

SIGNS IN THE cd -INDEX OF EULERIAN PARTIALLY ORDERED SETS

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ABSTRACT. A graded partially ordered set is Eulerian if every interval has the same number of elements of even rank and of odd rank. Face lattices of convex polytopes are Eulerian. For Eulerian partially ordered sets, the flag vector can be encoded efficiently in the cd -index. The cd -index of a polytope has all positive entries. An important open problem is to give the broadest natural class of Eulerian posets having nonnegative cd -index. This paper completely determines which entries of the cd -index are nonnegative for all Eulerian posets. It also shows that there are no other lower or upper bounds on cd -coefficients (except for the coefficient of c^n).

1. INTRODUCTION

In the past thirty years or more, there has been much interest in combinatorial questions about polytopes and other geometric complexes and partial orders. Of central importance is the flag vector of a partially ordered set (poset) and various combinatorial parameters derived from it. One of these parameters is the cd -index, defined for Eulerian posets, a class that contains face lattices of polytopes. The cd -index was discovered by Fine and introduced in the literature by Bayer and Klapper ([4]). It has captured the imagination, both for what is known and for what is not known about it. It embodies in an elegant way the linear relations of flag vectors of Eulerian posets (the generalized Dehn-Sommerville relations of Bayer and Billera [1]); the number of coefficients in the cd -index is a Fibonacci number. It is known to be nonnegative for polytopes (see Stanley [12]), but it is not known what it counts, except in special cases (see Purtill [11]). Among polytopes, the cd -index is minimized by the simplices (see Billera and Ehrenborg [5]). Novik ([10]) gives lower bounds for cd -coefficients of odd-dimensional simplicial manifolds (or, more generally, Eulerian Buchsbaum complexes).

Stanley ([12]) proved the nonnegativity of the cd -index for “S-shellable” regular CW-spheres (including polytopes). In [13] he proposes the following as the main open problem concerning the cd -index: Is the cd -index nonnegative for all Gorenstein* posets? (These are the Cohen-Macaulay Eulerian posets.) In fact, some parts of the cd -index are nonnegative for *all* Eulerian posets. In this paper we determine which cd -words have nonnegative coefficients for all Eulerian posets. For all other cd -words, we show how to construct Eulerian posets with arbitrarily

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large negative coefficients. The proofs grow out of the ideas of [2], which studies the cone of flag vectors of Eulerian posets.

2. DEFINITIONS

An *Eulerian poset* is a graded partially ordered set P in which every interval has the same number of elements of even and of odd rank. For P an Eulerian poset, the dual poset, obtained by reversing the order relation, is also Eulerian. The cd -index of an Eulerian poset is an invariant based on the numbers of chains in the poset. For P an Eulerian poset of rank $n + 1$ and $S \subseteq [1, n]$, $f_S(P)$ is the number of chains in P of the form $\hat{0} \prec x_1 \prec x_2 \prec \cdots \prec x_k \prec \hat{1}$, where $\{\text{rank}(x_i) : 1 \leq i \leq k\} = S$. The 2^n -tuple of flag numbers $f_S(P)$ (as S ranges over all subsets of $[1, n]$) is called the *flag vector* of P . The *flag h -vector* is obtained by performing inclusion-exclusion on the flag vector. Thus $h_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T$ or, equivalently, $f_S = \sum_{T \subseteq S} h_T$. Write a generating function in noncommuting variables, $\Psi(a, b) = \sum h_S u_S$, where $u_S = u_1 u_2 \cdots u_n$ with $u_i = a$ if $i \notin S$ and $u_i = b$ if $i \in S$. For every Eulerian poset, there is a polynomial $\Phi(c, d)$ in noncommuting variables c and d for which $\Psi(a, b) = \Phi(a + b, ab + ba)$. The polynomial $\Phi(c, d)$ (or $\Phi_P(c, d)$ when we need to specify the poset P) is called the *cd -index* of the poset. The cd -index of the dual of the Eulerian poset P is obtained by reversing every cd -word in the cd -index of P . The coefficient of a cd -word w is written as $[w]$ (or $[w]_P$). We think of each d as occupying two positions in a cd -word, namely, the positions of ab or ba in the corresponding ab -words. Let $\text{supp}(w)$ be the set of positions of d in w .

Stanley ([12]) notes a useful variation of the cd -index. The *ce -index* is obtained by replacing every d in $\Phi(c, d)$ by $(cc - ee)/2$. Alternatively, one gets the ce -index from $\Psi(a, b)$ —even for non-Eulerian posets—by letting $c = a + b$ and $e = a - b$. The ce -index is thus a polynomial in the noncommuting variables c and e , where for Eulerian posets the e 's occur only in pairs. Write L_Q for the coefficient of the word $v_Q = v_1 v_2 \cdots v_n$, where $v_i = c$ if $i \notin Q$ and $v_i = e$ if $i \in Q$. The vector of coefficients of the ce -index, $(L_Q(P))$, is also known as the *L -vector* of P .

For an Eulerian poset P , $L_Q(P) = 0$ unless Q is an *even set*, that is, Q is the union of disjoint intervals of even cardinality. We say Q *evenly contains* S , written $S \subseteq_e Q$, if S and Q are even sets, $S \subseteq Q$, and the difference set $Q \setminus S$ is also an even set. An “Eulerian” ce -word v_Q is converted to a sum of cd -words by replacing consecutive pairs of e 's in v_Q by $cc - 2d$ so that no e 's remain. This means that a cd -word w occurs in the expansion of a ce -word v_Q if and only if $\text{supp}(w) \subseteq_e Q$. Thus the coefficient in the cd -index of a cd -word w in which d occurs r times is

$$(1) \quad [w] = (-2)^r \sum_{\text{supp}(w) \subseteq_e Q} L_Q.$$

(See [2] for more information on L -vectors.)

In determining the cone of flag vectors of all graded posets ([6]), Billera and Hetyei construct sequences of posets with convergent (normalized) flag vectors. Bayer and Hetyei apply a doubling operation to some of these to get sequences of Eulerian posets. Given an interval $I = [i, j] \subseteq [1, n]$, a rank $n + 1$ poset P and a positive integer N , let $D_I^N(P)$ be the rank $n + 1$ poset obtained by replacing P_I , the subposet of P consisting of elements with ranks in I , by N copies of itself. The (horizontal) *double DP* of a poset P is the result of starting with P and successively applying the operators $D_{\{i\}}^2$, for $1 \leq i \leq n$. (In the Hasse diagram of P every edge

is replaced by \bowtie .) For \mathcal{I} a set of subintervals of $[1, n]$, \mathcal{I} is an *even interval system* if (1) no interval of \mathcal{I} is contained in another, (2) every interval of \mathcal{I} is of even cardinality, and (3) the intersection of any two intervals of \mathcal{I} is of even cardinality. For each even interval system \mathcal{I} over $[1, n]$, there exists a sequence of Eulerian posets, $DP(n, \mathcal{I}, N)$, whose normalized flag vectors (and hence, normalized cd -indices and ce -indices) converge. These are obtained by starting with a rank $n + 1$ chain, successively applying the operators D_I^N for the intervals $I \in \mathcal{I}$, and finally taking the horizontal double.

For \mathcal{I} an interval system of k intervals, write

$$L_S(DP(n, \mathcal{I})) = \lim_{N \rightarrow \infty} L_S(DP(n, \mathcal{I}, N))/N^k.$$

(Here $2^n N^k$ is the number of maximal chains in $DP(n, \mathcal{I}, N)$.) The symbol $DP(n, \mathcal{I})$ is referred to as a *limit poset*. These ce -index coefficients are given by the formula

$$(2) \quad L_S(DP(n, \mathcal{I})) = \sum_{j=0}^k (-1)^j |\{1 \leq i_1 < \dots < i_j \leq k : I_{i_1} \cup \dots \cup I_{i_j} = S\}|,$$

where $\mathcal{I} = \{I_1, I_2, \dots, I_k\}$. See [2] for details. The formula applies for non-Eulerian limit posets as well; in that case it can give nonzero L_Q for noneven sets Q .

We use one other result stated and proved in [2] (but implicit in [7]).

Proposition (Inequality Lemma). *Let T and V be subsets of $[1, n]$ with $T \subseteq V$, such that for every maximal interval I of V , $|I \cap T| \leq 1$. Write $S = [1, n] \setminus V$. For P any rank $n + 1$ Eulerian poset,*

$$\sum_{R \subseteq T} (-2)^{|T \setminus R|} f_{S \cup R}(P) \geq 0.$$

Equivalently,

$$(-1)^{|T|} \sum_{T \subseteq Q \subseteq V} L_Q(P) \geq 0.$$

3. THE MAIN RESULT

Theorem. 1. *For the following cd -words w , the coefficient of w as a function of Eulerian posets has greatest lower bound 0 and has no upper bound:*

- (a) $c^i d c^j$, with $\min\{i, j\} \leq 1$,
 - (b) $c^i d c d \dots c d c^j$ (at least two d 's alternating with c 's, i and j unrestricted).
2. *The coefficient of c^n in the cd -index of every Eulerian poset is 1.*
 3. *For all other cd -words w , the coefficient of w as a function of Eulerian posets has neither lower nor upper bound.*

Note. For $n \geq 5$, there are $\lfloor \binom{n-2}{2}/3 \rfloor + 4$ cd -words of the types described in part 1. This is a small portion of the cd words for large n .

Proof. The fact that the coefficient of c^n is 1 is immediate from the definition and is included only for completeness.

Let w be any cd -word containing r copies of d , with $r \geq 1$. Let \mathcal{I} be the set of two-element intervals of the positions of d in w . Compute the coefficient of w in the cd -index of $DP(n, \mathcal{I})$. If Q properly contains $\text{supp}(w)$, then by equation (2), $L_Q(DP(n, \mathcal{I})) = 0$. So by equations (1) and (2), for $DP(n, \mathcal{I})$, the coefficient of w is $[w] = (-2)^r L_{\text{supp}(w)}(DP(N, \mathcal{I})) = (-2)^r (-1)^r = 2^r$. This is the limit as N goes

to infinity of $1/N^r$ times $[w]$ for $DP(n, \mathcal{I}, N)$. So $(DP(n, \mathcal{I}, N))$ is a sequence of Eulerian posets with cd -coefficients $[w]$ not bounded above. (Earlier Stanley ([12]) showed this using a different sequence of posets.)

To show nonnegativity in part 1, we use equation (1) and the Inequality Lemma. If $w = dc^j$, let $S = \emptyset$ (so $V = [1, n]$) and $T = \{1\}$. Then the coefficient of w is $[w] = 2(-1) \sum_{T \subseteq Q \subseteq V} L_Q \geq 0$. If $w = cdc^j$ let $S = \{1\}$ (so $V = [2, n]$) and $T = \{2\}$. Then the coefficient of w is $[w] = (-2) \sum_{\{2,3\} \subseteq Q} L_Q = 2(-1) \sum_{T \subseteq Q \subseteq V} L_Q \geq 0$, because L_Q is zero unless Q is an even set. The cases of $w = c^i d$ and $w = c^i dc$ follow by duality.

Let $w = c^i dcd \cdots cdc^j$, with d occurring r times, $r \geq 2$; thus

$$\text{supp}(w) = \{i+1, i+2, i+4, i+5, \dots, i+3r-2, i+3r-1\}.$$

Let

$$S = \{i+3, i+6, \dots, i+3r-3\},$$

$$V = [1, n] \setminus S$$

and

$$T = \{i+2, i+4, i+7, \dots, i+3r-5, i+3r-2\}.$$

Here S is the set of positions of the c 's between d 's and T is a set of one position for each d , adjacent to the positions of the interior c 's. The coefficient of w is $[w] = (-2)^r \sum_{\text{supp}(w) \subseteq Q} L_Q$. The set Q evenly contains $\text{supp}(w)$ if and only if Q is an even set and $T \subseteq Q \subseteq V$. Since $L_Q = 0$ unless Q is an even set, $[w] = 2^r (-1)^r \sum_{T \subseteq Q \subseteq V} L_Q \geq 0$.

The double of the chain, DC^{n+1} , has cd -index c^n , so for the cd -coefficients in part 1, the lower bound of 0 is actually attained.

It remains to show that the coefficients of the cd words in part 3 can be arbitrarily negative. We use several lemmas.

Lemma 1. *For every even $n \geq 4$ the coefficient of $dc^{n-4}d$ as a function of Eulerian posets has no lower bound.*

Proof. Let $\mathcal{I} = \{[1, n]\}$. By equation (2) the only nonzero entries in the L -vector of $DP(n, \mathcal{I})$ are $L_\emptyset = 1$ and $L_{[1, n]} = -1$. By (1) the coefficient of $dc^{n-4}d$ in the cd -index of $DP(n, \mathcal{I})$ is $(-2)^2(-1) = -4$. This is the limit as N goes to infinity of $1/N^2$ times $[dc^{n-4}d]$ for $DP(n, \mathcal{I}, N)$. So $(DP(n, \mathcal{I}, N))$ is a sequence of Eulerian posets with cd -coefficients $[dc^{n-4}d]$ not bounded below. (A formula of Ehrenborg and Readdy ([8]) gives directly that the cd -index of $DP(n, \mathcal{I}, N)$ is $(N+1)c^n - N(cc-2d)^{n/2}$.) \square

In [3] Bayer and Hetyei discuss constructions of Eulerian posets whose normalized L -vectors converge to sums of L -vectors of non-Eulerian Billera-Hetyei limit posets. (A few examples are found in [2, Appendix A].)

Lemma 2. *For every odd $n \geq 7$ the coefficient of $dc^{n-4}d$ as a function of Eulerian posets has no lower bound.*

Proof. Write C^{n+1} for the chain of rank $n+1$. Let

$$P^I(N) = D_{[1,2]}^{N+1} D_{[3,n-3]}^{N+1} D_{[4,n-2]}^{N+1} D_{[n-1,n]}^{N+1} (C^{n+1});$$

let

$$P^{II}(N) = D_{[1,n-3]}^{N+1} D_{[3,n-2]}^{N^2} D_{[4,n]}^{N+1}(C^{n+1});$$

and let

$$P^{III}(N) = D_{[1,n]}^{N^4}(C^{n+1}).$$

Create a poset $P(N)$ from these three posets by identifying the elements of $P^{II}(N)$ with the elements of $P^I(N)$ at ranks 0, 1, 2, $n - 1$, n , and $n + 1$, and then identifying the elements of $P^{III}(N)$ with the elements of $P^I(N)$ and $P^{II}(N)$ only at ranks 0 and $n + 1$. The doubles $DP(N)$ of these posets are Eulerian, and the normalized L -vectors converge as N goes to infinity. Write $L_Q(DP) = \lim_{N \rightarrow \infty} L_Q(DP(N))/f_{[1,n]}(DP(N))$. Then

$$L_Q(DP) = L_Q(DP(n, \mathcal{I}_1)) + L_Q(DP(n, \mathcal{I}_2)) + L_Q(DP(n, \mathcal{I}_3)),$$

where $\mathcal{I}_1 = \{[1, 2], [3, n - 3], [4, n - 2], [n - 1, n]\}$, $\mathcal{I}_2 = \{[1, n - 3], [3, n - 2], [4, n]\}$, and $\mathcal{I}_3 = \{[1, n]\}$. The only nonzero L_Q for which $\{1, 2, n - 1, n\} \subseteq_e Q$ are $L_{\{1,2,n-1,n\}}(DP) = 1$, $L_{[1,n]\setminus\{3\}}(DP) = -1$ and $L_{[1,n]\setminus\{n-2\}}(DP) = -1$, so by equation (1) the coefficient of $dc^{n-4}d$ in the cd -index of DP is -4 . This is the limit as N goes to infinity of $1/f_{[1,n]}(DP(N))$ times $[dc^{n-4}d]$ for $DP(N)$. So $(DP(N))$ is a sequence of Eulerian posets with cd -coefficients $[dc^{n-4}d]$ not bounded below. (In fact, a flag vector calculation gives the coefficient of $dc^{n-4}d$ for $DP(N)$ as $4(N^2 - N^4)$.) □

The proof of Lemma 2 asserts that $DP(N)$ is Eulerian. It is easy to check by equation (2) that $L_Q(DP(n, \mathcal{I}_1)) + L_Q(DP(n, \mathcal{I}_2)) + L_Q(DP(n, \mathcal{I}_3)) = 0$ if Q is not an even set. This condition must hold if every $DP(N)$ is an Eulerian poset. But to prove that $DP(N)$ is Eulerian requires us to show that every interval of the poset has the same number of elements of even rank and of odd rank. We show the details in one case. Let $[x, y]$ be an interval of $P(N)$ with x of rank 2 and y of rank $n - 1$. For the Eulerian condition to hold on corresponding intervals in $DP(N)$, the interval $[x, y]$ of $P(N)$ must have one more element of even rank than of odd rank. If x and y are in the subposet $P^{III}(N)$, then $[x, y]$ has exactly one element of each rank, so the condition is met. Suppose x and y are identified elements of $P^I(N)$ and $P^{II}(N)$. In the open interval (x, y) in $P^I(N)$, ranks 3 and $n - 2$ each have $N + 1$ elements and each other rank has $(N + 1)^2$ elements. In the open interval (x, y) in $P^{II}(N)$, each rank has N^2 elements. So the number of even-rank elements in $[x, y]$ is $2 + ((N + 1)^2 + N^2)(n - 5)/2$, and the number of odd-rank elements in $[x, y]$ is $2(N + 1) + 2N^2 + ((N + 1)^2 + N^2)(n - 7)/2$. The difference is 1. Note that neither $P^I(N)$ nor $P^{II}(N)$ satisfies the Eulerian condition for $[x, y]$ by itself. The two subposets balance each other to achieve the Eulerian property. This works for all intervals.

Lemma 3. *The coefficient of $cdcc$ as a function of rank 7 Eulerian posets has no lower bound.*

Proof. The following limit poset is given in Appendix A of [2]. Let $P^I(N) = D_{[1,2]}^N D_{[2,6]}^N(C^7)$ and $P^{II}(N) = D_{[1,5]}^N D_{[5,6]}^N(C^7)$. Let $P(N)$ be formed from these two posets by identifying the elements at ranks 0, 1, 6, and 7. The double $DP(N)$ of this poset is Eulerian. In the limit, the normalized L -vector includes the following

values: $L_{34} = L_{1234} = L_{3456} = 0$, and $L_{123456} = 1$. These are the L_Q that contribute to the coefficient of $cdcc$ in the cd -index,

$$[cdcc] = -2(L_{34} + L_{1234} + L_{3456} + L_{123456}) = -2.$$

As argued before, this gives a sequence of Eulerian posets with cd -coefficients $[cdcc]$ not bounded below. (In fact, for $DP(N)$, $[cdcc] = -2(N-1)^2$.) \square

Lemma 4. *Let u and v be cd -words. If the coefficient of u as a function of Eulerian posets has no lower bound, then the coefficients of uv and vu as functions of Eulerian posets have no lower bounds.*

Proof. In [12] Stanley considers a “join” operation, which produces an Eulerian poset $P * Q$ from two Eulerian posets P and Q . He shows that the cd -indices satisfy $\Phi_{P*Q}(c, d) = \Phi_P(c, d)\Phi_Q(c, d)$. Let u be a cd -word of length m and v a cd -word of length n . Let B be the rank $n+1$ Boolean algebra. Every cd -word of length n has a positive coefficient in the cd -index of B . (This is proved most easily from the Ehrenborg-Readdy formula for the cd -index of a pyramid in [8].) Let P_N be a sequence of rank $m+1$ Eulerian posets for which $\lim_{N \rightarrow \infty} [u]_{P_N} = -\infty$. Then $\lim_{N \rightarrow \infty} [uv]_{P_N * B} = \lim_{N \rightarrow \infty} [vu]_{B * P_N} = -\infty$. \square

We now complete the proof of the theorem. Every cd -word not included in parts 1 and 2 of the Theorem contains the subword $cdcc$ or a subword of the form $dc^{n-4}d$ for $n-4 \neq 1$. Thus, by Lemmas 1 through 4, the coefficients of these cd -words as functions of Eulerian posets have no lower bounds. \square

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