VARIATIONAL REPRESENTATIONS OF VARADHAN FUNCTIONALS

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Abstract. Motivated by the theory of large deviations, we introduce a class of non-negative non-linear functionals that have a variational “rate function” representation.

1. Introduction

Let \((X, d)\) be a Polish space with metric \(d()\) and let \(C_b(X)\) denote the space of all bounded continuous functions \(F : X \rightarrow \mathbb{R}\). In his work on large deviations of probability measures \(\mu_n\), Varadhan \cite{12} introduced a class of non-linear functionals \(L\) defined by

\[
L(F) = \lim_{n \to \infty} \frac{1}{n} \log \int_X \exp(nF(x))d\mu_n
\]

and used the large deviations principle of \(\mu_n\) to prove the variational representation

\[
L(F) = L_0 + \sup_{x \in X} \{F(x) - I(x)\},
\]

where \(I : X \rightarrow [0, \infty]\) is the rate function governing the large deviations, and \(L_0 := L(0) = 0\).

Several authors \cite{1, 3, 4, 9, 10, 11} abstracted non-probabilistic components from the theory of large deviations. In particular, in \cite{3} (see also \cite{10 Theorem 3.1}) we give conditions which imply the rate function representation (2) when the limit (1) exists, and we show that the rate function is determined from the dual formula

\[
I(x) = L(0) + \sup_{F \in C_b(X)} \{F(x) - L(F)\}.
\]

In fact, one can reverse Varadhan’s approach, and show that large deviations of probability measures \(\mu_n\) follow from the variational representation (2) for (1) (see \cite{3 Theorem 1.2.3}). In this context we have \(\mu_n(X) = 1\) which implies \(L(0) = 0\) in (3) and correspondingly \(L_0 = 0\) in (2).

“Asymptotic values” in \cite{3} are essentially what we call Varadhan Functionals here; the theorems in that paper are not entirely satisfying because the assumptions are in terms of the underlying probability measures. In this paper we present a more...
satisfying approach which relies on the theory of probability for motivation purposes only.

**Definition 1.1.** A function $L : C_b(X) \to \mathbb{R}$ is a Varadhan Functional if the following conditions are satisfied:

(4) If $F \leq G$, then $L(F) \leq L(G)$ for all $F, G \in C_b(X)$,

(5) $L(F + \text{const}) = L(F) + \text{const}$ for all $F \in C_b(X)$, $\text{const} \in \mathbb{R}$.

Expression (4) provides an example of Varadhan Functional, if the limit exists.

Another example is given by variational representation (2).

Condition (4) is equivalent to $L(F) \geq L(F \vee G)$, where $a \land b$ denotes the maximum of two numbers. Varadhan Functionals like (1) satisfy a stronger condition.

**Definition 1.2.** A Varadhan Functional $L$ is maximal if $L(\cdot)$ is a lattice homomorphism

(6) $L(F \vee G) = L(F) \vee L(G)$.

It is easy to see that each Varadhan Functional $L(\cdot)$ satisfies the Lipschitz condition $|L(F) - L(G)| \leq \|F - G\|_\infty$; compare (9). Thus $L$ is a continuous mapping from the Banach space $C_b(X)$ of all bounded continuous functions into the real line. We will need the following stronger continuity assumption, motivated by the definition of the countable additivity of measures.

**Definition 1.3.** A Varadhan Functional is $\sigma$-continuous if the following condition is satisfied:

(7) If $F_n \searrow 0$, then $L(F_n) \to L(0)$.

Notice that if $X$ is compact, then by Dini’s theorem and the Lipschitz property, all Varadhan Functionals are $\sigma$-continuous.

Maximal Varadhan Functionals are convex; this follows from the proof of Theorem 2.1 which shows that formula (2) holds true for all Varadhan Functionals when the supremum is extended to all $x$ in the Cech-Stone compactification of $X$.

A simple example of convex and maximal but not $\sigma$-continuous Varadhan Functional is $L(F) = \limsup_{x \to -\infty} F(x)$, where $F \in C_b(\mathbb{R})$. This Varadhan Functional cannot be represented by variational formula (2). Indeed, (2) implies that $L(x) \geq F(x) - L(F) = F(x)$ for all $F \in C_b(\mathbb{R})$ that vanish at $\infty$; hence $L(x) = \infty$ for all $x \in \mathbb{R}$ and (2) gives $L(F) = -\infty$ for all $F \in C_b(\mathbb{R})$, a contradiction.

An example of a convex and $\sigma$-continuous but not maximal Varadhan Functional is $L(F) = \log \int_X F(x) \nu(dx)$, where $\nu$ is a finite non-negative measure.

2. **Variational representations**

The main result of this paper is the following.

**Theorem 2.1.** If a maximal Varadhan Functional $L : C_b(X) \to \mathbb{R}$ is $\sigma$-continuous, then there is $L_0 \in \mathbb{R}$ such that variational representation (2) holds true and the rate function $I : X \to [0, \infty]$ is given by the dual formula (3). Furthermore, $I(\cdot)$ is a tight rate function: sets $I^{-1}((0, a]) \subset X$ are compact for all $a > 0$.

The next result is closely related to Bryc [3] Theorem T.1.1] and Deuschel & Stroock [6, Theorem 5.1.6]. Denote by $\mathcal{P}(X)$ the metric space (with Prokhorov
metric) of all probability measures on a Polish space \( X \) with the Borel \( \sigma \)-field generated by all open sets.

**Theorem 2.2.** If a convex Varadhan Functional \( \mathbb{L} : C_b(X) \to \mathbb{R} \) is \( \sigma \)-continuous, then there is a lower semicontinuous function \( J : \mathcal{P}(X) \to [0, \infty] \) and a constant \( L_0 \) such that

\[
\mathbb{L}(F) = L_0 + \sup_{\mu \in \mathcal{P}} \left\{ \int F d\mu - J(\mu) \right\}
\]

for all bounded continuous functions \( F \).

A well-known example in large deviations is the convex \( \sigma \)-continuous functional \( \mathbb{L}(F) := \log \int \exp F(x) \nu(dx) \) with the rate function in (8) given by the relative entropy functional

\[
J(\mu) = \left\{ \begin{array}{ll}
\int \log \frac{d\mu}{d\nu} d\mu & \text{if } \mu \ll \nu \text{ is absolutely continuous,} \\
\infty & \text{otherwise.}
\end{array} \right.
\]

**Remark 2.1.** Deuschel & Stroock [6, Section 5.1] consider convex functionals : \( C_b(X) \to \mathbb{R} \) such that \( \Phi(\text{const}) = \text{const} \). Such functionals satisfy condition (5).

Indeed, write \( F + \text{const} \) as a convex combination \( F + \text{const} = (1 - \theta)F + \frac{\theta}{2}(2F) + \frac{\theta}{2}(2F) \), where \( 0 < \theta < 1 \). Using convexity and \( \Phi(\text{const}) = \text{const} \) we get \( \Phi(F + \text{const}) \leq \Phi(F) + \text{const} + \theta(2F - \Phi(F)) \). Since \( \theta > 0 \) is arbitrary this proves that \( \Phi(F + \text{const}) \leq \Phi(F) + \text{const} \). By routine symmetry considerations (replacing \( F \mapsto F - \text{const} \), and then \( \text{const} \mapsto -\text{const} \), [6] follows.

3. Proofs

Let \( L_0 := \mathbb{L}(0) \). Passing to \( \mathbb{L}'(F) := \mathbb{L}(F) - L_0 \) if necessary, without losing generality, we assume \( \mathbb{L}(0) = 0 \).

**Lemma 3.1.** Let \( \tilde{X} \) be a compact Hausdorff space. Suppose \( X \subset \tilde{X} \) is a separable metric space in the relative topology. If \( x_0 \in \tilde{X} \setminus X \), then there are bounded continuous functions \( F_n : \tilde{X} \to \mathbb{R} \) such that

(i) \( F_n(x) \searrow 0 \) for all \( x \in X \),
(ii) \( F_n(x_0) = 1 \) for all \( n \in \mathbb{N} \).

**Proof.** Since \( \tilde{X} \) is Hausdorff, for every \( x \in X \) there is an open set \( U_x \ni x \) such that its closure \( \bar{U}_x \) does not contain \( x_0 \).

By Lindelöf property for separable metric space \( X \), there is a countable subcover \( \{U_n\} \) of \( \{U_x\} \).

A compact Hausdorff space \( \tilde{X} \) is normal. So there are continuous functions \( \phi_n : \tilde{X} \to \mathbb{R} \) such that \( \phi_n |_{\bar{U}_n} = 0 \) and \( \phi_n(x_0) = 1 \).

To end the proof take \( F_n(x) = \min_{1 \leq k \leq n} \phi_k(x) \).

The following lemma is contained implicitly in [6, Theorem T.1.2].

**Lemma 3.2.** Theorem 2.1 holds true for compact \( X \).
Proof. Let $\mathbb{I}(\cdot)$ be defined by (3). Thus $\mathbb{I}(x) \geq F(x) - \mathbb{L}(F)$ which implies $\mathbb{L}(F) \geq \sup_{x \in X} \{ F(x) - I(x) \}$. To end the proof we need therefore to establish the converse inequality. Fix a bounded continuous function $F \in C_b(X)$ and $\epsilon > 0$. Let $s = \sup_{x \in X} \{ F(x) - I(x) \}$. Clearly $F(x) - I(x) \leq s \leq \mathbb{L}(F)$. By (3) again, for every $x \in X$, there is $F_x \in C_b(X)$ such that $I(x) < F_x(x) - \mathbb{L}(F_x) + \epsilon$. Therefore

$$F(x) \leq s + I(x) < s + \epsilon + F_x(x) - \mathbb{L}(F_x).$$

This means that the sets $U_x = \{ y \in X : F(y) - F_x(y) < s + \epsilon - \mathbb{L}(F_x) \}$ form an open covering of $X$. Using compactness of $X$, we choose a finite covering $U_{x(1)}, \ldots, U_{x(k)}$. Then, writing $F_i = F_{x(i)}$ we have

$$F(x) < \max_{1 \leq i \leq k} \{ F_i(x) - \mathbb{L}(F_i) \} + s + \epsilon$$

for all $x \in X$.

Using (1), (5), and (6) we have

$$\mathbb{L}(F) \leq \mathbb{L} \left( \max_{1 \leq i \leq k} \{ F_i - \mathbb{L}(F_i) \} + s + \epsilon \right)$$

$$= \mathbb{L} \left( \max_i \{ F_i - \mathbb{L}(F_i) \} \right) + s + \epsilon$$

$$= \max_i \{ \mathbb{L}(F_i) \} + s + \epsilon.$$

Since (5) implies $\mathbb{L}(F_i - \mathbb{L}(F_i)) = \mathbb{L}(F_i) - \mathbb{L}(F_i) = 0$ this shows that $s \leq \mathbb{L}(F) < s + \epsilon$. Therefore $\mathbb{L}(F) = s$, proving (2).

Proof of Theorem 2.7 Let $\hat{X}$ be the Čech–Stone compactification of $X$. Since the inclusion $X \subset \hat{X}$ is continuous, we define $\hat{\mathbb{I}} : C_b(\hat{X}) \to \mathbb{R}$ by $\hat{\mathbb{I}}(\hat{F}) := \mathbb{I}(\hat{F}|X)$. It is clear that $\hat{\mathbb{I}}$ is a maximal Varadhan Functional, so by Lemma 3.2 there is $\hat{\mathbb{I}} : \hat{X} \to [0, \infty]$ such that $\hat{\mathbb{I}}(\hat{F}) = \sup\{ \hat{F}(x) - I(x) : x \in \hat{X} \}$.

Using $\sigma$-continuity (7) it is easy to check that $\hat{\mathbb{I}}(x) = \infty$ for all $x \in \hat{X} \setminus X$. Indeed, given $x_0 \in \hat{X} \setminus X$ by Lemma 3.1 there are $F_n \in C_b(X)$ such that $F_n \upharpoonright X = F_{n-1} \upharpoonright X$ but $F_n(x_0) = C > 0$. Then from (3) we get $\hat{\mathbb{I}}(x_0) \geq \mathbb{I}(x_0) + F_n(x_0) - \mathbb{L}(F_n) \to \mathbb{I}(x_0) + C$. Since $C > 0$ is arbitrary, $\hat{\mathbb{I}}(x_0) = \infty$.

This shows that $\hat{\mathbb{I}}(\hat{F}) = \sup\{ \hat{F}(x) - I(x) : x \in X \}$ for all $\hat{F} \in C_b(\hat{X})$. It remains to observe that since $\hat{X}$ is a Čech-Stone compactification, every function $F \in C_b(X)$ is a restriction to $X$ of some $\hat{F} \in C_b(\hat{X})$ (see [21 IV.6.22]). Therefore (2) holds true for all $F \in C_b(X)$.

To prove that the rate function is tight, suppose that there is $a > 0$ such that $\mathbb{I}^{-1}[0,a]$ is not compact. Then there is $\delta > 0$ and a sequence $x_n \in X$ such that $\mathbb{I}(x_n) \leq a$, and $d(x_m,x_n) > \delta$ for all $m \neq n$. Since Polish spaces have Lindelöf property, there is a countable number of open balls of radius $\delta/2$ which cover $X$. For $k = 1,2,\ldots$, denote by $B_k \ni x_k$ one of the balls that contain $x_k$, and let $\phi_k$ be a bounded continuous function such that $\phi_k(x_k) = 2a$ and $\phi_k = 0$ on the complement of $B_k$. Then $F_n = \max_{k \geq n} \phi_k \upharpoonright 0$ pointwise. On the other hand (2) implies $\mathbb{L}(F_n) \geq L_0 + F_n(x_n) - I(x_n) \geq L_0 + a$, contradicting (7).

Lemma 3.3. If $\mathbb{I}(-)$ is a Varadhan Functional, then

$$\inf_{x \in X} \{ F(x) - G(x) \} \leq \mathbb{L}(F) - \mathbb{L}(G).$$
Proof. Let \( \text{const} = \inf_x \{ F(x) - G(x) \} \). Clearly, \( F \geq G + \text{const} \). By positivity condition (\ref{eq:positivity}) this implies \( L(F) \geq L(G + \text{const}) = L(G) + \text{const} \).

The next lemma is implicitly contained in the proof of [3, Theorem T.1.1]. Let \( \mathcal{P}_a(X) \) denote all regular finitely-additive probability measures on \( X \) with the Borel field.

**Lemma 3.4.** If \( L(\cdot) \) is a convex Varadhan Functional on \( C_b(X) \), then there exist a lower semicontinuous function \( J : \mathcal{P}_a(X) \to [0, \infty] \) such that

\[
L(F) = L(0) + \sup \{ \mu(F) - J(\mu) : \mu \in \mathcal{P}_a(X) \},
\]

and the supremum is attained.

**Proof.** Let \( J(\cdot) \) be defined by

\[
J(\mu) = L(0) + \sup \{ \mu(F) - L(F) : F \in C_b(X) \}
\]

and fix \( F_0 \in C_b(X) \). Recall that throughout this proof we assume \( L(0) = 0 \).

By the definition of \( J(\cdot) \), we need to show that

\[
L(F_0) = \sup_{\mu} \inf_F \{ \mu(F_0) - \mu(F) + L(F) \},
\]

where the supremum is taken over all \( \mu \in \mathcal{P}_a(X) \) and the infimum is taken over all \( F \in C_b(X) \). Moreover, since (\ref{eq:convexity}) implies that \( J(\mu) \geq \mu(F_0) - L(F_0) \) for all \( \mu \in \mathcal{P}_a(X) \), then \( L(F_0) \leq \sup_{\mu} \inf_F \{ \mu(F_0) - \mu(F) + L(F) \} \). Hence to prove (\ref{eq:supremum}), it remains to show that there is \( \nu \in \mathcal{P}_a(X) \) such that

\[
L(F_0) \leq \nu(F_0) - \nu(F) + L(F) \quad \text{for all } F \in C_b(X)
\]

(\text{also, for this } \nu, \text{ the supremum in (\ref{eq:convexity}) will be attained}). To find \( \nu \), consider the following sets. Let

\[
\mathcal{M} = \{ F \in C_b(X) : \inf_x [F(x) - F_0(x)] > 0 \}
\]

and let \( \mathcal{N} \) be a set of all finite convex combinations of functions \( g(x) \) of the form \( g(x) = F(x) + L(F_0) - L(F) \), where \( F \in C_b(X) \).

It is easily seen from the definitions that \( \mathcal{M} \) and \( \mathcal{N} \) are convex; also \( \mathcal{M} \subset C_b(X) \) is non-empty since \( 1 + F_0 \in \mathcal{M} \), and open since \( \{ F : \inf_x [F(x) - F_0(x)] \leq 0 \} \subset C_b(X) \) is closed. Furthermore, \( \mathcal{M} \) and \( \mathcal{N} \) are disjoint. Indeed, take arbitrary

\[
\mathcal{N} \ni g = \sum \alpha_k F_k + L(F_0) - \sum \alpha_k L(F_k).
\]

Then

\[
\inf_x \{ g(x) - F_0(x) \} = \inf_x \{ \sum \alpha_k F_k(x) - F_0(x) \} - \sum \alpha_k L(F_k) + L(F_0)
\]

\[
\leq \inf_x \{ \sum \alpha_k F_k(x) - F_0(x) \} - \sum \alpha_k L(F_k) + L(F_0) \leq 0,
\]

where the first inequality follows from the convexity of \( L(\cdot) \) and the second one follows from (\ref{eq:convexity}) applied to \( F = \sum \alpha_k F_k(x) \) and \( G = F_0 \).
Therefore $\mathcal{M}$ and $\mathcal{N}$ can be separated, i.e. there is a non-zero linear functional $f^* \in C_b^*(X)$ such that for some $\alpha \in \mathbb{R}$

\begin{equation}
 f^*(\mathcal{N}) \leq \alpha < f^*(\mathcal{M})
 \end{equation}

(see e.g. [7, V. 2. 8]).

Claim: $f^*$ is non-negative.

Indeed, it is easily seen that $F_0(\cdot)$ belongs to $\mathcal{N}$, and, as a limit of $\epsilon + F_0(x)$ as $\epsilon \to 0$, $F_0$ is also in the closure of $\mathcal{M}$. Therefore by (14) we have $\alpha = f^*(F_0)$. To end the proof take arbitrary $F$ with $\inf_x F(x) > 0$. Then $F + F_0 \in \mathcal{M}$ and by (14)

\begin{equation}
 f^*(F) = f^*(F + F_0) - f^*(F_0) > \alpha - f^*(F_0) = 0.
 \end{equation}

This ends the proof of the claim.

Without losing generality, we may assume $f^*(1) = 1$; then it is well known (see e.g. [2, Ch. 2 Section 4 Theorem 1]) that $f^*(F) = \nu(F)$ for some $\nu \in \mathcal{P}_b(X)$; for regularity of $\nu$ consult [7, IV.6.2]. It remains to check that $\nu$ satisfies (13). To this end observe that since $F + L(F_0) - L(F) \in \mathcal{N}$, by (14) we have $\nu(F) + L(F_0) - L(F) \leq \alpha = \nu(F_0)$ for all $F \in C_b(X)$. This ends the proof of (10).

Proof of Theorem 2.2. Lemma 3.4 gives the variational representation (10) with the supremum taken over a too large set. To end the proof we will show that $\mathcal{J}(\mu) = \infty$ on measures $\mu$ that fail to be countably-additive.

Suppose that $\mu$ is additive but not countably additive. Then Daniell-Stone theorem implies that there is $\delta > 0$ and a sequence $F_n \downarrow 0$ of bounded continuous functions such that $\int F_n d\mu > \delta > 0$ for all $n$. By (14) and $\sigma$-continuity, $\mathcal{J}(\mu) \geq L(0) + C \int F_n d\mu - L(CF_n) \geq L(0) + C\delta - L(CF_n) \to L(0) + C\delta$. Since $C > 0$ is arbitrary, then $\mathcal{J}(\mu) = \infty$ for all $\mu$ that are additive but not countably-additive. Thus (10) implies (8).

References


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