AN EQUIVALENT DEFINITION OF FUNCTIONS OF THE FIRST BAIRE CLASS

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Abstract. An equivalent definition of functions of the first Baire class in terms of \( \varepsilon - \delta \) is given.

Let \( X \) and \( Y \) be metric spaces. A function \( f : X \to Y \) is said to be of the first Baire class, or Baire-1, if \( \{ x : f(x) \in G \} \in \mathcal{F}_\sigma \) for every open set \( G \subset Y \). The study of functions of the first Baire class dates back to 1899 in Baire’s paper [1]. We refer the reader to \( [2] \) and \( [3] \) for some classical results on this class of functions.

Suppose \( X \) and \( Y \) are complete separable metric spaces. It is well known that a function \( f : X \to Y \) is Baire-1 if and only if the restriction \( f |_p \) of \( f \) to any non-empty closed subset \( P \) of \( X \) has a point of continuity in \( P \). (See, for instance, \( [2] \) and \( [3] \).) It is easy to see that a pointwise limit of a sequence of continuous functions from \( X \) to \( Y \) is Baire-1. The converse is not true. For instance, the function \( \chi_{[0,1]} : \mathbb{R} \to \{0,1\} \) is Baire-1 but not a pointwise limit of a sequence of continuous functions. However, if \( Y \) is a separable Banach space, then every function \( f : X \to Y \) that is Baire-1 is also a pointwise limit of a sequence of continuous functions.

These classifications do not involve \( \varepsilon - \delta \) as in the case of the definition of continuity of a function. In this note, we give an equivalent condition for a function to be of Baire class one in terms of \( \varepsilon - \delta \).

The elementary notion of continuity has numerous variants. Let \((X,d_X)\) and \((Y,d_Y)\) be complete separable metric spaces. A function \( f : X \to Y \) is continuous if, for each \( \varepsilon > 0 \), \( x \in X \), there exist \( \delta(x) > 0 \) such that
\[
    d_Y (f(x), f(y)) < \varepsilon
\]
whenever
\[
    d_X (x, y) < \delta(x) \quad \text{or} \quad d_X (x, y) < \delta(y).
\]

In this paper, we show that if the connective ‘or’ in (*) were replaced by ‘and’, i.e., if (*) were replaced by
\[
    d_X (x, y) < \min\{\delta(x), \delta(y)\},
\]
then we would obtain a necessary and sufficient condition for a function to be of the first Baire class.

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Theorem 1. Suppose $f : X \to Y$ is a mapping between complete separable metric spaces $(X, d_X)$ and $(Y, d_Y)$. Then the following statements are equivalent.

(1) For any $\varepsilon > 0$, there exists a positive function $\delta$ on $X$ such that
$$d_Y(f(x), f(y)) < \varepsilon$$
whenever
$$d_X(x, y) < \min\{\delta(x), \delta(y)\}.$$

(2) The function $f$ is of the first Baire class.

Proof. (1) $\Rightarrow$ (2). Let $F$ be a non-empty closed subset of $X$. It suffices to show that $f |_F$ has a point of continuity. To this end, we prove that, for some $x \in F$,
$$\omega_f(x, F) = \inf_{h > 0} \sup \{|f(u) - f(v)| : u, v \in N_h(x) \cap F\} = 0,$$
where $N_h(x) = \{u \in X : d_X(u, x) < h\}$.

Suppose, to the contrary,
$$\omega_f(x, F) > 0$$
for each $x \in F$.

Then we have
$$F = \bigcup_{n=1}^{\infty} C_n,$$
where $C_n$ is the closed set $\{x \in F : \omega_f(x, F) \geq \frac{1}{n}\}$. By Baire’s Category Theorem, there exists $n_0 \in \mathbb{N}$ such that $C_{n_0}$ contains a non-empty interior $U$ in $F$. Write $I = \overline{U} \subseteq C_{n_0}$.

Now let $\delta$ be a positive function as stated in (1) that corresponds to $\varepsilon = \frac{1}{n_0}$. Put $F_n = \{x \in I : \delta(x) > \frac{1}{n}\}$. Then we have
$$I = \bigcup_{n=1}^{\infty} F_n.$$

Again by Baire’s Category Theorem, there exists $K \in \mathbb{N}$, such that $\overline{F_K}$ has a non-empty interior in $I$, i.e., there exist $c \in F_K \cap U$ and $r > 0$, $r < \min\left\{\frac{1}{n_0}, \delta(c)\right\}$ such that
$$N_r(c) \cap I \subseteq \overline{F_K}.$$

The proof will be complete if we have shown that

$$\sup \{|f(u) - f(v)| : u, v \in N_r(c) \cap I\} \leq \frac{2}{3n_0};$$

Indeed, if this is so, then $\omega_f(c, F) \leq \frac{2}{3n_0}$, as $N_r(c) \cap I$ contains a neighbourhood of $c$ in $F$. This contradicts the fact that $c \in C_{n_0}$. Therefore $\omega_f(x, F) = 0$ for some $x \in F$.

To see (**), let $y \in N_r(c) \cap I \subseteq \overline{F_K}$. Then there exists
$$x \in N_r(c) \cap N_{\delta(y)}(y) \cap F_K.$$

We note that
$$d_X(x, y) < \min\{2r, \delta(y)\} \leq \min\left\{\frac{1}{K}, \delta(y)\right\} \leq \min\{\delta(x), \delta(y)\},$$
and
$$d_X(x, c) < r \leq \min\{\delta(x), \delta(c)\}.$$
Therefore, by our hypothesis,
\[ d_Y (f(y), f(c)) \leq d_Y (f(y), f(x)) + d_Y (f(x), f(c)) \leq \frac{1}{6n_0} + \frac{1}{6n_0} = \frac{1}{3n_0}. \]
Consequently,
\[ \sup \{ |f(u) - f(v)| : u, v \in N_r(c) \cap F \cap I \} \leq \frac{2}{3n_0}. \]

(2) \Rightarrow (1). Suppose \( f : X \to Y \) is a Baire-1 function between complete separable metric spaces \( X \) and \( Y \). Then \( Y \) is isometric to a subset of a separable Banach space \( Z \). Let \( h : (Y, d_Y) \to (Z, \| \cdot \|_Z) \) be an isometry. Then \( h \circ f : X \to Z \) is a Baire-1 function as well. Since \( Z \) is a Banach space, \( h \circ f \) is a pointwise limit of a sequence of continuous functions \( g_n : X \to Z, n \in \mathbb{N} \).

We show that \( f \) satisfies (1). Let \( \varepsilon > 0 \) and \( x \in X \) be given. For each \( n \in \mathbb{N} \), there exists a function \( \delta_n > 0 \) such that
\[ \|g_n(x) - g_n(y)\|_Z < \frac{\varepsilon}{3} \]
whenever \( d_X (x, y) < \delta_n (x) \). There also exists an integer \( n_x \) such that
\[ \|g_n(x) - h \circ f (x)\|_Z < \frac{\varepsilon}{3} \]
for all \( n \geq n_x \). Define a positive function \( \delta : X \to \mathbb{R}^+ \) by
\[ \delta (x) = \min_{1 \leq n \leq n_x} \delta_n (x) \text{ for each } x \in X. \]
Let \( x, y \in X \) such that
\[ d_X (x, y) < \min \{ \delta (x), \delta (y) \}. \]
Without loss of generality, we assume that \( n_x \leq n_y \); then
\[ \|h \circ f (x) - h \circ f (y)\|_Z < \|h \circ f (x) - g_{n_y} (x)\|_Z + \|g_{n_y} (x) - g_{n_y} (y)\|_Z + \|g_{n_y} (y) - h \circ f (y)\|_Z \]
\[ < \frac{\varepsilon}{3} + \|g_{n_y} (x) - g_{n_y} (y)\|_Z + \frac{\varepsilon}{3}. \]
But \( \|g_{n_y} (x) - g_{n_y} (y)\|_Z < \varepsilon/3 \) as \( d_X (x, y) \leq \delta (y) \leq \delta_{n_y} (y) \). Finally, since \( h \) is an isometry, \( d_Y (f(x), f(y)) = \|h \circ f (x) - h \circ f (y)\|_Z < \varepsilon. \)

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