

QUASILINEARITY BELOW THE 1ST EIGENVALUE

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ABSTRACT. This paper establishes the existence of two nontrivial weak solutions for a quasilinear Dirichlet problem below the first eigenvalue via the mountain pass theorem.

1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^N$ be a bounded open connected set, $N \geq 1$. Denote by D^α the differential operator

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

where $\alpha = (\alpha_1 \cdots \alpha_N)$ is a multi-index consisting of nonnegative integers, and $|\alpha| = \sum_{i=1}^N \alpha_i$ denotes the order of D^α . In order to write nonlinear partial differential operators in a convenient form, we introduce, as in [Br] the vector space \mathbf{R}^{s_m} whose elements are of the form

$$\xi_m = \{\xi_\alpha : |\alpha| \leq m\}$$

where m is a positive integer. For each $u \in W_0^{m,p}(\Omega)$, define $\xi_m(u)(x)$ to be the vector in \mathbf{R}^{s_m} given by

$$\{D^\alpha(u(x)) : |\alpha| \leq m\}.$$

(Observe that $D^{(0,\dots,0)}u = u$.)

In this paper we study the $2m$ th-order quasilinear differential operator in generalized divergence form

$$(1) \quad Q(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi_m(u)).$$

Also we deal with the superlinear differential equation

$$(2) \quad Q(u) = |u|^{q-2}u + \lambda|u|^{p-2}u$$

where λ is a real number strictly less than the first eigenvalue in $W_0^{m,p}(\Omega)$ and $1 < p < q$ where q has certain Sobolev restrictions. It will be clear that $u = 0$ is a weak solution of (2) in $W_0^{m,p}(\Omega)$. We shall show in this paper that under certain

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further assumptions on Q , there are two more nontrivial weak solutions of (2) in $W_0^{m,p}(\Omega)$. The result obtained here is to be viewed as a result in the same direction as in [Sh], a previous paper on the first eigenvalue in the space $W_0^{m,p}(\Omega)$. It will be clear, also, from the proof to be given that a $g(x, u)$ could be added to the right-hand side of (2), but we will leave these details to the interested reader.

The main thrust in this paper is to use the new idea of near p -homogeneity (assumption (A-4) below) in conjunction with variational techniques to obtain a new multiplicity result for a vast new set of equations not handled previously (e.g., examples (28) and (29) below). In particular, in (28) we get an equation involving the mean curvature [GT, p. 357] when $r = 1$, which is near p -homogeneous but not p -homogeneous.

There is a direct line from the theorem presented here using near p -homogeneity through the mountain pass theorem [Ra, p. 7], through the work of Leray and Lions (using (A-3) below, [LL, p. 103]) to the variational techniques of Euler and Lagrange.

We will assume that Q has a variational structure in the sense that there exists a potential function $F : \Omega \times \mathbf{R}^{sm} \rightarrow \mathbf{R}$ satisfying

(F-1) The map $x \rightarrow F(x, \xi_m)$ is measurable for each $\xi_m \in \mathbf{R}^{sm}$, and the map $x \rightarrow F(x, \xi_m)$ is continuously differentiable for a.e. $x \in \Omega$.

(F-2) There exist constants p and c_1 , with $1 < p < \infty$ and $c_1 > 0$, and a nonnegative function $h \in L^1(\Omega)$ such that

$$|F(x, \xi_m)| \leq h(x) + c_1 |\xi_m|^p$$

for a.e. $x \in \Omega$ and all $\xi_m \in \mathbf{R}^{sm}$.

(F-3) $F(x, 0) = 0$ for a.e. $x \in \Omega$, and for each α , with $0 \leq |\alpha| \leq m$,

$$\frac{\partial F}{\partial \xi_\alpha}(x, \xi_m) = A_\alpha(x, \xi_m) \text{ for } (x, \xi_m) \in \Omega \times \mathbf{R}^{sm}.$$

The functions $A_\alpha : \Omega \times \mathbf{R}^{sm} \rightarrow \mathbf{R}$ defined in (F-3) above will be assumed to satisfy the Caratheodory conditions (i.e., $A_\alpha(x, \xi_m)$ is measurable in x for all $\xi_m \in \mathbf{R}^{sm}$ and is continuous in ξ_m for a.e. $x \in \Omega$) as well as the the following four conditions:

(A-1) There exists a constant c_2 , with $c_2 > 0$, and a nonnegative function $\tilde{h} \in L^{p'}(\Omega)$, where $p' = p/(p - 1)$ and p is as in (F-2) above, such that

$$|A_\alpha(x, \xi_m)| \leq \tilde{h}(x) + c_2 |\xi_m|^{p-1}, \quad 0 \leq |\alpha| \leq m,$$

for a.e. $x \in \Omega$ and all $\xi_m \in \mathbf{R}^{sm}$.

(A-2) (Ellipticity) There exists a positive constant c_0 such that

$$\sum_{|\alpha| \leq m} A_\alpha(x, \xi_m) \xi_\alpha \geq c_0 \left\{ \sum_{|\alpha|=m} |\xi_\alpha|^2 \right\}^{p/2}$$

for a.e. $x \in \Omega$ and all $\xi_m \in \mathbf{R}^{sm}$ where p is as in (F-2).

(A-3) (Monotonicity) Let $\xi_m = (\eta_{m-1}, \varsigma_m)$ be the division of ξ_m into its m th order component and the corresponding $(m - 1)$ st order terms η_{m-1} ; i.e., $\eta_{m-1} = \{\xi_\beta : 0 \leq |\beta| \leq m - 1\} \in \mathbf{R}^{s_{m-1}}$, and $\varsigma_m = \{\xi_\alpha : |\alpha| = m\}$. Put $A_\alpha(x, \xi_m) = A_\alpha(x, \eta_{m-1}, \varsigma_m)$. Then for a.e. $x \in \Omega$ and each $\eta_{m-1} \in \mathbf{R}^{s_{m-1}}$,

$$\sum_{|\alpha|=m} [A_\alpha(x, \eta_{m-1}, \varsigma_m) - A_\alpha(x, \eta_{m-1}, \varsigma_m^*)] (\varsigma_\alpha - \varsigma_\alpha^*) > 0 \text{ for } \varsigma_m \neq \varsigma_m^*.$$

(A-4) (Near p -homogeneity) For $0 \leq |\alpha| \leq m$,

$$(i) \ A_\alpha(x, t\xi_m)t\xi_\alpha \leq |t|^p A_\alpha(x, \xi_m)\xi_\alpha, \ |t| \geq 1,$$

$$(ii) \ A_\alpha(x, t\xi_m)t\xi_\alpha \geq |t|^p A_\alpha(x, \xi_m)\xi_\alpha, \ |t| \leq 1,$$

for $t \in \mathbf{R}$, a.e. $x \in \Omega$, and all $\xi_m \in \mathbf{R}^{s_m}$ where p is as in (F-2).

We note, in particular, that (A-4)(ii) and the Caratheodory conditions imply that

$$A_\alpha(x, 0) = 0 \text{ for } 0 \leq |\alpha| \leq m \text{ and a.e. } x \in \Omega.$$

In the concluding section of this paper (i.e., §4), we shall give three examples of a Q defined by (1) which meets (F-1)–(F-3) and (A-1)–(A-4) above.

We define the following semilinear Dirichlet form:

$$(3) \quad \mathcal{Q}(u, v) = \sum_{|\alpha| \leq m} \int_{\Omega} A_\alpha(x, \xi_m(u)) D^\alpha v \ \forall u, v \in W_0^{m,p}(\Omega).$$

In view of (A-1) we see that \mathcal{Q} is well defined on $W_0^{m,p}(\Omega) \times W_0^{m,p}(\Omega)$.

Observe that by the definition of \mathcal{Q} in (3) and (A-2) we get

$$(4) \quad \mathcal{Q}(u, u) \geq c_0 \int_{\Omega} \left\{ \sum_{|\alpha|=m} |D^\alpha u|^2 \right\}^{p/2}$$

for all $u \in W_0^{m,p}(\Omega)$. Then it follows from [Sh, Proposition 1, p. 1822] that

$$\liminf_{\|u\|_{L^p} \rightarrow \infty} \frac{\mathcal{Q}(u, u)}{\|u\|_{L^p}^p} < \infty.$$

So we define as in [Sh, p. 1821]

$$\lambda_1 = \liminf_{\|u\|_{L^p} \rightarrow \infty} \frac{\mathcal{Q}(u, u)}{\|u\|_{L^p}^p}.$$

We will establish the following result where $\tilde{p} = Np/(N - mp)$ if $mp < N$.

Theorem. *Let $1 < p < \infty$ and let Ω be an open bounded connected set. Assume (F-1)–(F-3), and suppose $Q(u)$ is given by (1) where $A_\alpha(x, \xi_m)$ satisfies (A-1)–(A-4) for $0 \leq |\alpha| \leq m$. Suppose furthermore $\lambda < \lambda_1$. Also let q be such that $p < q < \tilde{p}$ for $mp < N$ with $p < q < \infty$ for $mp \geq N$. Then there are at least two nontrivial weak solutions of (2) in $W_0^{m,p}(\Omega)$.*

We observe that if $v \in W_0^{m,p}(\Omega)$ by the Sobolev imbedding theorem [Ad, p. 98], $v \in L^q(\Omega)$ where q is as stated in the theorem. Hence, if u also is in $W_0^{m,p}(\Omega)$, then $|u|^{q-2}uv$ is in $L^1(\Omega)$. Therefore, we say u is a weak solution of equation (2) in $W_0^{m,p}(\Omega)$ if

$$(8) \quad \mathcal{Q}(u, v) = \int_{\Omega} \left[|u|^{q-2}u + \lambda |u|^{p-2}u \right] v \ \forall v \in W_0^{m,p}(\Omega).$$

2. FUNDAMENTAL LEMMAS

We shall make use of the mountain pass theorem [Ra, p. 7], henceforth designated by (MPT), to establish our theorem. Let F be given as in (F-1)–(F-3). Define a functional $\mathcal{I} : W_0^{m,p}(\Omega) \rightarrow \mathbf{R}$ by

$$(9) \quad \mathcal{I}(u) = \int_{\Omega} [F(x, \xi_m(u)) - q^{-1}|u|^q - \lambda|u|^p/p].$$

Throughout this paper we will use the norm in $W_0^{m,p}(\Omega)$ given by

$$(10) \quad \|u\|_{m,p}^p = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p}^p,$$

where $\|\cdot\|_{L^p}$ denotes the L^p -norm. Also, we note that there are positive constants c_3 and c_4 such that

$$(11) \quad c_3 \|u\|_{m,p} \leq \|\xi_m(u)\|_{L^p} \leq c_4 \|u\|_{m,p} \quad \forall u \in W_0^{m,p}(\Omega)$$

and from Poincaré’s inequality (see [GT, p. 164]), there is a positive constant c_5 such that

$$(12) \quad \|\xi_m(u)\|_{L^p}^p \leq c_5 \int_{\Omega} \left\{ \sum_{|\alpha|=m} |D^\alpha u|^2 \right\}^{p/2} \quad \forall u \in W_0^{m,p}(\Omega).$$

In order to invoke the (MPT), we shall need the fact that \mathcal{I} defined in (9) satisfies the (PS)-condition, [Ra, p. 3]. This will follow from the next two lemmas.

Lemma 1. *Assume (F-1)–(F-3), (A-1)–(A-4), $\lambda < \lambda_1$, that p and q are as in the theorem, and that \mathcal{I} is given by (9). Suppose that $\{u_n\}$ is a sequence in $W_0^{m,p}(\Omega)$ satisfying (i) $\mathcal{I}(u_n)$ is uniformly bounded and (ii) $\mathcal{I}'(u_n) \rightarrow 0$ in $W_0^{m,p}(\Omega)^*$ as $n \rightarrow \infty$. Then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which is uniformly bounded in $W_0^{m,p}(\Omega)$.*

In the above, $W_0^{m,p}(\Omega)^*$ is understood to be the dual of $W_0^{m,p}(\Omega)$, i.e., the space of bounded linear functionals sending $W_0^{m,p}(\Omega)$ into \mathbf{R} . Also, $\mathcal{I}'(u)$ is the Frechet derivative of $\mathcal{I}(u)$.

Lemma 2. *Assume (F-1)–(F-3), (A-1)–(A-4), $\lambda < \lambda_1$, that p and q are as in the theorem and that \mathcal{I} is given by (9). Suppose that $\{u_n\}$ is a uniformly bounded sequence in $W_0^{m,p}(\Omega)$ and is also such that $\mathcal{I}'(u_n) \rightarrow 0$ in $W_0^{m,p}(\Omega)^*$ as $n \rightarrow \infty$. Then $\{u_n\}$ has a convergent subsequence; that is, there exist $u \in W_0^{m,p}(\Omega)$ and a subsequence $\{u_{n_k}\}$ satisfying $\lim_{k \rightarrow \infty} \|u_{n_k} - u\|_{m,p} = 0$.*

Since we are assuming throughout this paper that Ω is a bounded open connected set and that p and q are as in the theorem, it follows from [Ad, p. 144] that $W_0^{m,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$. Using this fact, we see that a proof similar to the one given in [RS, Lemma 4] (which hinges on (A-3), the monotonicity condition) also gives the proof of Lemma 2 above; so we will not give the proof of Lemma 2 here and will leave the details of the proof of said lemma to the reader.

We now give the proof of Lemma 1. Suppose to the contrary that the conclusion of the lemma is false. Then, we can suppose that

$$(13) \quad \lim_{n \rightarrow \infty} \|u_n\|_{m,p} = \infty.$$

We first show that under assumption (13), there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ and a positive constant c_6 such that

$$(14) \quad \|u_{n_k}\|_{m,p} \leq c_6 \|u_{n_k}\|_{L^p} \quad \forall k.$$

If (14) does not hold, then

$$(15) \quad \lim_{n \rightarrow \infty} \|u_n\|_{L^p} / \|u_n\|_{m,p} = 0.$$

Suppose (15) holds. Then it follows from (3), (F-3), (A-1), and Fubini's theorem that

$$(16) \quad \int_{\Omega} F(x, \xi_m(u)) = \int_0^1 \mathcal{Q}(tu, u) dt \geq \mathcal{Q}(u, u) / p$$

where we have also used (A-4)(ii).

Next, we observe from (9), (3), (F-3), and (A-1), that

$$(17) \quad \mathcal{I}'(u)(v) = \mathcal{Q}(u, v) - \int_{\Omega} [|u|^{q-2}u + \lambda |u|^{p-2}u]v \quad \forall v \in W_0^{m,p}(\Omega).$$

As a consequence, we see from (9), (16), and (17) that

$$q\mathcal{I}(u_n) - \mathcal{I}'(u_n)(u_n) \geq (q/p - 1) [\mathcal{Q}(u_n, u_n) - \lambda \|u_n\|_{L^p}^p].$$

So if (15) holds, we see on dividing both sides of this last inequality by $\|u_n\|_{m,p}^p$ and using the conditions in the hypothesis in Lemma 1 that

$$(18) \quad 0 \geq \limsup_{n \rightarrow \infty} \mathcal{Q}(u_n, u_n) / \|u_n\|_{m,p}^p.$$

But then (4) joined with (18) tells us that

$$\lim_{n \rightarrow \infty} \left[\int_{\Omega} \left\{ \sum_{|\alpha|=m} |D^\alpha u_n|^2 \right\}^{p/2} / \|u_n\|_{m,p}^p \right] = 0.$$

From (12) this fact in turn gives that

$$\lim_{n \rightarrow \infty} \|\xi_m(u_n)\|_{L^p} / \|u_n\|_{m,p} = 0,$$

which is a direct contradiction of (11). Hence (15) is false, and (14) is indeed true.

Since $1 < p < q$, we next obtain from (13) combined with (14) and Holder's inequality that

$$(19) \quad \lim_{n_k \rightarrow \infty} \|u_{n_k}\|_{L^q} = \infty.$$

Likewise, we see from the conditions in the hypothesis of Lemma 1 combined with (13) and (14) that

$$(20) \quad \lim_{n_k \rightarrow \infty} [p\mathcal{I}(u_{n_k}) - \mathcal{I}'(u_{n_k})(u_{n_k})] / \|u_{n_k}\|_{L^p} = 0.$$

Next, we observe from (9), (16), and (17) that

$$p\mathcal{I}(u_{n_k}) - \mathcal{I}'(u_{n_k})(u_{n_k}) \geq (1 - p/q) \int_{\Omega} |u_{n_k}|^q.$$

Dividing both sides of this last inequality by $\|u_{n_k}\|_{L^p}$ and passing to the limit as $n_k \rightarrow \infty$, we obtain from (20) that

$$(21) \quad \lim_{n_k \rightarrow \infty} \int_{\Omega} |u_{n_k}|^q / \left\{ \int_{\Omega} |u_{n_k}|^p \right\}^{1/p} = 0.$$

But by Holder’s inequality, there is a positive constant c_6 such that

$$(22) \quad c_6 \left\{ \int_{\Omega} |u|^p \right\}^{1/p} \leq \left\{ \int_{\Omega} |u|^q \right\}^{1/q} \quad \forall u \in W_0^{m,p}(\Omega).$$

Hence, we infer from (21) and (22) that

$$\lim_{n_k \rightarrow \infty} \left\{ \int_{\Omega} |u_{n_k}|^q \right\}^{1-1/q} = 0.$$

This last fact is a direct contradiction of (19). Therefore (13) is not true, and there is indeed a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ that is uniformly bounded in $W_0^{m,p}(\Omega)$, which in fact concludes the proof to Lemma 1.

3. PROOF OF THE THEOREM

Without loss of generality, we see from (2) that from the start we can assume that $\lambda_1 > 0$. We prove the theorem by observing that with $\mathcal{I}(u)$ for $u \in W_0^{m,p}(\Omega)$ defined by (9), the Frechet derivative $\mathcal{I}'(u)$ evaluated at v (i.e., $\mathcal{I}'(u)(v)$) is given by (17). Consequently, if $u_o \in W_0^{m,p}(\Omega)$ is a critical point of \mathcal{I} —that is, if $\mathcal{I}'(u_o)(v) = 0$ for all $v \in W_0^{m,p}(\Omega)$ —it is clear from (17) and (8) that u_o constitutes a weak solution of (2) in $W_0^{m,p}(\Omega)$. Furthermore, from (F-3) and (9), it follows that $\mathcal{I}(0) = 0$. Hence, if we can show that the critical point u_o is such that $\mathcal{I}(u_o) \neq 0$, we will have that u_o is a nontrivial weak solution of (2). Also, we observe from (A-4)(i) and (ii) on setting $t = -1$ that

$$-A_{\alpha}(x, -\xi_m)\xi_{\alpha} = A_{\alpha}(x, \xi_m)\xi_{\alpha} \text{ for } |\alpha| \leq m.$$

Hence, $-A_{\alpha}(x, -\xi_m) = A_{\alpha}(x, \xi_m)$ for $|\alpha| \leq m$. Therefore from (3),

$$\mathcal{Q}(-u, v) = -\mathcal{Q}(u, v) \quad \forall u, v \in W_0^{m,p}(\Omega),$$

and we see from (17) that $\mathcal{I}'(-u)(v) = -\mathcal{I}'(u)(v) \quad \forall u, v \in W_0^{m,p}(\Omega)$. Therefore, if u_o is a nontrivial critical point of \mathcal{I} , so is $-u_o$. As a consequence, to prove the theorem, we need only show the existence of one nontrivial critical point of \mathcal{I} . We shall do this by invoking the (MPT), [Ra, p. 7], and showing that the conditions in the hypothesis of the theorem imply that \mathcal{I} satisfies the conditions in the hypothesis of the (MPT) which in turn implies that \mathcal{I} possesses a nontrivial critical point.

To see that \mathcal{I} defined by (9) does indeed satisfy the conditions in the hypothesis of the (MPT), we take for our real Banach space $E = W_0^{m,p}(\Omega)$ and observe from (17), (F-1)–(F-3), (A-1) and (A-2), and the compactness of the imbedding of E in $L^q(\Omega)$ that $\mathcal{I} \in C^1(E, \mathbf{R})$. Also, it follows from Lemmas 1 and 2 that \mathcal{I} satisfies (PS) and from (9) and (F-3) that $\mathcal{I}(0) = 0$.

To complete the proof of the theorem, we see from the (MPT) that it remains to prove

$$(23) \quad \exists \rho, \gamma > 0 \text{ such that } \mathcal{I}(u) \geq \gamma \text{ for } \|u\|_{m,p} = \rho;$$

$$(24) \quad \exists \psi \in E \text{ with } \|\psi\|_{m,p} > \rho \text{ such that } \mathcal{I}(\psi) \leq 0.$$

To establish (23), we observe from (A-4) and (3) that

$$(25) \quad \mathcal{Q}(ru, ru) \leq |r|^p \mathcal{Q}(u, u) \text{ for } |r| \geq 1 \text{ and } u \in E.$$

Hence, from our definition of λ_1 and (25), it follows that for $u \in E$,

$$\frac{\mathcal{Q}(u, u)}{\|u\|_{L^p}^p} \geq \liminf_{r \rightarrow \infty} \frac{\mathcal{Q}(ru, ru)}{\|ru\|_{L^p}^p} \geq \lambda_1.$$

As a consequence of this last set of inequalities with $\lambda_1 = \lambda + 2\delta$ where $\delta > 0$, we obtain from (9) and (16) that

$$\mathcal{I}(u) \geq \frac{\delta}{p} \|u\|_{L^p}^p + \frac{\delta}{\lambda_1 p} \mathcal{Q}(u, u) - \frac{1}{q} \|u\|_{L^q}^q,$$

and hence from (4), (11), and (12) that

$$(26) \quad \mathcal{I}(u) \geq \frac{\delta}{p} \|u\|_{L^p}^p + \frac{\delta c_0 c_3^p}{\lambda_1 p c_5} \|u\|_{m,p}^p - \frac{1}{q} \|u\|_{L^q}^q.$$

Also, we have from the hypothesis on q and from [Ad, p. 98] that there exists $c_7 > 0$ such that $\|u\|_{L^q} \leq c_7 \|u\|_{m,p} \forall u \in W_0^{m,p}(\Omega)$. Hence, we obtain from (26) that

$$\mathcal{I}(u) \geq \frac{\delta}{p} \|u\|_{L^p}^p + \left[\frac{\delta c_0 c_3^p}{\lambda_1 p c_5} - \frac{c_7^q}{q} \|u\|_{m,p}^{q-p} \right] \|u\|_{m,p}^p.$$

Since $q > p$, it is clear that the expression in brackets in this last inequality is positive for $\|u\|_{m,p}$ sufficiently small, and (23) follows immediately from this observation.

To show that (24) is valid, take $\phi \in E$ such that $\|\phi\|_{L^q} \neq 0$. Then it follows from (9), the equality in (16), and (25) that for $r \geq 1$,

$$\mathcal{I}(r\phi) \leq r^p \int_0^1 \mathcal{Q}(t\phi, \phi) dt - r^q \int_{\Omega} |\phi|^q / q - \lambda r^p \int_{\Omega} |\phi|^p / p.$$

Since $\|\phi\|_{L^q} \neq 0$ and $p < q$, it is clear from this last inequality that $\lim_{r \rightarrow \infty} \mathcal{I}(r\phi) = -\infty$. Choose $r_0 > 0$ such that $\mathcal{I}(r_0\phi) \leq -1$ and $\|r_0\phi\|_{m,p} \geq 2\rho$. Set $\psi = r_0\phi$. Then ψ satisfies (24), and (24) is indeed valid. Hence all the conditions in the hypothesis of the (MTP) are met. Consequently, \mathcal{I} possesses a nontrivial critical point, and the proof of the theorem is complete.

4. EXAMPLES

We conclude this paper by presenting a few examples of quasilinear operators $Q(u)$ for which the theorem obtained here applies. The first example is for $m = 1$ with

$$(27) \quad Q(u) = - \sum_{|\alpha|=1} D^\alpha \left[\left(\sum_{|\beta|=1} |D^\beta u|^{2k} \right)^{(p-2k)/2k} (D^\alpha u)^{2(k-1)} D^\alpha u \right],$$

where k is a positive integer and $p > 1$. For this case, the potential function $F : \mathbf{R}^{N+1} \rightarrow \mathbf{R}$ is then

$$F(\xi_1) = \frac{1}{p} \left(\sum_{|\alpha|=1} \xi_\alpha^{2k} \right)^{p/2k}.$$

The second example is also for $m = 1$ with

$$(28) \quad Q(u) = \sum_{|\alpha|=1} (-1)^{|\alpha|} D^\alpha \left[|\xi'_1(u)|^{p-2} + \left(1 + |\xi'_1(u)|^2 \right)^{-(2-r)/2} \right] D^\alpha u$$

where $p \geq 2, 1 \leq r < 2$, and $|\xi'_1(u)|^2 = \sum_{|\alpha|=1} |D^\alpha u|^2$. The potential function $F : \mathbf{R}^{N+1} \rightarrow \mathbf{R}$ in this case is

$$F(\xi_1) = \frac{1}{p} \left(\sum_{|\alpha|=1} \xi_\alpha^2 \right)^{p/2} + \frac{1}{r} \left(1 + \sum_{|\alpha|=1} \xi_\alpha^2 \right)^{r/2} - \frac{1}{r}.$$

For our final example, we let m be an arbitrary positive integer and set

$$(29) \quad Q(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha [c_\alpha |D^\alpha u|^{p-2} + c_\alpha^* \frac{|D^\alpha u|^{2(k-1)}}{(1 + |D^\alpha u|^{2k})^{(2k-r)/2k}}] D^\alpha u$$

where k is a positive integer, $1 \leq r < 2k$, and $p \geq 2k$. Also, $c_\alpha > 0$ and $c_\alpha^* \geq 0$ for $|\alpha| \leq m$. The potential function $F : \mathbf{R}^{s_m} \rightarrow \mathbf{R}$ is

$$F(\xi_m) = \sum_{|\alpha| \leq m} [c_\alpha |\xi_\alpha|^p / p + c_\alpha^* (1 + \xi_\alpha^{2k})^{r/2k} / r] - \sum_{|\alpha| \leq m} c_\alpha^* / r.$$

In all of the above three cases, it is an easy matter to see that F meets (F-1)–(F-3) and that $A_\alpha(\xi_m) = \frac{\partial F}{\partial \xi_\alpha}(\xi_m)$ satisfies (A-1)–(A-4) where, in particular, for the monotonicity condition (A-3) we make use of [KS, p. 16]. We also observe that the last two examples actually are near p -homogeneous and not p -homogeneous.

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