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DOMINATION BY POSITIVE DISJOINTLY STRICTLY SINGULAR OPERATORS

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Abstract. We prove that each positive operator from a Banach lattice E to a Banach lattice F with a disjointly strictly singular majorant is itself disjointly strictly singular provided the norm on F is order continuous. We prove as well that if $S : E \to E$ is dominated by a disjointly strictly singular operator, then S^2 is disjointly strictly singular.

1. Introduction

The classical problem of domination for positive compact operators on Banach lattices was solved by Dodds and Fremlin ([5]) for a pair of positive operators $0 \leq S \leq T$ defined on a Banach lattice E with order continuous dual norm and taking values in a Banach lattice F with order continuous norm: we can guarantee that S is compact if T is so. A full answer to this problem was given by Aliprantis and Burkinshaw in [2], namely if $E = F$ and either the norm on E or the norm on E' is order continuous, then the compactness of T is inherited by the operator S^2. They also show that for an arbitrary Banach lattice, T compact always implies S^3 compact. More recently Wickstead has given in [14] not only sufficient but necessary conditions for the problem of domination for positive compact operators to have a solution.

The problem of domination for weakly compact operators was first considered by Abramovich in [1], giving a positive solution for a Banach lattice E and a KB-space F. Later on, a general result was obtained by Wickstead in [13] where it was shown that the problem has a positive answer if and only if either the norm on E' or F is order continuous. Again Aliprantis and Burkinshaw settled the question by considering the case $E = F$ and showing that T weakly compact implies S^2 weakly compact ([3]).

The aim of this paper is to study the problem of domination for positive disjointly strictly singular operators. We recall that an operator T between a Banach lattice E and a Banach space Y is said to be *disjointly strictly singular* (DSS) if there is no disjoint sequence of non-null vectors $(x_n)_n$ in E such that the restriction of T to the subspace $[x_n]$ spanned by the vectors $(x_n)_n$ is an isomorphism. DSS operators were introduced by Rodríguez-Salinas and the second author in [9]. This class of operators, a generalization of the class of strictly singular (or Kato) operators, is a

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useful tool to compare the lattice structure of Banach function lattices ([7]). Recall that an operator T between two Banach spaces X and Y is said to be *strictly singular* if the restriction of T to any infinite dimensional closed subspace is not an isomorphism. Every strictly singular operator is DSS but the converse is not true (f.i. take the natural inclusion $i : L^p[0,1] \to L^q[0,1]$ with $p > q \geq 1$). However if E is a Banach lattice with a Schauder basis of mutually disjoint vectors or a $C(K)$-space, then every DSS operator from E to Y is strictly singular. The set of all DSS operators between E and Y is a vector space which is stable under the composition by the left but not by the right ([8, Prop. 1]).

The main results of the paper are presented now.

Theorem 1.1. *Let E and F be Banach lattices and $0 \leq S \leq T : E \to F$ two positive operators. If the norm on F is order continuous and T is DSS, then S is also DSS.*

Moreover, if F is σ-Dedekind complete and every positive operator from E to F dominated by a DSS operator is DSS, then either the norm on F or the norm on E' is order continuous.

In the case $E = F$ we obtain the following:

Theorem 1.2. *Let $0 \leq S \leq T$ be two operators on a Banach lattice E. If T is DSS, then S^2 is DSS.*

For any unexplained terms from Banach lattices and regular operators theory we refer to [4], [12] or [15].

2. Proofs

Let us start by recalling a couple of well-known facts.

Lemma 2.1. *Let E be an L-space. Then every weakly-null sequence of positive vectors is convergent to zero.*

Lemma 2.2. *Let (Ω, Σ, μ) be a finite measure space and $(f_n)_n$ be a weakly convergent sequence in $L^1(\mu)$. If $(f_n)_n$ converges to zero in μ-measure, then $(f_n)_n$ converges to zero in norm.*

Proof. We can assume w.l.o.g. that $\mu(\Omega) = 1$. The sequence $(f_n)_n$ is uniformly absolutely continuous since it is weakly convergent (cf. [6, Cor. IV.8.11]). Hence for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|\chi_B f_n\|_1 < \varepsilon/2$ for every integer n and every $B \in \Sigma$ with $\mu(B) < \delta$. Consider $B_n = \{t \in \Omega : |f_n(t)| > \varepsilon/2\}$. By assumption there exists an integer n_0 such that $\mu(B_n) < \delta$ for $n \geq n_0$. Thus, for $n \geq n_0$ we have

$$\|f_n\|_1 = \int_{B_n} |f_n| + \int_{B_n^c} |f_n| \leq \|\chi_{B_n} f_n\|_1 + \varepsilon/2 < \varepsilon.$$

□

The following result will be used in the sequel (cf. [12, Cor. 3.4.14 and Thm. 3.4.17]).

Proposition 2.3. *Let E and F be two Banach lattices such that the norm on F is order continuous. If $T : E \to F$ is a positive operator, then T preserves an isomorphic copy of l^1 if and only if T preserves a lattice isomorphic copy of l^1.*

The proposition given next is essential in the proof of Theorem 1.1. Notice that in the definition of a DSS operator the disjoint vectors are not in general positive.

Proposition 2.4. *Let T be a positive operator defined on a Banach lattice E and with values in a Banach lattice F with order continuous norm. Then T is DSS if and only if there is no disjoint sequence of non-null positive vectors $(y_n)_n$ in E such that the restriction of T to the span $[y_n]$ is an isomorphism.*

Proof. We just need to prove the non-trivial implication. Suppose that T is not DSS; then there exists a (normalized) sequence in E of pairwise disjoint elements $(x_n)_n$ such that the restriction of T to the span $[x_n]$ is an isomorphism, that is,

$$\left\| T\left(\sum_{n=1}^{\infty} a_n x_n\right)\right\| \geq \alpha \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \quad \text{for some } \alpha > 0$$

(note that $(x_n)_n$ is an unconditional basic sequence, being disjoint). Since the span of the sequence $(Tx_n)_n$ is a separable subspace of the Banach lattice F, we can find a closed order ideal J of F with a weak unit which contains $[Tx_n]$ (cf. [11, Prop. 1.a.9]); furthermore J is complemented in F by a positive projection P (cf. [11, Prop. 1.a.11]). Consider the operator $PT : E \to J$. Clearly the restriction of PT to the span $[x_n]$ is not DSS. On the other hand the assumption on T shows that the operator PT is not invertible on the span of any disjoint sequence of *positive* vectors. Therefore there is no loss of generality in assuming that F itself has a weak unit. In such a case we can represent F as an (in general not closed) order ideal of $L^1(\Omega, \Sigma, \mu)$ for some probability space (Ω, Σ, μ), which is continuously included in $L^1(\Omega, \Sigma, \mu)$ (cf. [11, Thm. 1.b.14]).

$(*)$ Let us consider the sublattice $[\,|x_n|\,]$; we claim that $[\,|x_n|\,]$ contains no lattice isomorphic copy of c_0.

To prove this we will assume the contrary and get a contradiction by showing that T is invertible on a sublattice of $[|x_n|]$. Suppose then that there exists a normalized sequence $(z_k)_k \subset [\,|x_n|\,]$ of mutually disjoint positive vectors equivalent to the unit basis $\{e_k\}_k$ of c_0. Let us write $z_k = \sum_{j=1}^{\infty} a_j^k |x_j|$ with $a_j^k \geq 0$ for all j. If $\|\,T\,(z_{k_l})\| \underset{l}{\to} 0$ for some subsequence $(k_l)_l$, then $\left\| T\left(\sum_{j=1}^{\infty} a_j^{k_l} x_j\right)\right\| \underset{l}{\to} 0$ and hence $\|z_{k_l}\| = \left\| \sum_{j=1}^{\infty} a_j^{k_l} x_j \right\| \underset{l}{\to} 0$ since the restriction of T to the span $[x_n]$ is an isomorphism; however this is a contradiction with $\|z_k\| = 1$ for all k. Thus we may assume that $\inf_k \|Tz_k\| \geq \delta > 0$ for some $\delta > 0$. The sequence $(z_k)_k$ is $\sigma([z_k], [z_k]')$-null being equivalent to the unit basis of c_0. Hence $(Tz_k)_k$ is weakly null in F being T bounded. Note that $(Tz_k)_k$ is also convergent to zero in the weak topology of $L^1(\mu)$ since F is continuously included in $L^1(\mu)$. In fact $(T\,z_k)_k$ converges to zero in $L^1(\mu)$ since $Tz_k \geq 0$ for all k (cf. Lemma 2.1).

Let us consider the *Kadec-Pełczynski* set $M(\varepsilon) = \{y \in F : \mu(\sigma(y, \varepsilon)) \geq \varepsilon\}$, where $\varepsilon > 0$ and $\sigma(y, \varepsilon) = \{t \in \Omega : |y(t)| \geq \varepsilon \|y\|\}$. If $(Tz_k)_k \subset M(\varepsilon)$ for some $\varepsilon > 0$, then $\|\,T\,z_k\|_1 \geq \varepsilon^2 \|Tz_k\|$ for all k; hence $\|Tz_k\| \underset{k}{\to} 0$, which is a contradiction with $\inf_k \|Tz_k\| \geq \delta > 0$. Thus we may assume that $(Tz_k)_k \not\subseteq M(\varepsilon)$ for every $\varepsilon > 0$; then, by Kadec-Pełczynsky's disjointification process (cf. [11, Prop. 1.c.8]) we may choose a subsequence $(Tz_{k_j})_j$ equivalent to a disjoint sequence in F; it follows that $(Tz_{k_j})_j$ is an unconditional basic sequence with unconditional constant, say $\beta > 0$.

For every integer j we have

$$\Big\|\sum_{j=1}^{\infty} a_j\, T\, z_{k_j}\Big\| \geq \beta^{-1}\Big\|\sum_{j=1}^{\infty} |a_j|\, T\, z_{k_j}\Big\| \geq \beta^{-1}|a_j|\,\|\,T\, z_{k_j}\| \geq \beta^{-1}|a_j|\delta$$

(note that in the previous inequalities we use that $Tz_{k_j} \geq 0$ for all j). Thus

$$\Big\|\,T\Big(\sum_{j=1}^{\infty} a_j z_{k_j}\Big)\Big\| \geq \beta^{-1}\delta\Big(\bigvee_{j=1}^{\infty}|a_j|\Big) \geq K\Big\|\sum_{j=1}^{\infty} a_j z_{k_j}\Big\|,$$

where K is a positive constant. Hence the operator T preserves a lattice copy of c_0, which is a contradiction.

Now if we apply Rosenthal's dichotomy theorem (cf. [10, Thm. 2.e.5]) to the sequence $(|x_n|)_n$, we obtain a subsequence $(|x_{n_j}|)_j$ satisfying either (1) $(|x_{n_j}|)_j$ is equivalent to the unit basis of l^1 or (2) $(|x_{n_j}|)_j$ is a weakly Cauchy sequence. Suppose first that (1) holds. Then

$$\sum_j a_j x_{n_j} < \infty \Leftrightarrow \sum_j a_j |x_{n_j}| < \infty \Leftrightarrow \sum_j |a_j| < \infty;$$

hence T preserves an isomorphic copy of l^1 or, equivalently by Proposition 2.3, T preserves a lattice copy of l^1. Contradiction.

Finally let us show that case (2) also leads to contradiction. Indeed, once the statement $(*)$ has been proved we may assume that the Banach lattice $[\,|x_n|\,]$ is weakly sequentially complete or equivalently a KB-space (cf. [4, Thm. 14.12]); hence the subsequence $(|x_{n_j}|)_j$ must be weakly convergent. Thus the separable lattice $[\,|x_n|\,]$ has an order continuous norm and a weak unit, and hence it can be considered as a continuously-included order ideal in $L^1(\Omega', \Sigma', \mu')$ for some probability space (Ω', Σ', μ') (cf. [11, Thm. 1.b.14]). It follows that $(|x_{n_j}|)_j$ is convergent in the weak topology of $L^1(\mu')$ to a function f. In fact $f = 0$ since the sequence $(|x_{n_j}|)_j$ converges to zero in μ'-measure being pairwise disjoint (cf. Lemma 2.2). Since T is bounded, the sequence $(T|x_{n_j}|)_j$ converges to zero in the weak topologies of F and $L^1(\mu)$; hence $\|\,T|x_{n_j}|\,\|_1 \underset{j}{\to} 0$ by Lemma 2.1.

We apply again the Kadec-Pełczynski method: if $T(|x_{n_j}|)_j \subset M(\varepsilon)$ for some $\varepsilon > 0$, then $\|\,T\,|x_{n_j}|\,\|_1 \underset{j}{\to} 0$ implies $\|\,T\,|x_{n_j}|\,\| \underset{j}{\to} 0$ and $\|x_{n_j}\| \underset{j}{\to} 0$ follows; this is a contradiction with the initial choice of $(x_n)_n$. Thus we may assume $(T\,|x_{n_j}|)_j \nsubseteq M(\varepsilon)$ for all $\varepsilon > 0$; in this case we may choose a subsequence, still denoted by $(T|x_{n_j}|)_j$, which is equivalent to a disjoint sequence in F. It follows that $(T\,|x_{n_j}|)_j$ is an unconditional basic sequence with unconditional constant, say $K > 0$. And

$$\begin{aligned}\alpha\Big\|\sum_{j=1}^{\infty} a_j|x_{n_j}|\,\Big\| &= \alpha\Big\|\sum_{j=1}^{\infty} a_j x_{n_j}\Big\| \leq \Big\|\sum_{j=1}^{\infty} a_j\, T\, x_{n_j}\Big\| \\ &\leq \Big\|\sum_{j=1}^{\infty} |a_j|\, T\,|x_{n_j}|\,\Big\| \leq K\,\Big\|\sum_{j=1}^{\infty} a_j\, T\,|x_{n_j}|\,\Big\| = K\,\Big\|\,T\,\Big(\sum_{j=1}^{\infty} a_j|x_{n_j}|\Big)\Big\|,\end{aligned}$$

that is, T is invertible on the span $[\,|x_{n_j}|\,]$. This contradiction concludes the proof. □

Remark 2.5. If the dual norm on E' and the norm on F are simultaneously order continuous, then the previous proof becomes shorter by noticing that the sequence $(|x_n|)_n$ is weakly null (cf. [12, Thm. 2.4.14]).

We recall that an operator T between a Banach lattice E and a Banach space Y is said to be *order-weakly compact* if $T[-x,x]$ is relatively weakly compact for every $x \in E_+$. It is known that $T : E \to Y$ is order-weakly compact if and only if T does not preserve a sublattice isomorphic to c_0 whose unit ball is order bounded in E (cf. [12, Cor. 3.4.5]). Consequently every DSS operator is order-weakly compact. The following characterization will be used in the sequel (cf. [12, Prop. 3.4.9]).

Proposition 2.6. *Let E be a Banach lattice, F be a Banach space and $T : E \to F$ a bounded operator. Then T is order-weakly compact if and only if T transforms every order bounded and weakly null sequence of positive vectors in E in a sequence convergent to zero.*

We pass now to prove the main result of the paper.

Proof of Theorem 1.1. We prove first that if the norm on F as well as the norm on E' are order continuous, then T DSS implies S DSS.

Assume that this is not the case. By Proposition 2.4 there exists a disjoint (normalized) sequence $(x_n)_n$ of positive vectors in E and $\alpha > 0$ such that $\|Sx\| \geq \alpha\|x\|$ for all $x \in [x_n]$. As in the proof of Proposition 2.4 the Banach lattice F can be considered to have a weak unit. Hence there exist a probability space (Ω, Σ, μ), an order ideal I of $L^1(\Omega, \Sigma, \mu)$, a lattice norm $\| \, \|_I$ on I and an order isometry ψ between F and $(I, \| \, \|_I)$, such that the canonical inclusion from I in $L^1(\mu)$ is continuous with $\|f\|_1 \leq \|f\|_I$ (cf. [11, Thm. 1.b.14]). Note that $\psi T : E \to I$ is DSS and that $0 \leq \psi S \leq \psi T$. Note too that if ψS were DSS, then S would also be DSS. This observation allows us to reduce the proof to the case that F is an order ideal in $L^1(\Omega, \Sigma, \mu)$.

We claim that $(Tx_n)_n \nsubseteq M(\varepsilon)$ for all $\varepsilon > 0$ where $M(\varepsilon)$ denotes a Kadec-Pełczynski set as above. Indeed, the norm-bounded disjoint sequence $(x_n)_n$ is weakly null since the norm on E' is order continuous (cf. [12, Thm. 2.4.14]). Hence $(Tx_n)_n$ is a weakly null sequence in $L^1(\mu)$; in fact $\|Tx_n\|_1 \to 0$ by Lemma 2.1. If $(Tx_n)_n \subseteq M(\varepsilon)$ for some $\varepsilon > 0$, then $\|Tx_n\|_1 \geq \varepsilon^2 \|Tx_n\|$; hence $(Tx_n)_n$ converges to zero in F. The inequalities $0 \leq Sx_n \leq Tx_n$ for all n show that $(Sx_n)_n$ converges to zero in F, and hence $\|x_n\| \underset{n}{\to} 0$ since $\|Sx_n\| \geq \alpha\|x_n\|$ for all n; however this is a contradiction with the choice of $(x_n)_n$. Now, by Kadec-Pełczynski's disjointification process (cf. [11, Prop. 1.c.8]), we may choose a subsequence, still denoted by $(Tx_n)_n$, equivalent to a pairwise disjoint sequence in F; hence $(Tx_n)_n$ is an unconditional basic sequence with unconditional constant, say $K > 0$. We have

$$\begin{aligned}\left\|T\left(\sum_{n=1}^{\infty} a_n x_n\right)\right\| &= \left\|\sum_{n=1}^{\infty} a_n T x_n\right\| \geq K^{-1}\left\|\sum_{n=1}^{\infty} |a_n| T x_n\right\| \geq K^{-1}\left\|\sum_{n=1}^{\infty} |a_n| S x_n\right\| \\ &\geq \alpha K^{-1}\left\|\sum_{n=1}^{\infty} |a_n| x_n\right\| = \alpha K^{-1}\left\|\sum_{n=1}^{\infty} a_n x_n\right\|,\end{aligned}$$

for all $x \in [x_n]$. However this is impossible since T is a DSS operator.

We can now prove the first part of Theorem 1.1 in the general case. Assume the opposite, that is, there is a disjoint sequence $(x_n)_n$ of positive vectors in E such that the restriction of S to the sublattice $[x_n]$ is an isomorphism while T is DSS. By the lines above we can assume that the dual norm on the sublattice $[x_n]'$ is not order continuous. Then $[x_n]$ contains a lattice copy of l^1 (cf. [12, Thm. 2.4.14]) which is preserved by S. It follows that the adjoint operator S' is not order-weakly compact (cf. [12, Cor. 3.4.14]) and so is T' by Proposition 2.6 (note that $0 \le S' \le T'$); hence T preserves a copy of l^1 (cf. [12, Cor. 3.4.14]) or equivalently T preserves a lattice copy of l^1 (cf. Proposition 2.3). Contradiction.

To prove the second part of Theorem 1.1 assume that the norms on E' and F are simultaneously not order continuous. Since the dual norm on E' is not order continuous, we can find in E a lattice copy of l^1 complemented by a positive projection (cf. [12, Thm. 2.4.14 and Prop. 2.3.11]); let H_1 be the sublattice of E lattice isomorphic to l^1, ϕ_1 the lattice isomorphism between H_1 and l^1, and P_1 the positive projection from E onto H_1. On the other hand, since the norm on F is not order continuous and F is σ-Dedekind complete, there exist a sublattice H_2 in F and a lattice isomorphism ϕ_2 between H_2 and l^∞ (cf. [12, Cor. 2.4.3]). Consider the operators defined from l^1 into l^∞ by

$$T(a=(a_n)) = \left(\sum_{n=1}^{\infty} a_n\right)(1,1,\cdots) \quad \text{and} \quad S(a=(a_n)) = \left(\left(\sum_{n=1}^{\infty} x_{k,n}a_n\right)_{k=1}^{\infty}\right),$$

where $S \equiv (x_{k,n})$ is the infinite matrix with $\{0,1,-1\}$-entries defined as follows:

$$S \equiv (x_{k,n}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & -1 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Note that for a given $a \in l^1$ there exists an integer k with $2+2^2+\cdots+2^{n-1} < k \le 2+2^2+\cdots+2^n$ satisfying

$$\sum_{m=1}^{n} |a_m| = \sum_{j=1}^{n} x_{k,j}a_j.$$

Clearly S is a linear isometry from l^1 into l^∞. Indeed,

$$\|S(a)\|_\infty = \sup_k \left(\left| \sum_{n=1}^{\infty} x_{k,n}a_n \right| \right) \le \sum_{n=1}^{\infty} |a_n| = \|a\|_1.$$

On the other hand, for a given $\varepsilon > 0$ there exists an integer n such that $\sum_{m=1}^{n} |a_m| \ge (\|a\|_1 - \varepsilon)$; hence there is an integer k satisfying $\sum\limits_{j=1}^{n} x_{k,j}a_j = \sum\limits_{m=1}^{n} |a_m| \ge (\|a\|_1 - \varepsilon)$. The relations $\|Sa\|_\infty \ge \|a\|_1 - \varepsilon$ and $\|Sa\|_\infty = \|a\|_1$ follow.

Consider now the operators S^+ and S^- defined by the sequences $(x'_{k,n})_{n,k}$ and $(x''_{k,n})_{n,k}$ where

$$x'_{k,n} = \begin{cases} x_{k,n} & \text{if } x_{k,n} > 0, \\ 0 & \text{if } x_{k,n} \le 0, \end{cases} \qquad x''_{k,n} = \begin{cases} -x_{k,n} & \text{if } x_{k,n} < 0, \\ 0 & \text{if } x_{k,n} \ge 0. \end{cases}$$

The operator $\widetilde{S}(a) = (\sum_{j=1}^{\infty} |x_{k,j}| a_j)_{k=1}^{\infty}$ from l^1 into l^∞ clearly factorizes through the space c of all convergent sequences (which is isomorphic to c_0); thus $\widetilde{S}$ is strictly singular. It follows from the equalities $\widetilde{S} = S^+ + S^-$ and $S = 2S^+ - \widetilde{S}$ that S^+ is not strictly singular. Hence S^+ is neither DSS as an operator from l^1 into l^∞ (in fact the inequality $\|S^+a\| \geq 1/2\|a\|_1$ holds for all $a \in l^1$). Finally it is clear that $S^+ \leq T$ and that T is DSS being a rank-one operator.

Consider the operators $S' = \phi_2 S^+ \phi_1 P_1$ and $T' = \phi_2 T \phi_1 P_1$ defined on E and with values in F. Clearly $0 \leq S' \leq T'$; moreover $\phi_2 S^+ \phi_1$ is not strictly singular since ϕ_1 and ϕ_2 are isomorphisms and S^+ is not strictly singular. The inequalities

$$m\|h\|_E \leq \|S^+\phi_1(h)\|_{l^\infty} \leq \|\phi_2 S^+ \phi_1(h)\|_F \leq M\|h\|_H$$

show that S' is invertible on H, or equivalently that S' is not DSS. □

Remark 2.7. The proof actually shows that if the norms on E' and F are simultaneously not order continuous and F is σ-Dedekind complete, then the problem of domination for strictly singular operators has in general a negative answer. This problem requires its own study which will be carried out elsewhere.

We consider next the problem of domination in the case $E = F$. To this end we recall the following *factorization* result due to Aliprantis and Burkinshaw (cf. [4, Thm. 18.7] or [12, Thm. 3.4.6]): Let E and Y be a Banach lattice and a Banach space respectively and $T : E \to Y$ an order-weakly compact operator. Then there exist a Banach lattice F with order continuous norm, a lattice homomorphism Q from E into F and a bounded operator S from F into Y such that $T = SQ$. Moreover, if Y is a Banach lattice and $0 \leq T_1 \leq T$, then $0 \leq S_1 \leq S$.

Proposition 2.8. *Let E_i, $i = 1, 2, 3$, be Banach lattices and $0 \leq S_i \leq T_i$ operators defined on E_i and taking values in E_{i+1} for $i = 1, 2$. If T_1 is DSS and T_2 is order-weakly compact, then $S_2 S_1$ is DSS.*

Proof. Given $0 \leq S_2 \leq T_2 : E_2 \to E_3$, we may find, by the above result, a Banach lattice G with order continuous norm, a lattice homomorphism Q from E_2 into G and two positive operators $\widetilde{S_2} \leq \widetilde{T_2}$ from G into E_3 such that $T_2 = \widetilde{T_2} Q$ and $S_2 = \widetilde{S_2} Q$. Consider the operators $S = QS_1$ and $T = QT_1$ from E_1 into G. Note that T is DSS since T_1 is so and we are composing by the left; hence S is DSS by Theorem 1.1 and so is $\widetilde{S_2} S$. Finally the equality $S_2 S_1 = \widetilde{S_2} S$ concludes that $S_2 S_1$ is DSS. □

Now Theorem 1.2 is a direct consequence of Proposition 2.8.

Remark 2.9. Theorem 1.2 is the best possible. Indeed, consider $E = l^1 \oplus l^\infty$ and the operators $0 \leq \widetilde{S} \leq \widetilde{T}$ on E defined via the matrices

$$\widetilde{S} = \begin{pmatrix} 0 & 0 \\ S^+ & 0 \end{pmatrix}, \qquad \widetilde{T} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$$

where S^+ and T are the operators defined in the proof of Theorem 1.1. Clearly $\widetilde{T}$ is DSS and $\widetilde{S}$ is not.

Recall that an *orthomorphism* on a Banach lattice E is a band preserving operator which is also order bounded. An easy consequence of Theorem 1.2, which is obtained by reasoning as in [4, Thm. 16.21], is

Corollary 2.10. *Let S and T be two positive operators on a Dedekind complete Banach lattice E such that $0 \leq S \leq T$ holds. If T is DSS and S is an orthomorphism, then S is DSS.*

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