A SHORT PROOF THAT HYPERSPACES OF PEANO CONTINUA ARE ABSOLUTE RETRACTS

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Abstract. We give a short proof of Wojdyslawski’s famous theorem.

Theorem (Wojdyslawski [6]). Let $X$ be a Peano continuum. Then the hyperspace $2^X$ of all nonempty compact subsets of $X$ is an absolute retract for metric spaces.

This result is an essential step in the proof of the Curtis-Schori-West hyperspace Theorem to the effect that $2^X$ is a Hilbert cube for any Peano continuum $X$ (see, e.g., the book of van Mill [5, §8.4]). Wojdyslawski’s original proof is rather complicated [6]. A simpler proof was suggested later on by Kelley [4], which is, however, based on a difficult Lefschetz-Dugundji characterization of metric ANR’s (see [5, Theorem 5.2.1]). Yet another proof, also based on the Lefschetz-Dugundji characterization, can be found in [5, §5.3]. Our proof is elementary and it does not rely on the Lefschetz-Dugundji criterion.

Proof. Let $d$ be any compatible metric on $X$ and let $d_H$ be the Hausdorff metric on $2^X$. Assume that $(Y, \rho)$ is a metric space, $A$ is a closed subset of $Y$ and $f : A \to 2^X$ is a continuous map. Following [3], choose a canonical cover $\omega$ of $Y \setminus A$ in $Y$, that is to say: (1) $\omega$ is an open cover of $Y \setminus A$, locally finite in $Y \setminus A$; (2) for each neighborhood $V$ of a point $a \in A$ in $Y$ there exists a neighborhood $S$ of $a$ in $Y$ contained in $V$, such that every element $U \in \omega$ which meets $S$ is contained in $V$. We note that the second condition implies that every neighborhood of any boundary point of $A$ in $Y$ contains infinitely many open sets in $\omega$ (see [2, Ch. III, §1]).

Let $N(\omega)$ denote the nerve of $\omega$ endowed with the CW topology. We will denote by $p_U$ the vertex of $N(\omega)$ corresponding to $U \in \omega$. Then according to [3], there exist a Hausdorff space $Z$ and a continuous map $\mu : Y \to Z$ with the following properties:

(a) $Z$ as a set coincides with the disjoint union $A \cup N(\omega)$;
(b) $A$ is closed in $Z$ and the restriction $\mu|_A$ is the identical homeomorphism;
(c) $Z \setminus A = N(\omega)$ is taken with its CW topology and $\mu(Y \setminus A) \subset Z \setminus \mu(A)$;
(d) a base of neighborhoods of $a \in A$ in $Z$ is determined by selecting a neighborhood $W$ of $a$ in $Y$ and taking in $Z$ the set $W \cap A$ together with the closed
star of every vertex $p_U$ of $N(\omega)$ corresponding to a set $U \in \omega$ with $U \subset W$. This neighborhood is denoted by $\tilde{W}$.

It is sufficient to prove that $f$ extends to a continuous map $F : Z \to 2^X$; then the map $\Phi = F_{[\mu]} : Y \to 2^X$ will be the desired extension of $f$.

Let $N_k(\omega)$ denote the $k$-skeleton of $N(\omega)$. First we extend $f$ to a map $f_0 : A \cup N_0(\omega) \to 2^X$ as follows: in every set $U \in \omega$ we select a point $x_U$ and then choose a point $a_U \in A$ such that $\rho(x_U, a_U) < 2\rho(x_U, A)$. Set $f_0(p_U) = f(a_U)$ and $f_0(a) = f(a)$ for $a \in A$. It is readily seen that $f_0$ is continuous. Now we will extend $f_0$ over each simplex of $N(\omega)$ and thus we obtain the desired map $F$. Since $2^X$ is a Peano continuum [5, Proposition 5.3.10], it is path-connected and locally path-connected by a well-known result of Mazurkiewicz (see [5 Theorem 5.3.13]). For any two points $B, C \in 2^X$ we select a path $l_{B,C} : [0, 1] \to 2^X$ such that $l_{B,C}(0) = B$, $l_{B,C}(1) = C$ and

$$diam(l_{B,C}([0, 1])) < 2\inf\{\text{diam } \gamma([0, 1]) : \gamma \text{ is a path from } B \text{ to } C\}.$$

We now extend $f_0$ to a map $f_1 : A \cup N_1(\omega) \to 2^X$ by the rule: $f_1(a) = f_0(a)$ for $a \in A$ and $f_1(t(p_U) + (1 - t)p_U) = l_{f_0(p_U), f_0(p_U)}(t)$, $0 \leq t \leq 1$. One needs to prove $f_1$ continuous only at points of $A$. Let $a \in A, \varepsilon > 0$ and $O(f(a), \delta)$ be the $\delta$-neighborhood of $f(a)$ in $2^X$. By the local path-connectedness of $2^X$, there is a path-connected neighborhood $Q$ of $f_0(a) = f(a)$ contained in $O(f(a), \varepsilon/8)$. By continuity of $f_0$, there exists a neighborhood of $a$ in $Z$ of the form $\tilde{W}$ such that $f_0(\tilde{W} \cap (A \cup N_0(\omega))) \subset Q$. Then $f_1(\tilde{W} \cap (A \cup N_1(\omega))) \subset O(f(a), \varepsilon)$. Indeed, if $z = t(p_U) + (1 - t)p_U \in \tilde{W} \cap N_1(\omega)$, then $f_0(p_U), f_0(p_V) \in Q$; so $Q$ contains a path $\gamma$, connecting $f_0(p_U)$ and $f_0(p_V)$. Hence $\text{diam } \gamma([0, 1]) < \varepsilon/2$, which implies that $\text{diam } l_{f_0(p_U), f_0(p_V)}([0, 1]) < \varepsilon/2$. Then $d_H(f_1(z), f_1(a)) < \varepsilon$ because $f_1(z) \in l_{f_0(p_U), f_0(p_V)}([0, 1])$.

Now suppose that a continuous extension $f_k : A \cup N_k(\omega) \to 2^X$ of $f_{k-1}$, $k \geq 1$ has already been constructed. We shall construct an extension $f_{k+1} : A \cup N_{k+1}(\omega) \to 2^X$ of $f_k$. Let $\sigma$ be any $(k + 1)$-dimensional simplex in $N(\omega)$. Let $B^{k+1}$ be the $(k + 1)$-dimensional Euclidean closed unit ball and $S^k$ be its boundary space. We aim at approximating the following well-known easy fact: for every $k \geq 1$ there exists a continuous function $r : B^{k+1} \to S^k$ such that $r(y) = \{y\}$ for all $y \in S^k$ (see, e.g., [5, Proposition 5.3.11]). To this end, it is convenient to identify the pair $(\sigma, \partial \sigma)$ with $(B^{k+1}, S^k)$. Then the preceding fact insures the existence of a continuous map $r_\sigma : \sigma \to \partial \sigma$ such that $r_\sigma(z) = \{z\}$ for every $z \in \partial \sigma$. The map $g_\sigma : 2^{\partial \sigma} \to 2^X$ defined by $g_\sigma(C) = \bigcup_{c \in C} f_k(c)$ is continuous [5 Corollary 5.3.7]. Then $f_\sigma = g_\sigma r_\sigma : \sigma \to 2^X$ is a continuous extension of $f_k|\partial \sigma$. Now we set $f_{k+1}(z) = f_\sigma(z)$ if $z \in \sigma$, and $f_{k+1}(a) = f_k(a)$ if $a \in A$. Then $f_{k+1}$ extends $f_k$ and is continuous on $N_{k+1}(\omega)$. We define the map $F : Z \to 2^X$ as follows: $F(z) = f_k(z)$ whenever $z \in A \cup N_k(\omega)$. Clearly, $F$ is continuous on $N(\omega)$. Let us check its continuity at points of $A$. Let $a \in A$ and $\varepsilon > 0$. By continuity of $f_1$, there is a neighborhood of $a$ in $Z$ of the form $\tilde{W}$ such that $f_1(\tilde{W} \cap (A \cup N_1(\omega))) \subset O(f(a), \varepsilon)$. We claim that $F(\tilde{W}) \subset O(f(a), \varepsilon)$. We shall prove by induction on the dimension of $\sigma$ that $F(\sigma) \subset O(f(a), \varepsilon)$ for every simplex $\sigma \subset \tilde{W}$. If $\dim \sigma = 1$, then $F(\sigma) = f_1(\sigma) \subset O(f(a), \varepsilon)$. Assume that the claim is true for all simplices $s \subset \tilde{W}$ with $\dim s \leq k$. Let $\sigma \subset \tilde{W}$, $\dim \sigma = k + 1$ and $z \in \sigma$. As $F(z) = f_{k+1}(z) = g_\sigma(r_\sigma(z))$, we have $F(z) = \bigcup_{c \in r_\sigma(z)} f_k(c)$. But $d_H(f_k(c), f(a)) < \varepsilon$ for all $c \in \partial \sigma$, and in particular, for all $c \in r_\sigma(z)$. This
yields that \( d_H \left( \bigcup_{c \in r_n(z)} f_k(c), f(a) \right) < \varepsilon \), i.e., \( d_H(F(z), f(a)) < \varepsilon \), completing the inductive step.

The reader can easily observe that the same proof serves also for Curtis’ theorem [1, Theorem 1.6] on growth hyperspaces \( G \subset 2^X \), where \( X \) is any connected and locally continuum-connected metrizable space.

References