A FORMULA FOR $k$-HYPONORMALITY OF BACKSTEP EXTENSIONS OF SUBNORMAL WEIGHTED SHIFTS

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ABSTRACT. Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence of positive real numbers and let $W_\alpha$ be a subnormal weighted shift with a weight sequence $\alpha$. Consider an extended weight sequence $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ with $0 < x \leq \alpha_0$ and let $HE(\alpha, k) := \{ x > 0 : W_\alpha(x) \text{ is } k\text{-hyponormal} \}$ for $k \in \mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ is the set of natural numbers. We obtain a formula to find the interval $HE(\alpha, k) \setminus HE(\alpha, k + 1)$, which provides several examples to distinguish the classes of $k$-hyponormal operators from one another.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $A, B \in \mathcal{L}(\mathcal{H})$, let $[A, B] = AB - BA$. We say that an $n$-tuple $T = (T_1, \cdots, T_n)$ of operators in $\mathcal{L}(\mathcal{H})$ is hyponormal if the operator matrix $([T_j^* T_i])_{i,j=1}^n$ is positive on the direct sum of $n$ copies of $\mathcal{H}$. For arbitrary positive integer $k$, $T \in \mathcal{L}(\mathcal{H})$ is $k$-hyponormal if $(I, T, \cdots, T^k)$ is hyponormal. It is well known that $T$ is subnormal if and only if $T$ is $1$-hyponormal (cf. [Br], [Hal]).

Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a sequence of positive real numbers. Let $x > 0$ and let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be an augmented weight sequence. For $k \in \mathbb{N} \cup \{\infty\}$, we write $HE(\alpha, k)$ for the set of all positive real variable $x$ such that $W_\alpha(x)$ is $k$-hyponormal (cf. [Ch1, Ch2]). It follows from [Ch1] that if $\alpha(x) : x, \sqrt{\frac{3}{2}}, \sqrt{\frac{4}{3}}, \sqrt{\frac{5}{4}}, \cdots$, then there exists a sequence $\{\lambda_k\}_{k=1}^\infty$ of positive numbers with $\lim_{k \to \infty} \lambda_k = \sqrt{\frac{3}{2}}$ such that $\lambda_k > \lambda_{k+1}$ ($k \geq 1$) and $HE(\alpha, k) = (0, \lambda_k]$, where $\lambda_1 = \sqrt{\frac{3}{2}}, \lambda_2 = \sqrt{\frac{3}{2}}, \lambda_3 = \sqrt{\frac{5}{3}}, \lambda_4 = \sqrt{\frac{5}{4}}, \cdots$ and $HE(\alpha, \infty) = (0, \sqrt{\frac{3}{2}}]$, which gives an example that distinguishes the classes of $k$-hyponormal operators from one another. In this paper, we obtain a formula that captures such examples.

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Note that for a unilateral weighted shift $W_\alpha$ with $\alpha_n = \alpha_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, 2-hyponormality immediately forces the weight sequence $\alpha$ to be flat, that is, $\alpha_1 = \alpha_2 = \cdots$ (cf. [Cu1]). In [Sta], J. Stampfli had previously established this for subnormal shifts, so if the subnormal weighted shift is not flat, its weight sequence $\alpha$ satisfies $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$. Throughout this paper we may assume that the subnormal weighted shift $W_\alpha$ satisfies $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ to escape the trivial case.

A weighted shift $W_\alpha$ is said to be recursively generated if there exist an integer $r \geq 1$ and a vector $\psi = (\psi_0, \cdots, \psi_{r-1}) \in \mathbb{C}^r$ such that $\gamma_n = \psi_{r-1} \gamma_{n-1} + \cdots + \psi_0 \gamma_{n-r}$ ($n \geq r$), where $\gamma_n$ ($n \geq 0$) is the moment of $W_\alpha$, i.e., $\gamma_0 = 1$, $\gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2$ ($n \geq 1$), equivalently,

\[
\alpha_n^2 = \psi_{r-1} + \frac{\psi_{r-2}}{\alpha_{n-1}^2} + \cdots + \frac{\psi_0}{\alpha_{n-r+1}^2} \quad (n \geq r).
\]

The smallest such integer $r$ is called the rank of $\gamma$. A weighted shift $W_\alpha$ is non-recursively generated if it is not recursively generated. Note that a subnormal weighted shift is recursively generated if and only if the corresponding probability measure has finite support (cf. [ShT], p. 6) or [CuF1].

For the moment sequence $(\gamma_n)_{n=0}^{\infty}$ of $W_\alpha$, we denote

\[
A(i,j) := \begin{bmatrix}
\gamma_i & \gamma_{i+1} & \cdots & \gamma_{i+j} \\
\gamma_{i+1} & \gamma_{i+2} & \cdots & \gamma_{i+j+1} \\
& \ddots & \ddots & \ddots \\
\gamma_{i+j} & \gamma_{i+j+1} & \cdots & \gamma_{i+2j}
\end{bmatrix}
\]

If a subnormal weighted shift $W_\alpha$ is recursively generated and rank$\gamma = r$, then $\det A(i, r-1) \neq 0$ and $\det A(i, j) = 0$ for any $i \geq 1, j \geq r$. Note that if a subnormal weighted shift $W_\alpha$ is non-recursively generated, then $\det A(i, j) > 0$ for any positive integers $i$ and $j$ (cf. [Cu3]).

2. A FORMULA FOR $k$-HYPONORMALITY

2.1. Non-recursively generated type. Let $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ be a sequence of positive real numbers. Let $x > 0$ and let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be an augmented weight sequence. Assume that $W_\alpha$ is a non-recursively generated subnormal weighted shift. For brevity, let us put $t := \frac{1}{2x}$. Then it follows from [Cu1] Theorem 4] that $W_{\alpha(x)}$ is $k$-hyponormal if and only if

\[
D_k(t) := \begin{bmatrix}
t & \gamma_0 & \gamma_1 & \cdots & \gamma_{k-1} \\
\gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\
& \ddots & \ddots & \ddots & \ddots \\
\gamma_{k-1} & \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1}
\end{bmatrix}
\]

is non-negative, where $\gamma_0 := 1, \gamma_{n+1} := \alpha_n^2 \gamma_n$ ($n \geq 0$). Note that $d_k(t) := \det D_k(t)$ is a polynomial in $t$ of degree $1$. Since $W_\alpha$ is non-recursively generated subnormal, the coefficient of $t$ in $d_k(t)$, $\det A(1, k-1)$, is positive. Hence $d_k(t)$ has a unique zero. We write $t_k := t_k(\alpha)$ for the unique zero of $d_k(t)$. 
Theorem 2.1. Let \( \alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots \) be a sequence of positive real numbers. Assume that \( W_\alpha \) is a non-recursively generated subnormal weighted shift. Let \( x > 0 \) and let \( \alpha(x) : x, \alpha_0, \alpha_1, \cdots \) be the associated augmented weight sequence. Let \( t_k := t_k(\alpha) \) be the unique zero of \( \det D_k(t) \), where \( t := \frac{1}{\alpha} \). Then

\[
(2.1) \quad t_{k+1}(\alpha) = t_k(\alpha) + \frac{[\det A(0, k)]^2}{\det A(1, k-1) \cdot \det A(1, k)}
\]

for all \( k = 1, 2, \cdots \).

For an \( n \times n \) matrix \( A = [a_{ij}]_{1 \leq i,j \leq n} \), we write

\[
A \begin{pmatrix} i_1 & i_2 & \cdots & i_p \end{pmatrix} = \begin{pmatrix} a_{i_1, k_1} & a_{i_1, k_2} & \cdots & a_{i_1, k_p} \\ a_{i_2, k_1} & a_{i_2, k_2} & \cdots & a_{i_2, k_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_p, k_1} & a_{i_p, k_2} & \cdots & a_{i_p, k_p} \end{pmatrix}
\]

for a minor of \( A \) of order \( p \). We recall a fundamental result from [Gan, p. 22] as follows.

Lemma 2.2. Let \( A = [a_{ij}]_{1 \leq i,j \leq n} \) be an \( n \times n \) matrix. Then

\[
\det A = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} A \begin{pmatrix} k_1 & k_2 & \cdots & k_p \end{pmatrix} \begin{pmatrix} i_1 & i_2 & \cdots & i_p \end{pmatrix} (-1)^{\sum_{e=1}^p i_e + \sum_{e=1}^p k_e} \cdot A \begin{pmatrix} k'_1 & k'_2 & \cdots & k'_{n-p} \end{pmatrix} \begin{pmatrix} i'_1 & i'_2 & \cdots & i'_{n-p} \end{pmatrix}
\]

(2.2)

where \( i_1 < i_2 < \cdots < i_p \) and \( i'_1 < i'_2 < \cdots < i'_{n-p} \) form a complete system of indices \( 1, 2, \cdots, n \), as do \( k_1 < k_2 < \cdots < k_p \) and \( k'_1 < k'_2 < \cdots < k'_{n-p} \).

Proof of Theorem 2.1. Let \( M^{(i)}_k \) be the \( k \times k \) matrix obtained by removing the \( (i+1) \)-th column from the matrix

\[
\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{k-1} & \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} \end{bmatrix}
\]

and let \( d^{(i)}_k := \det M^{(i)}_k \) for \( i = 0, 1, 2, \cdots, k \). Let us expand \( d_k(t) = \det D_k(t) \) by the first row to obtain

\[
d_k(t) = td^{(0)}_k - \gamma_0 d^{(1)}_k + \gamma_1 d^{(2)}_k - \cdots + (-1)^k \gamma_{k-1} d^{(k)}_k.
\]

Let \( t_k := t_k(\alpha) \) be the zero of \( d_k(t) \). Since \( d^{(0)}_k > 0 \) for \( k = 1, 2, \cdots, \), we have

\[
t_k = \frac{\gamma_0 d^{(1)}_k}{d^{(0)}_k} - \frac{\gamma_1 d^{(2)}_k}{d^{(0)}_k} + \cdots + (-1)^{k+1} \frac{\gamma_{k-1} d^{(k)}_k}{d^{(0)}_k} \quad (k \in \mathbb{N}).
\]
Hence
\[ d_k^{(0)} d_{k+1}^{(0)} (t_{k+1} - t_k) = \gamma_0 (d_k^{(0)} d_{k+1}^{(1)} - d_k^{(1)} d_{k+1}^{(0)}) - \gamma_1 (d_k^{(0)} d_{k+1}^{(2)} - d_k^{(2)} d_{k+1}^{(0)}) + \cdots + (-1)^{k-1} \gamma_{k-1} (d_k^{(0)} d_{k+1}^{(k-1)} - d_k^{(k-1)} d_{k+1}^{(0)}) + (-1)^k \gamma_k d_k^{(0)} d_{k+1}^{(k+1)}. \]

(2.3)

We first denote
\[ \overline{A}^{(i)}_{(2k+1) \times (2k+1)} := \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{i+1} & \cdots & \gamma_k & \gamma_{k+1} \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{i+2} & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{i+k} & \cdots & \gamma_{2k} & \gamma_{2k+1} \\ \gamma_{i+1} \\ \gamma_{i+2} \\ \vdots \\ \gamma_{i+k} \\ O_{k \times i} & \gamma_{i+1} & \gamma_{i+2} & \gamma_{k+1} \end{bmatrix}, \]

where \( O_{i \times j} \) is the \( i \times j \) zero matrix and \( B_{k+1}^{(i+1)} \) is the \( k \times k \) submatrix obtained by removing the \((i+1)\)-column from the matrix

\[ \begin{bmatrix} \gamma_1 & \cdots & \gamma_{k+1} \\ \vdots & \ddots & \vdots \\ \gamma_k & \cdots & \gamma_{2k} \end{bmatrix}, \]

\((i = 0, 1, \ldots, k)\). Let \( v(i, j) = [\gamma_i, \gamma_{i+1}, \ldots, \gamma_{i+j}]^T \). Since the \( k+2 \) columns of the submatrix

\[ \begin{bmatrix} \gamma_0 & \cdots & \gamma_{k-1} & \gamma_k & \gamma_{k+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \gamma_k & \cdots & \gamma_{2k} & \gamma_{2k+1} \end{bmatrix}, \]

in the upper half submatrix of the matrix \( \overline{A}^{(i)}_{(2k+1) \times (2k+1)} \) are linearly dependent, there exist real numbers \( \phi_i, \ i = 0, 1, \ldots, k \), such that \( v(k+1, k) = \phi_0 v(0, k) + \cdots + \phi_k v(k, k) \), which proves easily that the columns of \( \overline{A}^{(i)}_{(2k+1) \times (2k+1)} \) are linearly dependent. Hence

\[ \det \overline{A}^{(i)}_{(2k+1) \times (2k+1)} = 0 \quad \text{for} \quad i = 0, 1, \ldots, k. \]

(2.4)
For brevity we write $\tilde{A} := A^{(i)}_{(2k+1) \times (2k+1)}$. Then, applying Lemma 2.2 (using $\tilde{A}$ and the last $k$ rows of $A$), we have that

$$\det \tilde{A} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2k+1} A^{(k+2 \cdots 2k+1)}_{i_1 \cdots i_k} \cdot (-1)^{\sum_{v=1}^{k} i_v + \sum_{v=k+1}^{2k+1} v} \cdot \tilde{A}^{(1 \cdots k+1)}_{i_1' \cdots i_{k+1}'}$$

where $i_1 < i_2 < \cdots < i_k$ and $i_1' < i_2' < \cdots < i_{k+1}'$

form a complete system of indices $1, 2, \cdots, 2k+1$.

$$\begin{aligned}
&= (-1)^{1+\sum_{v=k+2}^{2k+1} v + \sum_{v=k+2}^{2k+1} v} \cdot d_k^{(i+1)} d_{k+1}^{(0)} \\
&\quad + (-1)^{k+2+\sum_{v=k+2}^{2k+1} v + \sum_{v=k+2}^{2k+1} v} \cdot \det B_{k+1}^{(i+1)} \cdot \det A(0, k) \\
&\quad + (-1)^{(i+2) + \sum_{v=k+2}^{2k+1} v + \sum_{v=k+2}^{2k+1} v} \cdot d_k^{(i+1)} d_{k+1}^{(0)} \\
&= d_k^{(i+1)} d_{k+1}^{(0)} + \det B_{k+1}^{(i+1)} \cdot \det A(0, k) - d_k^{(i+1)} d_{k+1}^{(0)}.
\end{aligned}$$

Hence by (2.4) we have

$$\text{(2.5)} \quad d_k^{(i)} d_{k+1}^{(0)} d_k^{(i+1)} d_{k+1}^{(0)} = \det A(0, k) \cdot \det B_{k+1}^{(i+1)} \quad (i = 0, \cdots, k-1).$$

Since $d_{k+1}^{(k+1)} = \det A(0, k)$, by (2.3) and (2.5) we have

$$d_k^{(0)} d_{k+1}^{(0)} (t_{k+1} - t_k) = [\det A(0, k)] \cdot \sum_{i=0}^{k-1} (-1)^i \gamma_i \det B_{k+1}^{(i+1)} + (-1)^k \gamma_k d_k^{(0)}$$

which proves the theorem.

Since $\det A(1, k-1) > 0$ for $k = 1, 2, \cdots$, $d_k(t) \geq 0 \iff t \geq t_k(\alpha)$ for all $k = 1, 2, \cdots$. Since $\det A(0, k) > 0$ for $k = 1, 2, \cdots$ and $t_1(\alpha) = \frac{\alpha^2}{\gamma_1}$, by (2.1), $0 < t_1(\alpha) < t_2(\alpha) < \cdots$. Hence we have the following corollary.

**Corollary 2.3.** Let $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ be a sequence of positive real numbers. Let $x > 0$ and let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be the associated augmented weight sequence. Assume that $W_\alpha$ is a non-recursively subnormal weighted shift.
Let \( t_k := t_k(\alpha) \) be the unique zero of \( \det D_k(t) \), where \( t := \frac{1}{t_k} \). Then for any \( k \in \mathbb{N} \),
\[
(2.6) \quad HE(\alpha, k) \setminus HE(\alpha, k + 1) = \left( \frac{1}{\sqrt{t_{k+1}(\alpha)}}, \frac{1}{\sqrt{t_k(\alpha)}} \right].
\]
In particular, \( HE(\alpha, \infty) = \bigcap_{k=1}^{\infty} (0, \frac{1}{\sqrt{t_k(\alpha)}}] \).

2.2. Recursively generated type. Let \( W_\alpha \) be a recursively generated subnormal weighted shift and let \( \text{rank} \gamma = r \). Then \( \det A(1, i) > 0 \) for \( i = 1, \ldots, r-1 \) and \( \det A(1, i) = 0 \) for \( i \geq r \). According to the proof of Theorem 2.1,
\[
(2.7) \quad t_{k+1}(\alpha) = t_k(\alpha) + \frac{[\det A(0, k)]^2}{\det A(1, k-1) \cdot \det A(1, k)}
\]
for all \( k = 1, 2, \ldots, r-1 \). Since \( \text{rank} D_{k+1}(t) = \text{rank} D_k(t) \) \( (k \geq r) \) by [CuF1], we have that \( D_{k+1}(t) \geq 0 \iff D_k(t) \geq 0 \) \( (k \geq r) \). So \( t_{k+1}(\alpha) = t_k(\alpha) \) for all \( k \geq r \). Hence we have the following proposition.

Proposition 2.4. Assume \( W_\alpha \) is a recursively generated subnormal weighted shift and let \( \text{rank} \gamma = r \). Let \( x > 0 \) and let \( \alpha(x) : x, \alpha_0, \alpha_1, \ldots \) be an augmented weight sequence. Let \( t_k := t_k(\alpha) \) be the unique zero of \( \det D_k(t) \), where \( t := \frac{1}{t_k} \). Then we have
(i) for \( p \leq r-1 \),
\[
HE(\alpha, p) \setminus HE(\alpha, p + 1) = \left( \frac{1}{\sqrt{t_{p+1}(\alpha)}}, \frac{1}{\sqrt{t_p(\alpha)}} \right],
\]
(ii) for any \( p \geq r \), \( HE(\alpha, p) = HE(\alpha, \infty) \).

3. Examples

3.1. Non-recursively generated type. Given any non-recursively generated subnormal weighted shift \( W_\alpha \), by Theorem 2.1 and Corollary 2.3 the one step extension of \( W_\alpha \) provides several examples to distinguish the classes of \( k \)-hyponormal operators. For example, we may recapture Curto’s example [Cu1] Proposition 7] as follows.

Example 3.1. Let \( \alpha_n := \sqrt{\frac{4 + \sqrt{n+3}}{n+3}} \) \( (n \geq 0) \). It follows from [Cu1] (or Example 3.2) that \( W_{\alpha_n(x)} \) is subnormal if and only if \( 0 < x \leq \sqrt{\frac{2}{3}} \). Since the support of Berger measure corresponded by \( W_\alpha \) is not finite, \( W_\alpha \) is non-recursively generated. Applying Theorem 2.1, we have \( t_1 = \frac{2}{5}, t_2 = \frac{16}{25}, t_3 = \frac{16}{25}, t_4 = \frac{48}{25}, t_5 = \frac{48}{25}, \ldots \). Hence \( HE(\alpha, k) \setminus HE(\alpha, k + 1) = (\lambda_{k+1}, \lambda_k) \), where \( \lambda_1 = \sqrt{\frac{2}{3}}, \lambda_2 = \frac{4}{5}, \lambda_3 = \sqrt{\frac{2}{5}}, \lambda_4 = \sqrt{\frac{2}{5}}, \lambda_5 = \sqrt{\frac{48}{25}}, \ldots \), and \( HE(\alpha, \infty) = (0, \sqrt{\frac{2}{3}}] \).

Let \( W_\alpha \) be a weighted shift whose restriction to \( \sqrt{\{e_1, e_2, \ldots \}} \) is subnormal, with associated Berger measure \( \mu \). Then it follows from [Cu1] Proposition 8] that \( W_\alpha \) is subnormal iff
\[
(3.1) \quad \frac{1}{t} \in L^1(\mu) \quad \text{and} \quad \alpha_0^2 \cdot \left\| \frac{1}{t} \right\|_{L^1(\mu)} \leq 1.
\]
In particular, \( W_\alpha \) is never subnormal when \( \mu(\{0\}) > 0 \). The following example is useful when considering the behavior of Bergman shift extensions.
Example 3.2. Let
\[ \alpha(x_1, \ldots, x_n) = x_n, \ldots, x_1, \sqrt{m \over m+1}, \ldots, \sqrt{m+k-1 \over m+k}, \ldots. \]

(i) If \(1 \leq n \leq m-1\), then
\[ HE(\alpha, \infty) = \{(x_1, \ldots, x_n) | W_{\alpha(x_1, \ldots, x_n)} \text{ is subnormal}\} \]
\[ = \left\{ \left( \sqrt{m-1 \over m}, \sqrt{m-2 \over m-1}, \ldots, \sqrt{m-n+1 \over m-n+2}, x_n \right) \mid 0 < x_n \leq \sqrt{m-n \over m-n+1} \right\}. \]

(ii) If \(n \geq m\), then \(HE(\alpha, \infty) = \emptyset\).

Proof. (i) First we will find the range of \(x_1\) needed for the subnormality of \(W_{\alpha(x_1)}\). Let \(\mu_1\) be the probability measure corresponding to the subnormal weighted shift with the weights \(\sqrt{m \over m+1}, \sqrt{m+1 \over m+2}, \ldots, \sqrt{m+k-1 \over m+k}, \ldots\). Then
\[ \int_{[0,1]} t^k d\mu_1(t) = \frac{m}{m+1} \frac{m+k-1}{m+k} = \frac{m}{m+k} = \int_0^1 mt^{k+m-1} dt. \]
So \(d\mu_1 = mt^{m-1} dt\). Since \(\|1/t\|^1_{L^1(\mu_1)} = \frac{m}{m-1}\), by (3.1) we have \(x_1 \leq \sqrt{m-1 \over m}\). Let \(\mu_2\) be the probability measure corresponding to the weighted shift with the weights \(x_1, \sqrt{m \over m+1}, \ldots, \sqrt{m+k-1 \over m+k}, \ldots\). Since
\[ x_1^2 \cdot \frac{m}{m+1} \ldots \frac{m+k-2}{m+k-1} = \int_0^1 t^k d\mu_2, \]
using a method similar to that described above, we have \(d\mu_2 = x_1^2 mt^{m-2} dt\). Furthermore, since \(\mu_2[0,1] = 1\), we have \(x_1 = \sqrt{m-1 \over m}\). Hence \(d\mu_2 = (m-1)t^{m-2} dt\). In general, let \(\mu_i\) be the probability measure corresponding to the weighted shift with the weights \(x_{i-1}, \ldots, x_1, \sqrt{m \over m+1}, \ldots, \sqrt{m+k-1 \over m+k}, \ldots\). Then it follows easily from mathematical induction that \(d\mu_i = (m-i+1)t^{m-i} dt, x_i = \sqrt{m-i \over m-i+1} (1 \leq i \leq n-1)\) and \(x_n \leq \sqrt{m-n \over m-n+1}\).

(ii) Let \(\mu_n\) be the probability measure corresponding to the weighted shift with the weights \(x_{n-1}, \ldots, x_1, \sqrt{m \over m+1}, \ldots, \sqrt{m+k-1 \over m+k}, \ldots\). Then
\[ d\mu_n = (m-n+1)t^{m-n} dt. \]
Since \(n \geq m\),
\[ \int_0^1 \frac{1}{t} d\mu_n = (m-n+1) \cdot \int_0^1 t^{m-n-1} dt = \infty, \]
which implies that \(\frac{1}{t} \notin L^1(\mu_n)\). Hence \(HE(\alpha, \infty) = \emptyset\).

Example 3.3. Let \(W_\alpha\) be the weighted shift whose weight sequence is given by \(\alpha_n := \sqrt{m+1 \over m+2} (n \geq 0)\). By (ii) of Example 3.2, \(W_{\alpha(x)}\) is not subnormal for any \(x > 0\). By Theorem 2.1, we have \(t_1 = 2, t_2 = 3, t_3 = \frac{6}{5}, t_4 = \frac{14}{9}, t_5 = \frac{30}{17}, \ldots\). Hence \(HE(\alpha, k) \setminus HE(\alpha, k+1) = (\lambda_{k+1}, \lambda_k)\), where \(\lambda_1 = \sqrt{2 \over 3}, \lambda_2 = \sqrt{2 \over 5}, \lambda_3 = \sqrt{3 \over 11}, \lambda_4 = \sqrt{6 \over 25}, \lambda_5 = \sqrt{30 \over 137}, \ldots, \) and \(HE(\alpha, \infty) = \emptyset\).
In addition, we consider an example of non-Bergman shift type.

**Example 3.4.** Let $W_{\alpha}$ be the weighted shift whose weight sequence is given by

$$
\alpha_n := \sqrt{\frac{n+1}{n+2}} \cdot \frac{1}{2} \cdot \frac{2^{n+2} - 1}{2^{n+1} - 1} \quad (n \geq 0).
$$

Let $d\mu := 2\lambda(\frac{1}{4} - 1)dt$. Since $\gamma_n = 2 \int_\frac{1}{4}^1 t^n dt = \frac{1}{n+1} \cdot \frac{1}{2} \cdot (2^{n+1} - 1)$, $\mu$ is the probability measure corresponding to the weighted shift with a weight sequence $\alpha := \{\alpha_n\}_{n=0}^\infty$.

Hence $W_{\alpha}$ is subnormal, and, by (3.1), $W_{\alpha(x)}$ is subnormal if and only if $x^2 \int_0^1 \frac{1}{4} d\mu \leq 1$, which is equivalent to $0 < x \leq \frac{1}{\sqrt{2}\lambda_2}$. By Theorem 2.1, we have $t_1 = \frac{1}{4}$, $t_2 = \frac{18}{15}$, $t_3 = \frac{262}{189}$, $t_4 = \frac{445}{327}$, $t_5 = \frac{34997}{25245}$, \ldots, and $HE(\alpha, k) \setminus HE(\alpha, k + 1) = (\lambda_k - 1, \lambda_k]$, where $\lambda_3 = \frac{189}{262} \approx 0.849337$, $\lambda_4 = \frac{321}{445} \approx 0.849322$, $\lambda_5 = \frac{25245}{34997} \approx 0.849322$, \ldots, and $HE(\alpha, \infty) = (0, \frac{1}{\sqrt{2}\lambda_2}]$, with $\frac{1}{\sqrt{2}\lambda_2} \approx 0.849322$.

**3.2. Recursively generated type.** Using Proposition 2.4, we can recapture a well-known result (cf. [CuP1], [CuP2] or [ChLi]).

**Example 3.5.** We consider $\alpha := (a, b, c)^\wedge$, where $0 < a < b < c$. Since $\text{rank} \gamma = 2$, by (2.7) $t_1 = \frac{1}{\sqrt{a}}$ and $t_2 = \frac{a^4 - 2a^2 b^2 + b^2 c^2}{a^2 b^2 - b^2}$. Hence

(i) $HE(\alpha, 1) = \{x : 0 < x \leq a\}$,

(ii) $HE(\alpha, 2) = \cdots = HE(\alpha, \infty) = \{x : 0 < x \leq ab\sqrt{\frac{x^2 - b^2}{x^2 - a^2}}\}$.

We close the paper with the following example.

**Example 3.6.** Let $\alpha := (a, b, c, d, e)^\wedge$ with $0 < a < b < c < d < e$ satisfying

$$
(3.2) \quad \frac{b^2}{c^2} \cdot \frac{e^4 - 2a^2 c^2 + a^2 b^2}{c^2} \leq d^2 \quad \text{and} \quad \frac{c^2}{d^2} \cdot \frac{d^4 - 2b^2 d^2 + b^2 e^2}{d^2 - b^2} \leq e^2.
$$

Then by [Li] Corollary 2.12, $W_{\alpha}$ is subnormal. Assume that $\text{rank} \gamma = 3$. Then by (1.1),

$$
\alpha_n^2 = \psi_2 + \frac{\psi_1}{\alpha_{n-1}^2} + \frac{\psi_0}{\alpha_{n-2}^2} \quad (n \geq 3)
$$

with

$$
\psi_0 = \frac{a^2 b^4 c^2 (b^2 c^4 - 2b^2 c^2 d^2 + c^2 d^4 + b^2 d^2 e^2 - c^2 e^2) + a^4 b^2 c^2 + b^2 d^2 e^2 - c^2 e^2 - c^2 d^2 e^2}{a^4 b^4 - 2a^2 b^2 c^2 + b^2 c^4 + a^2 c^2 d^2 - b^2 c^2 d^2},
$$

$$
\psi_1 = \frac{b^2 c^2 (a^2 b^2 c^2 - a^2 b^2 d^2 - a^2 c^2 d^2 + c^2 d^4 + a^2 c^2 d^2 - c^2 d^2 e^2)}{a^4 b^4 - 2a^2 b^2 c^2 + b^2 c^4 + a^2 c^2 d^2 - b^2 c^2 d^2},
$$

$$
\psi_2 = \frac{c^2 (a^2 b^2 c^2 - a^2 b^2 d^2 + b^2 c^2 d^2 + a^2 c^2 d^2 - b^2 d^2 e^2)}{a^4 b^4 - 2a^2 b^2 c^2 + b^2 c^4 + a^2 c^2 d^2 - b^2 c^2 d^2}.
$$

Hence by (2.7), $t_1 = \frac{1}{\sqrt{a}}$, $t_2 = \frac{a^4 - 2a^2 b^2 + b^2 c^2}{a^2 b^2 - b^2}$, and $t_3 = \frac{A}{B}$, where

$$
A = a^4 b^6 - 3a^2 b^4 c^2 + a^4 b^2 c^4 + 2a^2 b^2 c^4 + 2a^4 b^2 c^2 d^2 - 2a^2 b^4 c^2 d^2 - 2a^2 b^2 c^4 d^2 + b^2 c^2 d^4 - a^4 c^2 d^4 e^2 + 2a^2 b^2 c^2 d^2 e^2 - b^2 c^4 d^2 e^2,
$$

$$
B = a^2 b^2 c^2 (b^2 c^4 - 2b^2 c^2 d^2 + c^2 d^4 + b^2 d^2 e^2 - c^2 e^2).
$$
Hence

(i) \( HE(\alpha, 1) = \{ x : 0 < x \leq a \} \),

(ii) \( HE(\alpha, 2) = \{ x : 0 < x \leq \sqrt{\frac{c^2-b^2}{a^2-2ab+b^2+c^2}} \} \),

(iii) \( HE(\alpha, 3) = \cdots = HE(\alpha, \infty) = \{ x : 0 < x \leq \sqrt{\frac{b}{a}} \} \).

REFERENCES


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