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ITERATION PROCESSES FOR APPROXIMATING FIXED POINTS OF OPERATORS OF MONOTONE TYPE

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Abstract. In this paper, the unique fixed points of multi-valued and single-valued operators of monotone type are approximated by Ishikawa and Mann iteration processes with errors in real Banach spaces. The operators may not satisfy the Lipschitzian conditions. The results presented improve and extend some recent results.

1. Introduction and preliminaries

Throughout this paper, we always assume that X is a real Banach space, X^* is the dual space of X, and $\langle\cdot,\ \cdot\rangle$ is the pairing between X and X^*. If D is any set, 2^D denotes the family of all non-empty subsets of D. The mapping $J\colon X \to 2^{X^*}$ denotes the *normalised duality mapping* defined by

$$J(x) = \{f \in X^* : \langle x,\ f\rangle = \| f \| \cdot \| x \|,\ \| f \| = \| x \|\},\ \ x \in X.$$

Definition 1.1. Let D be a non-empty subset of X and $T\colon D \to 2^X$ be a multi-valued operator.

(1) T is said to be of *monotone type* if there exist a constant $k \in (0,1)$ and an $x^* \in D$ such that for each $x \in D$ there exists $j(x-x^*) \in J(x-x^*)$ satisfying

$$\langle \xi - x^*,\ j(x-x^*)\rangle \le k \cdot \| x - x^* \|^2,\ \text{ for all } \xi \in Tx. \tag{1.1}$$

(2) T is said to be *strongly* (*or strictly*) *accretive* if there is a constant $k \in (0,1)$, called the strong accretive constant for T, such that for each $x, y \in D$, for each $\xi \in Tx,\ \eta \in Ty$, there exists $j(x-y) \in J(x-y)$ with

$$\langle \xi - \eta,\ j(x-y)\rangle \ge k \cdot \| x - y \|^2. \tag{1.2}$$

(3) T is *strongly (or strictly) pseudo-contractive* if $I - T$ is strongly accretive, where I is the identity operator on D.

Lemma 1.1 ([1]). *Let D be a non-empty subset of X. If $T{:}D \to 2^X$ is strongly pseudo-contractive, then for each $x, y \in X$, for each $\xi \in Tx$ and $\eta \in Ty$, there*

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exists $j(x-y) \in J(x-y)$ *such that*

$$\langle \xi - \eta,\ j(x-y)\rangle \leq (1-k)\cdot \| x-y \|^2$$

where $k \in (0,1)$ *is the strongly accretive constant of* $I-T$.

Remark 1. From Lemma 1.1, it is easy to see that a strongly (strictly) pseudo-contractive mapping with a fixed point is of monotone type. For results on strongly accretive mappings or strongly pseudo-contractive mappings, we refer to Chidume [4, 5], Deng and Ding [6], Osilike [9], Tan and Xu [10] and the references therein.

The purpose of this paper is to study the approximate problems of fixed points for multi-valued and single-valued operators of monotone type by Ishikawa and Mann iteration processes with errors in real Banach spaces. The operator may not satisfy a Lipschitz condition. Our results presented in this paper improve and extend some recent results of Chang [1, 2], Chidume [3, 4, 5], Deng and Ding [6], Dunn [7] and Tan and Xu [10].

For the sake of convenience, we first give some definitions and conclusions.

Definition 1.2. Let D be a non-empty subset of X, let $T\colon D \to 2^X$ be a multi-valued mapping and $x_0 \in D$ be a given point. If the sequences $\{x_n\}, \{y_n\} \subset D$ are defined by

$$\begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n \in (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \end{cases} \tag{1.3}$$

then $\{x_n\}$ is called an *Ishikawa iteration process* for T where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying some additive conditions.

In particular, if $\beta_n = 0$ for all $n \geq 0$, then the $\{x_n\} \subset D$ defined by

$$x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{1.4}$$

is called a *Mann iteration process* for T.

The above two iteration processes have been extensively studied by many authors for approximating either fixed points of nonlinear mappings or solutions of nonlinear operator equations in Banach spaces.

Definition 1.3. Let D be a non-empty subset of X, let $T\colon D \to 2^X$ be a multi-valued mapping and $x_0 \in D$ be a given point. If the sequences $\{x_n\}$, $\{y_n\} \subset D$ are defined by

$$\begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n \in (1-\beta_n)x_n + \beta_n T x_n + v_n, \quad n \geq 0, \end{cases} \tag{1.5}$$

then $\{x_n\}$ is called an *Ishikawa iteration process with errors* for T where $\{u_n\}$ and $\{v_n\}$ are two sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying some additive conditions.

In particular, if $\beta_n = 0$, $v_n = 0$ for all $n \geq 0$, then the sequence $\{x_n\} \subset D$ defined by

$$x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T x_n + u_n, \quad n \geq 0, \tag{1.6}$$

is called a *Mann iteration process with errors* for T.

It is obvious that the Mann and Ishikawa iteration processes are both special cases of the Ishikawa iteration process with errors.

The following result is [2, Lemma 2.1]:

Lemma 1.2. *For any $x, y \in X$, we have*

$$\| x+y \|^2 \leq \| x \|^2 + 2\langle y,\ j(x+y)\rangle \ \text{ for all } \ j(x+y) \in J(x+y). \tag{1.7}$$

The following result is [8, Lemma 2]:

Lemma 1.3. *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ be three non-negative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \ \text{ for all } \ n \geq n_0 \tag{1.8}$$

where n_0 is some positive integer, $\{t_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

2. Ishikawa iteration process of fixed points for operators of monotone type

Theorem 2.1. *Let D be a non-empty subset of X and $T\colon D \to 2^X$ be a multi-valued operator of monotone type. If the set $F(T)$ of fixed points of T is non-empty, then:*

(1) *for any $q \in F(T)$, we have $q = x^*$, where x^* is the point appearing in* (1.1), *and so T has a unique fixed point in D;*

(2) *suppose that $T(D)$ is bounded in X and there exists $x_0 \in D$ such that the Ishikawa iteration process with errors $\{x_n\}$, $\{y_n\} \subset D$ defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n\eta_n + u_n, \ \text{ where } \ \eta_n \in Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n\xi_n + v_n, \ \text{ where } \ \xi_n \in Tx_n, \quad n \geq 0, \end{cases} \tag{2.1}$$

where $\{u_n\}$, $\{v_n\}$ are two sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n \to 0\ (n \to \infty)$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\| u_n \| = o(\alpha_n)$;

(iii) $\| \eta_n - \xi_{n+1} \| \to 0\ (n \to \infty)$.

Then $\{x_n\}$ converges strongly to the unique fixed point x^*.

Proof. (1) Let $q \in D$ be a fixed point of T. Since T is of monotone type, from (1.1) we have

$$\langle q - x^*,\ j(q-x^*)\rangle \leq k \cdot \| q - x^* \|^2, \quad \text{where} \quad j(q-x^*) \in J(q-x^*).$$

Therefore, we have

$$(1-k) \cdot \| q - x^* \|^2 \leq 0.$$

Since $k \in (0,1)$, we have $\| q - x^* \|^2 = 0$, that is, $q = x^*$. This implies that T has at most one fixed point in D.

(2) Since $T(D)$ is bounded, let

$$d = sup\{\| \xi \|\colon\ \xi \in Tx,\ x \in D\} + \| x^* \|, \tag{2.2}$$

$$M = d + \| x_0 - x^* \| + 1. \tag{2.3}$$

Again since $\| u_n \| = o(\alpha_n)$, there exist $\varepsilon_n \geq 0$ and $\varepsilon_n \to 0$ $(n \to \infty)$, such that $\| u_n \| = \alpha_n \cdot \varepsilon_n$. Without loss of generality, we can assume that $\varepsilon_n \in (0,1)$. Now we shall prove that

$$\| x_{n+1} - x^* \| \leq M, \quad \text{for each} \quad n \geq 0. \tag{2.4}$$

Indeed, when $n = 0$, we have

$$\begin{aligned} \| x_1 - x^* \| &= \| (1-\alpha_0)(x_0 - x^*) + \alpha_0(\eta_0 - x^*) + u_0 \| \\ &\leq (1-\alpha_0) \| x_0 - x^* \| + \alpha_0(\| \eta_0 \| + \| x^* \|) + \varepsilon_0 \cdot \alpha_0 \\ &\leq \| x_0 - x^* \| + d + \varepsilon_0 \leq M. \end{aligned}$$

Suppose (2.4) is true for $n = k - 1 \geq 0$. Then

$$\begin{aligned} \| x_{k+1} - x^* \| &= \| (1-\alpha_k)(x_k - x^*) + \alpha_k(\eta_k - x^*) + u_k \| \\ &\leq (1-\alpha_k) \| x_k - x^* \| + \alpha_k(\| \eta_k \| + \| x^* \|) + \varepsilon_k \cdot \alpha_k \\ &\leq (1-\alpha_k) \cdot M + \alpha_k(\| \eta_k \| + \| x^* \| + \varepsilon_k) \\ &\leq (1-\alpha_k) \cdot M + \alpha_k \cdot M = M. \end{aligned}$$

Thus (2.4) holds by induction.

On the other hand, by (2.1), Lemma 1.2 and (2.4), we have

$$\begin{aligned} \| x_{n+1} - x^* \|^2 &= \| (1-\alpha_n)(x_n - x^*) + \alpha_n(\eta_n - x^*) + u_n \|^2 \\ &\leq (1-\alpha_n)^2 \| x_n - x^* \|^2 + 2\alpha_n \langle \eta_n - x^*,\ j(x_{n+1} - x^*) \rangle \\ &\quad + 2\langle u_n,\ j(x_{n+1} - x^*) \rangle \\ &\leq (1-\alpha_n)^2 \| x_n - x^* \|^2 + 2\alpha_n \langle \eta_n - \xi_{n+1},\ j(x_{n+1} - x^*) \rangle \\ &\quad + 2\alpha_n \langle \xi_{n+1} - x^*,\ j(x_{n+1} - x^*) \rangle + 2 \| u_n \| \cdot M \qquad (2.5) \\ &\leq (1-\alpha_n)^2 \| x_n - x^* \|^2 + 2\alpha_n \cdot B_n \\ &\quad + 2\alpha_n \langle \xi_{n+1} - x^*,\ j(x_{n+1} - x^*) \rangle \\ &\quad + 2\alpha_n \varepsilon_n \cdot M, \quad \text{for all} \quad j(x_{n+1} - x^*) \in J(x_{n+1} - x^*) \end{aligned}$$

where $B_n = \| \eta_n - \xi_{n+1} \| \cdot \| x_{n+1} - x^* \|$. It follows from (2.4) and condition (iii) that $B_n \to 0$ $(n \to \infty)$.

Since T is of monotone type, $\xi_{n+1} \in Tx_{n+1}$, from (1.1) we know that there exists a $\tilde{j}(x_{n+1} - x^*) \in J(x_{n+1} - x^*)$ such that

$$\langle \xi_{n+1} - x^*,\ \tilde{j}(x_{n+1} - x^*) \rangle \leq k \cdot \| x_{n+1} - x^* \|^2. \qquad (2.6)$$

Substituting (2.6) into (2.5), we have

$$\begin{aligned} \| x_{n+1} - x^* \|^2 &\leq (1-\alpha_n)^2 \| x_n - x^* \|^2 \\ &\quad + 2\alpha_n \cdot B_n + 2\alpha_n \cdot k \| x_{n+1} - x^* \|^2 + 2\alpha_n \varepsilon_n \cdot M. \qquad (2.7) \end{aligned}$$

Since $\alpha_n \to 0$ $(n \to \infty)$, there exists a positive integer n_0 such that when $n \geq n_0$, $2\alpha_n < 1$. Hence $2\alpha_n k < 1$, $1 - 2\alpha_n k > 1 - k$ for all $n \geq n_0$. After simplifying it follows from (2.7) that

$$\begin{aligned} \| x_{n+1} - x^* \|^2 &\leq \frac{(1-\alpha_n)^2}{1 - 2\alpha_n \cdot k} \| x_n - x^* \|^2 + \frac{2\alpha_n}{1 - 2\alpha_n \cdot k}(B_n + \varepsilon_n \cdot M) \\ &\leq \frac{(1-\alpha_n)^2}{1 - 2\alpha_n \cdot k} \| x_n - x^* \|^2 + \frac{2\alpha_n}{1-k} \lambda_n \qquad (2.8) \end{aligned}$$

where $\lambda_n = B_n + \varepsilon_n M \to 0$ $(n \to \infty)$. Since

$$\frac{(1-\alpha_n)^2}{1 - 2\alpha_n \cdot k} = 1 - \frac{2(1-k) - \alpha_n}{1 - 2\alpha_n \cdot k} \alpha_n$$

and

$$\frac{2(1-k)-\alpha_n}{1-2\alpha_n\cdot k} \to 2(1-k)\ (n\to\infty),$$

therefore there exists $n_1 \geq n_0$ such that for any $n \geq n_1$ the following holds:

$$\frac{2(1-k)-\alpha_n}{1-2\alpha_n\cdot k} > 1-k.$$

Hence for any $n \geq n_1$ from (2.8) we have

$$\| x_{n+1}-x^* \|^2 \leq [1-(1-k)\alpha_n]\cdot \| x_n - x^* \|^2 + \frac{2\alpha_n}{1-k}\lambda_n.$$

Letting $a_n = \| x_n - x^* \|^2,\ t_n = (1-k)\alpha_n,\ b_n = \frac{2\alpha_n}{1-k}\lambda_n,\ c_n = 0$, we have

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \quad \text{for all} \quad n \geq n_1$$

and they satisfy all conditions in Lemma 1.3. Therefore $a_n \to 0\ (n\to\infty)$, i.e., $x_n \to x^*\ (n\to\infty)$. This completes the proof.

Remark 2. When T is a multi-valued uniformly continuous mapping with Hausdorff metric induced by norm $\|\cdot\|$, condition (iii) of Theorem 2.1 can be satisfied.

Theorem 2.1 improves and extends Chang [1, Theorem 3.1], in its three aspects: (a) it abolishes the condition that X is uniformly smooth; (b) that $\{u_n\}$ is a summable sequence is replaced by $\| u_n \| = o(\alpha_n)$; (c) it abolishes the condition that $\| v_n \| \to 0\ (n\to\infty)$.

In Theorem 2.1, if $T\colon D\to X$ is a single-valued mapping, and so $\eta_n = Ty_n, \xi_n = Tx_n, n = 0,1,2,\cdots$, we can obtain the following Theorem 2.2.

Theorem 2.2. *Let D be a non-empty subset of X and $T\colon D\to D$ be a single-valued uniformly continuous operator of monotone type such that $F(T)\neq\emptyset$.*

(1) *If $q\in F(T)$, then $q = x^*$, where x^* is the point appearing in* (1.1), *so that T has a unique fixed point in D.*

(2) *Suppose that $T(D)$ is bounded in X and there exists $x_0\in D$ such that the Ishikawa iteration process with errors $\{x_n\}$, $\{y_n\}\subset D$ defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n + u_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n + v_n, \quad n\geq 0, \end{cases} \tag{2.9}$$

where $\{u_n\}$, $\{v_n\}$ are two sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n\to 0,\ \beta_n\to 0\ (n\to\infty)$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\| u_n \| = o(\alpha_n)$ *and* $\| v_n \| \to 0\ (n\to\infty)$.

Then $\{x_n\}$ converges strongly to the unique fixed point x^*.

Proof. It is sufficient to prove that condition (iii) in Theorem 2.1 is satisfied. In fact, from (2.9) we have

$$\begin{aligned} \| y_n - x_{n+1} \| &= \| (\alpha_n-\beta_n)x_n + \beta_n Tx_n - \alpha_n Ty_n + v_n - u_n \| \\ &\leq \| (\alpha_n-\beta_n)x_n + \beta_n Tx_n - \alpha_n Ty_n \| + \| u_n \| + \| v_n \| . \end{aligned} \tag{2.10}$$

Since $T(D)$ is bounded, from (2.2) and (2.3) it is easy to see that $\{x_n\}, \{Tx_n\}, \{Ty_n\}$ all are bounded sequences in D. It follows from (2.10) and conditions: $\alpha_n \to 0,\ \beta_n\to 0,\ \| u_n \|\to 0,\ \| v_n \|\to 0\ (n\to\infty)$ that $\| y_n - x_{n+1} \|\to 0\ (n\to\infty)$. By the uniform continuity of T, we have $\| Ty_n - Tx_{n+1} \|\to 0\ (n\to\infty)$. This shows that condition (iii) in Theorem 2.1 is satisfied. This completes the proof.

Theorem 2.3. *Let D be a non-empty subset of X and $T{:}D \to 2^X$ be a multi-valued operator of monotone type such that $F(T) \neq \emptyset$.*

(1) *If $q \in F(T)$, then $q = x^*$, where x^* is the point appearing in* (1.1)*, so that T has a unique fixed point in D;*

(2) *Suppose that $T(D)$ is bounded in X and there exists $x_0 \in D$ such that the Ishikawa iteration process $\{x_n\}$, $\{y_n\} \subset D$ defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n \eta_n, \ \text{ where } \ \eta_n \in Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n \xi_n, \ \text{ where } \ \xi_n \in Tx_n, \quad n \geq 0, \end{cases} \tag{2.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n \to 0$ $(n \to \infty)$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(ii) $\| \eta_n - \xi_{n+1} \| \to 0$ $(n \to \infty)$.

Then $\{x_n\}$ converges strongly to the unique fixed point x^*.

Proof. Taking $u_n = v_n = 0$ for all $n \geq 0$ in Theorem 2.1, the conclusions of Theorem 2.3 can be obtained from Theorem 2.1 immediately.

Theorem 2.4. *Let D be a non-empty subset of X and $T\colon D \to D$ be a single-valued uniformly continuous operator of monotone type such that $F(T) \neq \emptyset$.*

(1) *If $q \in F(T)$, then $q = x^*$, where x^* is the point appearing in* (1.1)*, so that T has a unique fixed point in D.*

(2) *Suppose that $T(D)$ is bounded in X and there exists $x_0 \in D$ such that the Ishikawa iteration process $\{x_n\}$, $\{y_n\} \subset D$ defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \end{cases} \tag{2.12}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n \to 0$, $\beta_n \to 0$ $(n \to \infty)$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to the unique fixed point x^*.

Proof. Taking $u_n = v_n = 0$ for all $n \geq 0$ in Theorem 2.2, the conclusions of Theorem 2.4 can be obtained from Theorem 2.2 immediately.

Remark 3. (1) Theorem 2.1—Theorem 2.4 improve and extend Chang [1, Theorem 2.1, 3.1 and 3.2], Chidume [3, Theorem 1], and Dunn [7, Theorem 1] in several aspects.

(2) Since every strongly pseudo-contractive mapping with a fixed point is a special case of the monotone type mapping, Theorem 2.1—Theorem 2.4 also improve and extend Liu [8, Theorem 2]. Morever, Theorem 2.1—Theorem 2.4 also improve and extend the corresponding results of Chang [2, Theorem 3.3], Chidume [4, Theorem 2], Chidume [5, Theorem 4], Deng and Ding [6, Theorem 1], Osilike [9, Theorem 1], and Tan and Xu [10, Theorem 3.2 and Theorem 4.2] in several aspects.

3. Mann iteration process of fixed points for operators of monotone type

Theorem 3.1. *Let D be a non-empty subset of X and $T{:}D \to 2^X$ be a multi-valued operator of monotone type such that $F(T) \neq \emptyset$.*

(1) *If $q \in F(T)$, then $q = x^*$, where x^* is the point appearing in* (1.1)*, so that T has a unique fixed point in D.*

(2) *Suppose that $T(D)$ is bounded in X and there exists $x_0 \in D$ such that the Mann iteration process with errors $\{x_n\} \subset D$ defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \xi_n + u_n, \text{ where } \xi_n \in Tx_n, \quad n \geq 0, \tag{3.1}$$

where $\{u_n\}$ is a sequence in X and $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n \to 0$ $(n \to \infty)$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\| u_n \| = o(\alpha_n)$;
(iii) $\| \xi_n - \xi_{n+1} \| \to 0$ $(n \to \infty)$.

Then $\{x_n\}$ converges strongly to the unique fixed point x^*.

Proof. Taking $\beta_n = 0$ and $v_n = 0$ for all $n \geq 0$ in Theorem 2.1, the conclusions of Theorem 3.1 can be obtained from Theorem 2.1 immediately.

Theorem 3.2. *Let D be a non-empty subset of X and $T\colon D \to D$ be a single-valued uniformly continuous operator of monotone type such that $F(T) \neq \emptyset$.*

(1) *If $q \in F(T)$, then $q = x^*$, where x^* is the point appearing in* (1.1)*, so that T has a unique fixed point in D.*

(2) *Suppose that $T(D)$ is bounded in X and there exists $x_0 \in D$ such that the Mann iteration process with errors $\{x_n\} \subset D$ defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + u_n, \quad n \geq 0, \tag{3.2}$$

where $\{u_n\}$ is a sequence in X and $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n \to 0$ $(n \to \infty)$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\| u_n \| = o(\alpha_n)$.

Then $\{x_n\}$ converges strongly to the unique fixed point x^*.

Proof. Taking $\beta_n = 0$ and $v_n = 0$ for all $n \geq 0$ in Theorem 2.2, the conclusions of Theorem 3.2 can be obtained from Theorem 2.2 immediately.

Theorem 3.3. *Let D be a non-empty subset of X and $T\colon D \to 2^X$ be a multi-valued operator of monotone type such that $F(T) \neq \emptyset$.*

(1) *If $q \in F(T)$, then $q = x^*$, where x^* is the point appearing in* (1.1)*, so that T has a unique fixed point in D.*

(2) *Suppose that $T(D)$ is bounded in X and there exists $x_0 \in D$ such that the Mann iteration process $\{x_n\} \subset D$ defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \xi_n, \text{ where } \xi_n \in Tx_n, \quad n \geq 0, \tag{3.3}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n \to 0$ $(n \to \infty)$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\| \xi_n - \xi_{n+1} \| \to 0$ $(n \to \infty)$.

Then $\{x_n\}$ converges strongly to the unique fixed point x^*.

Proof. Taking $u_n = 0$ for all $n \geq 0$ in Theorem 3.1, the conclusions of Theorem 3.3 can be obtained from Theorem 3.1 immediately.

Theorem 3.4. *Let D be a non-empty subset of X and $T\colon D \to D$ be a single-valued uniformly continuous operator of monotone type such that $F(T) \neq \emptyset$.*

(1) *If $q \in F(T)$, then $q = x^*$, where x^* is the point appearing in* (1.1)*, and so T has a unique fixed point in D.*

(2) *Suppose that $T(D)$ is bounded in X and there exists $x_0 \in D$ such that the Mann iteration process $\{x_n\} \subset D$ defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{3.4}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying the following condition:

(i) $\alpha_n \to 0$ $(n \to \infty)$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to the unique fixed point x^*.

Proof. Taking $u_n = v_n = 0$ for all $n \geq 0$ in Theorem 3.2, the conclusions of Theorem 3.4 can be obtained from Theorem 3.2 immediately.

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