NON-INVERTIBILITY OF CERTAIN ALMOST MATHIEU OPERATORS

R. BALASUBRAMANIAN, S. H. KULKARNI, AND R. RADHA

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Abstract. It is shown that the almost Mathieu operators of the type
\[ T(e_n) = e_{n-1} + \lambda \sin(2nr)e_n + e_{n+1} \]
where \( \lambda \) is real and \( r \) is a rational multiple of \( \pi \) and \( \{ e_n : n = 1, 2, 3, \ldots \} \), an orthonormal basis for a Hilbert space, is not invertible.

Let \( H \) be a Hilbert space with an orthonormal basis \( \{ e_n : n = 1, 2, 3, \ldots \} \). An important class of tridiagonal operators used in mathematical physics are almost Mathieu operators which are defined by
\[ T(e_n) = e_{n-1} + \lambda \cos(2n\pi \alpha + \theta)e_n + e_{n+1}, \]
where \( \alpha, \lambda, \theta \) are real. Certain questions regarding the Lebesgue measure of the spectra of such operators seem to have received a good deal of attention in the literature. (See [1], [2], [4].) However, the question of invertibility of such operators seems to be unexplored. In this note we prove that the almost Mathieu operators of the type
\[ T(e_n) = e_{n-1} + \lambda \sin(2nr)e_n + e_{n+1}, \]
are not invertible. Since every separable Hilbert space is isometrically isomorphic to \( \ell^2 \), the main theorem is proved for operators on \( \ell^2 \).

Theorem 0.1. Let \( V \) be an infinite tridiagonal matrix whose diagonal elements are \( d_1, d_2, \ldots, d_m, 0, -d_m, \ldots, -d_1, 0 \) repeated in the same order and off diagonal entries are 1. Then \( V \) defines a bounded linear operator on \( \ell^2 \) and \( V \) is not invertible.

Proof. That \( V \) defines a bounded linear operator on \( \ell^2 \) is straightforward. To show that \( V \) is not invertible, we prove that \( V \) is not onto. In particular we aim to show that \( e_1 \) is not in the range of \( V \). Let \( x = (\alpha_1, \alpha_2, \ldots) \in \ell^2 \) such that \( Vx = (1, 0, 0, \ldots) \). Then \( \alpha_1 d_1 + \alpha_2 = 1 \) and
\[ \alpha_{n+1} + \alpha_n \lambda_n + \alpha_{n+1} = 0, \quad n = 1, 2, 3, \ldots, \]
where \( \lambda_n \) are the diagonal elements of the matrix, viz. \( d_1, d_2, \ldots, d_m, 0, -d_m, \ldots, -d_1, 0 \). We first consider a block of \( 2m + 3 \) equations for \( n = m + 1 \) to \( 3m + 3 \).

For \( n = 2m + 2, \lambda_n = 0 \), we have \( \alpha_{2m+3} = -\alpha_{2m+1} \). Next we consider the two
equations adjacent to the above for \( n = 2m + 1 \) (with \( \lambda_n = -d_1 \)) and \( n = 2m + 3 \) (with \( \lambda_n = d_1 \))

\[
\begin{align*}
\alpha_{2m} - d_1 \alpha_{2m+1} + \alpha_{2m+2} &= 0, \\
\alpha_{2m+2} + d_1 \alpha_{2m+3} + \alpha_{2m+4} &= 0,
\end{align*}
\]

This yields (using \( \alpha_{2m+1} = -\alpha_{2m+3} \)) \( \alpha_{2m+4} = \alpha_{2m} \). Proceeding in this way we can prove by induction \( \alpha_{2m+1 + k} = (-1)^k \alpha_{2m+2 - k} \) for \( k = 0, 1, 2, \ldots, m + 1 \). In particular \( \alpha_{3m+3} = (-1)^{m+1} \alpha_{m+1} \) and \( \alpha_{3m+2} = (-1)^m \alpha_{m+2} \). Now, the next block of 2m + 3 equations for \( n = 3m + 3 \) to \( 5m + 5 \) is exactly same as the previous block. Hence as above, \( \alpha_{5m+5} = (-1)^{m+1} \alpha_{3m+3} = \alpha_{m+1} \). Thus \( \alpha_n = \pm \alpha_{m+1} \) for \( n = m + 1, 3m + 3, 5m + 5, \ldots \). Since \( x \in \ell^2 \), we have \( \alpha_{m+1} = 0 \). Similarly \( \alpha_{m+2} = 0 \). However, then \( x = 0 \) and so \( Vx = e_1 \) is impossible.

Now we consider a separable Hilbert space \( H \) with an orthonormal basis \( \{ e_n : n = 1, 2, 3, \ldots \} \) and the almost Mathieu operator of the type

\[
T e_n = e_{n-1} + \lambda \sin(2nr) e_n + e_{n+1}
\]

where \( \lambda \) is real, \( r \) is a rational multiple of \( \pi \) say \( \frac{p}{q} \). Then using the properties of the sine function, we see that \( T \) is a matrix of the type defined in Theorem 0.1 with a suitable choice of \( m \). Thus we conclude the following result.

**Corollary 0.2.** Let \( T e_n = e_{n-1} + \lambda \sin(2nr) e_n + e_{n+1} \), \( \lambda \) real and \( r \) a rational multiple of \( \pi \). Then \( T \) is not invertible.

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**References**


The Institute of Mathematical Sciences, C.I.T. Campus, Madras-600 113, India
E-mail address: balu@imsc.ernet.in

Department of Mathematics, Indian Institute of Technology, Madras-600 036, India
E-mail address: shk@acer.iitm.ernet.in

Department of Mathematics, Anna University, Madras-600 025, India
E-mail address: radharam@annauniv.edu
E-mail address: radharam@imsc.ernet.in