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AN EQUIVALENT DEFINITION OF FUNCTIONS OF THE FIRST BAIRE CLASS

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Abstract. An equivalent definition of functions of the first Baire class in terms of $\varepsilon - \delta$ is given.

Let X and Y be metric spaces. A function $f : X \to Y$ is said to be of the *first Baire class*, or *Baire-1*, if $\{x : f(x) \in G\}$ is $\mathcal{F}_\sigma$ for every open set $G \subset Y$. The study of functions of the first Baire class dates back to 1899 in Baire's paper [1]. We refer the reader to [2] and [3] for some classical results on this class of functions.

Suppose X and Y are complete separable metric spaces. It is well known that a function $f : X \to Y$ is Baire-1 if and only if the restriction $f \restriction_P$ of f to any non-empty closed subset P of X has a point of continuity in P. (See, for instance, [2] and [3].) It is easy to see that a pointwise limit of a sequence of continuous functions from X to Y is Baire-1. The converse is not true. For instance, the function $\chi_{[0,1]} : \mathbb{R} \to \{0, 1\}$ is Baire-1 but not a pointwise limit of a sequence of continuous functions. However, if Y is a separable Banach space, then every function $f : X \to Y$ that is Baire-1 is also a pointwise limit of a sequence of continuous functions.

These classifications do not involve $\varepsilon - \delta$ as in the case of the definition of continuity of a function. In this note, we give an equivalent condition for a function to be of Baire class one in terms of $\varepsilon - \delta$.

The elementary notion of continuity has numerous variants. Let (X, d_X) and (Y, d_Y) be complete separable metric spaces. A function $f : X \to Y$ is continuous if, for each $\varepsilon > 0$, $x \in X$, there exist $\delta(x) > 0$ such that

$$d_Y(f(x), f(y)) < \varepsilon$$

whenever

$$(*) \qquad d_X(x, y) < \delta(x) \textbf{ or } d_X(x, y) < \delta(y).$$

In this paper, we show that if the connective '**or**' in (*) were replaced by '**and**', i.e., if (*) were replaced by

$$d_X(x, y) < \min\{\delta(x), \delta(y)\},$$

then we would obtain a necessary and sufficient condition for a function to be of the first Baire class.

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Theorem 1. *Suppose $f : X \to Y$ is a mapping between complete separable metric spaces (X, d_X) and (Y, d_Y). Then the following statements are equivalent.*

(1) For any $\varepsilon > 0$, there exists a positive function δ on X such that

$$d_Y(f(x), f(y)) < \varepsilon$$

whenever

$$d_X(x, y) < \min\{\delta(x), \delta(y)\}.$$

(2) The function f is of the first Baire class.

Proof. (1) $\Rightarrow$ (2). Let F be a non-empty closed subset of X. It suffices to show that $f\restriction_F$ has a point of continuity. To this end, we prove that, for some $x \in F$,

$$\omega_f(x, F) = \inf_{h>0} \sup\{|f(u) - f(v)| : u, v \in N_h(x) \cap F\} = 0,$$

where $N_h(x) = \{u \in X : d_X(u, x) < h\}$.

Suppose, to the contrary,

$$\omega_f(x, F) > 0 \text{ for each } x \in F.$$

Then we have

$$F = \bigcup_{n=1}^{\infty} C_n,$$

where C_n is the closed set $\left\{x \in F : \omega_f(x, F) \geq \frac{1}{n}\right\}$. By Baire's Category Theorem, there exists $n_0 \in \mathbb{N}$ such that C_{n_0} contains a non-empty interior U in F. Write $I = \overline{U} \subset C_{n_0}$.

Now let δ be a positive function as stated in (1) that corresponds to $\varepsilon = \frac{1}{6n_0}$. Put $F_n = \left\{x \in I : \delta(x) > \frac{1}{n}\right\}$. Then we have

$$I = \bigcup_{n=1}^{\infty} F_n.$$

Again by Baire's Category Theorem, there exists $K \in \mathbb{N}$, such that $\overline{F_K}$ has a non-empty interior in I, i.e., there exist $c \in F_K \cap U$ and $r > 0$, $r < \min\left\{\frac{1}{2K}, \delta(c)\right\}$ such that

$$N_r(c) \cap I \subset \overline{F_K}.$$

The proof will be complete if we have shown that

$$(**) \qquad \sup\{|f(u) - f(v)| : u, v \in N_r(c) \cap I\} \leq \frac{2}{3n_0}.$$

Indeed, if this is so, then $\omega_f(c, F) \leq \dfrac{2}{3n_0}$, as $N_r(c) \cap I$ contains a neighbourhood of c in F. This contradicts the fact that $c \in C_{n_0}$. Therefore $\omega_f(x, F) = 0$ for some $x \in F$.

To see (**), let $y \in N_r(c) \cap I \subset \overline{F_K}$. Then there exists

$$x \in N_r(c) \cap N_{\delta(y)}(y) \cap F_K.$$

We note that

$$d_X(x, y) < \min\{2r, \delta(y)\} \leq \min\left\{\frac{1}{K}, \delta(y)\right\} \leq \min\{\delta(x), \delta(y)\}, \text{ and}$$

$$d_X(x, c) < r \leq \min\{\delta(x), \delta(c)\}.$$

Therefore, by our hypothesis,

$$d_Y(f(y), f(c)) \leq d_Y(f(y), f(x)) + d_Y(f(x), f(c)) \leq \frac{1}{6n_0} + \frac{1}{6n_0} = \frac{1}{3n_0}.$$

Consequently,

$$\sup\{|f(u) - f(v)| : u, v \in N_r(c) \cap F \cap I\} \leq \frac{2}{3n_0}.$$

(2) $\Rightarrow$ (1). Suppose $f : X \to Y$ is a Baire-1 function between complete separable metric spaces X and Y. Then Y is isometric to a subset of a separable Banach space Z. Let $h : (Y, d_Y) \to (Z, ||\cdot||_Z)$ be an isometry. Then $h \circ f : X \to Z$ is a Baire-1 function as well. Since Z is a Banach space, $h \circ f$ is a pointwise limit of a sequence of continuous functions $g_n : X \to Z$, $n \in \mathbb{N}$.

We show that f satisfies (1). Let $\varepsilon > 0$ and $x \in X$ be given. For each $n \in \mathbb{N}$, there exists a function $\delta_n > 0$ such that

$$\|g_n(x) - g_n(y)\|_Z < \frac{\varepsilon}{3}$$

whenever $d_X(x, y) < \delta_n(x)$. There also exists an integer n_x such that

$$\|g_n(x) - h \circ f(x)\|_Z < \frac{\varepsilon}{3}$$

for all $n \geq n_x$. Define a positive function $\delta : X \to \mathbb{R}^+$ by

$$\delta(x) = \min_{1 \leq n \leq n_x} \delta_n(x) \text{ for each } x \in X.$$

Let $x, y \in X$ such that

$$d_X(x, y) < \min\{\delta(x), \delta(y)\}.$$

Without loss of generality, we assume that $n_x \leq n_y$; then

$$\begin{aligned}
&||h \circ f(x) - h \circ f(y)||_Z \\
&< \left|\left|h \circ f(x) - g_{n_y}(x)\right|\right|_Z + \left|\left|g_{n_y}(x) - g_{n_y}(y)\right|\right|_Z + \left|\left|g_{n_y}(y) - h \circ f(y)\right|\right|_Z \\
&< \tfrac{\varepsilon}{3} + \left|\left|g_{n_y}(x) - g_{n_y}(y)\right|\right|_Z + \tfrac{\varepsilon}{3}.
\end{aligned}$$

But $\left|\left|g_{n_y}(x) - g_{n_y}(y)\right|\right|_Z < \varepsilon/3$ as $d_X(x, y) \leq \delta(y) \leq \delta_{n_y}(y)$. Finally, since h is an isometry, $d_Y(f(x), f(y)) = ||h \circ f(x) - h \circ f(y)||_Z < \varepsilon$. □

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