero integer solution in which \( |\sum d_j| \leq 2^n \). This result was motivated by research in the theory of finite measurement [2,3,4] and confirms conjectures that arose in that context. We will illustrate our main result, Theorem 1, with examples from the measurement-theory context.

Each equation in \( S_n \) has the form
\[
\sum_{j \in A} x_j - \sum_{j \in B} x_j = 0,
\]
where \( A, B \subseteq \{1, 2, \ldots, n + 1\} \), \( A \cap B = \emptyset \) and \( A \cup B \neq \emptyset \). We say that \( S_n \) is an interval system if the indices \( 1, 2, \ldots, n + 1 \) can be arranged singly around a circle so that every non-empty \( A \) and \( B \) of \( \{1\} \) is an interval or arc of contiguous indices on the circle. Figure 1 illustrates equations of an interval system for \( n = 8 \) under the natural clockwise order of 1, 2, \ldots, 9.
Theorem 1. Every interval system $S_n$ has a non-zero integer solution $d = (d_1, d_2, \ldots, d_{n+1})$ with

$$
\left| \sum_{j=1}^{n+1} d_j \right| \leq 2^n .
$$

The theorem is proved in the next section.

Equations like (1) arise in the theory of measurement [1, 6, 8] from qualitative equality comparisons in assessments of subjective probability, subset evaluation, and comparable preference differences. Examples include equally likely events $A$ and $B$, equally valuable subsets $A$ and $B$, and, for a finite set $\{1, 2, \ldots\}$ of items that increase in preference in the order $1, 2, \ldots$ qualitatively equal differences in preference between $i$ and $j > i$, and between $k$ and $l > k$. The finite-measurement references [2, 3, 4] focus on qualitative equality comparisons that translate into $S_n$ sets that yield additive numerical subjective probabilities or item utilities that are unique up to multiplication by a non-zero constant. Most of this work is concerned with minimal integer solutions in which all $d_j$ are positive, so we will illustrate Theorem 1 in the positive-solution mode.

The simplest interval system is $\{x_1 = x_2, x_2 = x_3, \ldots, x_n = x_{n+1}\}$ with solution $d = (1, 1, \ldots, 1)$. The first two of the following three examples of interval systems attain the upper bound of $2^n$ for $\sum d_j$.

1. $\{x_2 = x_1, x_3 = x_1 + x_2, x_4 = x_1 + x_2 + x_3, \ldots, x_{n+1} = x_1 + x_2 + \cdots + x_n\}$.
2. $d = (5, 5, 6, 4, 4, 8)$ with $\sum d_j = 2^5$ is a solution to

$$
\begin{align*}
x_1 &= x_2, \\
x_1 + x_2 &= x_3 + x_4, \\
x_1 + x_2 + x_3 &= x_4 + x_5 + x_6, \\
x_4 &= x_5, \\
x_4 + x_5 &= x_6 .
\end{align*}
$$

These equations and those of example 1 have the feature that $A$ and $B$ of (1) are adjacent intervals, or that $A \cup B$ itself is an interval under the natural order of $1, 2, \ldots, n + 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{interval_system_equations.png}
\caption{Interval system equations for $n = 8$.}
\end{figure}
3. \( d = (7, 6, 1, 2, 3, 9) \) with \( \sum d_j = 28 < 2^5 \) is the positive minimal integer solution of
\[
x_1 = x_2 + x_3 \\
x_1 + x_2 + x_3 = x_4 + x_5 + x_6 \\
x_2 = x_3 + x_4 + x_5 \\
x_2 + x_3 + x_4 = x_6 \\
x_3 + x_4 = x_5 .
\]

A and B (\( \{2, 3, 4\} \) and \( \{6\} \)) are not adjacent in the fourth equation, and there is no way to reorder the indices around a circle to yield an equivalent \( S_5 \) that is an interval system in which every \( A \cup B \) is an interval.

Violations of (2) occur with \( S_n \)'s that are not interval systems.

4. The simplest case is \( d = (4, 1, 2, 3) \) for
\[
x_1 = x_2 + x_4 \\
x_1 + x_2 = x_3 + x_4 \\
x_2 + x_3 = x_4 .
\]
The first equation departs from the interval format under the natural order 1, 2, 3, 4, and there is no way to reorder 1, 2, 3 and 4 around a circle so that the three equations form an interval system.

5. The largest \( \sum d_j \) presently known [3] for a positive minimal integer solution at \( n = 6 \) in which one \( d_j \) equals 1 is \( d = (1, 5, 14, 18, 36, 44, 74) \) for
\[
x_1 + x_4 = x_2 + x_3 \\
x_1 + x_5 = x_2 + x_3 + x_4 \\
x_1 + x_3 + x_6 = x_2 + x_4 + x_5 \\
x_1 + x_2 + x_6 = x_3 + x_5 \\
x_1 + x_2 + x_7 = x_5 + x_6 \\
x_1 + x_2 + x_8 + x_4 + x_5 = x_7 .
\]
This has \( \sum d_j = 192 \) in contrast to the upper bound of 64 for an interval system.

It is proved in [3] that
\[
2^n (\min d_j)/\sum d_j \to 0 \quad \text{as} \quad n \to \infty
\]
for some sequence of \( S_n \)'s with strictly positive \( d \) solutions. However, we could find no interval system with a strictly positive \( d \) solution in which \( 2^n (\min d_j)/\sum d_j < 1 \), and conjectured [2, 4] that all such cases have
\[
\sum d_j/(\min d_j) \leq 2^n .
\]
A stronger conjecture asserts that, regardless of the magnitude of \( \min d_j \), every interval system with a strictly positive minimal integer solution \( d \) satisfies \( \sum d_j \leq 2^n \). This is now confirmed by Theorem 1.

2. Proof of Theorem 1

We assume that \( S_n \) is an interval system under the natural clockwise order of 1, 2, \ldots, \( n + 1 \). Let \( C \) be the \( n \times (n + 1) \) coefficient matrix of \( S_n \) with equations as in (1), let \( C_j \) be the \( j^{th} \) column of \( C \), so \( C = [C_1 C_2 \cdots C_{n+1}] \), and let \( C^{-j} \) be the \( n \times n \) matrix obtained by removing \( C_j \) from \( C \). It follows from [3] pp. 22–23 that
one non-zero integer solution of $S_n$ is
\[ d_j = (-1)^{j+1}|C^{-j}|, \quad j = 1, 2, \ldots, n + 1. \]
We show that this implies (2).

Elementary operations on determinants yield
\[ \sum_{j=1}^{n+1} d_j = |C_2 - C_1 C_3 - C_2 \cdots C_{n+1} - C_n|. \]
For example,
\[
\sum d_j = |C_2 C_3 \cdots C_{n+1}| - |C_1 C_3 \cdots C_{n+1}| + |C_1 C_2 C_4 \cdots C_{n+1}| \\
- |C_1 C_2 C_3 C_5 \cdots C_{n+1}| + \cdots \\
= |C_2 - C_1 C_3 \cdots C_{n+1}| - |C_2 - C_1 C_2 C_4 \cdots C_{n+1}| \\
+ |C_2 - C_1 C_2 C_3 C_5 \cdots C_{n+1}| - \cdots \\
= |C_2 - C_1 C_3 - C_2 C_4 \cdots C_{n+1}| - |C_2 - C_1 C_3 - C_2 C_3 \cdots C_{n+1}| + \cdots \\
= |C_2 - C_1 C_3 - C_2 C_4 - C_3 C_5 \cdots C_{n+1}| - \cdots.
\]
Let $\Delta C = [C_2 - C_1 C_3 - C_2 \cdots C_{n+1} - C_n]$.

Because each row of $C$ has a block of 1’s and/or a block of -1’s elsewhere, with possible circular wrap-around, the left-to-right non-zero entries in every row of $\Delta C$ is a member of $R \cup (-R)$, where, with $c^* = -c$,
\[ R = \{2^2, 21^1, 12^2, 11^2, 21^*, 12^*, 2, 111^*1^*, 11^11^1, 111^*, 11^11^*, 11^*, 1\}. \]
We remark that $11^*11^*$ and $11^11^*$ and their negatives can be added to $R \cup (-R)$ without affecting ensuing conclusions.

It follows that every row of $\Delta C$ can be written as the sum of one or two vectors in $\{1, 0, -1\}^n$ whose non-zero entries are $11^*$, $1^*1^*$, 1 or 1*. For example,
\[
00202^*0 = 00101^*0 + 00101^*0 \\
201^*01^*0 = 101^*000 + 10001^*0 \\
01^*0110 = 01^*0100 + 000010,
\]
and so forth. Then, by sequential row splits [5, p. 10], $|\Delta C|$ equals the sum of $2^n$ or fewer determinants in which every row’s non-zero entries are $11^*$, $1^*1^*$, 1 or 1*. A straightforward induction on $k$ for $k \times k$ matrices with such rows shows that each of the latter determinants is in $\{1, 0, -1\}$. Because $\sum d_j = |\Delta C|$, it follows that $|\sum d_j| \leq 2^n$.

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REFERENCES


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