INEQUALITIES FOR PRODUCTS OF SPECTRAL RADII

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Abstract. It is shown that submultiplicative inequalities for spectral radii often imply supermultiplicative inequalities, and vice versa.

1. Introduction

The spectral radius of a linear transformation on a finite-dimensional complex space, or of a bounded linear operator on a complex Banach space, or, more generally, of an element of a complex Banach algebra, is the supremum of the moduli of numbers in the spectrum of the element. The famous spectral radius formula of Gelfand states that the spectral radius, $r(A)$, is given by

$$r(A) = \lim_{n \to \infty} \|A^n\|^{1/n}.$$ 

In this note we consider semigroups (i.e., subsets closed under multiplication) of operators or elements of a Banach algebra on which the spectral radius is $k$-submultiplicative, in the sense that there is a positive real number $k$ such that $r(AB) \leq kr(A)r(B)$ for all $A$ and $B$ in the semigroup. Our main results give sufficient conditions under which $k$-submultiplicativity implies that

$$r(AB) \geq \frac{1}{k}r(A)r(B) \quad \text{for all } A \text{ and } B.$$ 

2. A general result in Banach algebras

Recall that $r(AB) = r(BA)$ for any elements of a Banach algebra (in fact, the spectra of $AB$ and $BA$ differ at most by $\{0\}$). Our basic lemmas hold for any non-negative functions agreeing on $AB$ and $BA$.

Lemma 1. If $k > 0$ and $f$ is a non-negative function on a semigroup satisfying $f(AB) = f(BA)$ and $f(AB) \leq kf(A)f(B)$ for all $A, B$, then $f(A^nB^n) \leq k^{n-1}(f(AB))^n$ for every positive integer $n$. 

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Proof. The proof is by induction on $n$. The case $n = 1$ is trivial, so assume the inequality for $n - 1$. Then
\begin{align*}
f(A^n B^n) &= f(AA^{n-1}B^{n-1}B) \\
&= f(A^{n-1}B^{n-1}BA) \\
&\leq kf(A^{n-1}B^{n-1})f(BA) \\
&\leq k^{n-2}(f(AB))^{n-1}f(AB) \\
&= k^{n-1}(f(AB))^n.
\end{align*}

Lemma 2. If $\varepsilon > 0$ and $f$ is a non-negative function on a semigroup satisfying $f(AB) = f(BA)$ and $f(AB) \geq \varepsilon f(A)f(B)$ for all $A, B$, then $f(A^n B^n) \geq \varepsilon^{n-1}(f(AB))^n$ for every positive integer $n$.

Proof. The proof of this lemma is exactly the same as the preceding one except that the inequalities are reversed.

Lemma 3. If $f$ satisfies the hypothesis of Lemma 2 (respectively, Lemma 1), and if $A$ and $B$ are elements of the semigroup such that $\{f(A^n B^n)\}$ is bounded and bounded away from 0, then $f(AB) \geq \frac{1}{k}$ (respectively, $f(AB) \leq \frac{1}{k}$).

Proof. Suppose $f$ satisfies the hypothesis of Lemma 1. Choose a subsequence so that $f(A^n B^n)$ converges to some $t$; then $t > 0$. Since
\[ f(A^n B^n) \leq k^{n-1}(f(AB))^n, \]

taking $n_1$th roots gives
\[ (f(A^n B^n))^{\frac{1}{n_1}} \leq k^{1-\frac{1}{n_1}}f(AB). \]

Taking the limit as $\{n_i\} \to \infty$ yields
\[ 1 \leq kf(AB). \]

The proof of the consequence of Lemma 2 is exactly the same except that the inequalities are reversed.

Theorem 1. If $S$ is a semigroup contained in a Banach algebra and there exist $\varepsilon > 0$ and $k > 0$ such that $\varepsilon r(A)r(B) \leq r(AB) \leq kr(A)r(B)$ for all $\{A, B\} \subset S$, then
\[ \frac{1}{k}r(A)r(B) \leq r(AB) \leq \frac{1}{\varepsilon}r(A)r(B) \]
for all $\{A, B\} \subset S$.

Proof. The inequalities of the hypothesis clearly extend to the semigroup $\mathbb{R}^+S = \{tS : t \geq 0, S \in S\}$.

Fix $A$ and $B$ in $\mathbb{R}^+S$; since multiplying each of $A$ and $B$ by positive numbers does not change any of the inequalities, we can assume that $r(A) = r(B) = 1$. We must then show that $\frac{1}{k} \leq r(AB) \leq \frac{1}{\varepsilon}$.

By Lemma 3 it suffices to show that $r(A^n B^n)$ is bounded and is bounded away from 0.

By hypothesis,
\[ \varepsilon r(A^n)r(B^n) \leq r(A^n B^n) \leq kr(A^n)r(B^n) \]
for all $n$. Since $r(A) = r(B) = 1$, $r(A^n) = r(B^n) = 1$, and
\[ \varepsilon \leq r(A^n B^n) \leq k \]
for all $n$.

**Corollary 1.** Let $S$ be a semigroup contained in a Banach algebra. If $r$ is submultiplicative on $S$ and if there is an $\varepsilon > 0$ such that $r(AB) \geq \varepsilon r(A)r(B)$ for all $A$ and $B$ in $S$, then $r$ is multiplicative on $S$.

**Proof.** Submultiplicativity means that $k = 1$ in the hypothesis of Theorem 1 and $\frac{1}{k} = 1$.

3. LINEAR TRANSFORMATIONS ON FINITE-DIMENSIONAL SPACES

In the finite-dimensional case, the existence of an $\varepsilon$ such that $r(AB) \geq \varepsilon r(A)r(B)$ can be inferred under various hypotheses.

**Theorem 2.** Let $S$ be a semigroup of linear transformations on a finite-dimensional space which is closed under multiplication by positive real numbers and is also closed in the topological sense. If $S$ has no non-zero divisors and $r(AB) \leq kr(A)r(B)$ for all $A$ and $B$ in $S$, then $r(AB) \geq \frac{1}{k}r(A)r(B)$ for all $A$ and $B$ in $S$.

**Proof.** This will follow from Theorem 1 if we show the existence of an $\varepsilon > 0$ such that $r(AB) \geq \varepsilon r(A)r(B)$, for every $A$ and $B$ in $S$. If there were no such $\varepsilon$, then for every positive integer $n$ there would be $A_n$ and $B_n$ in $S$ satisfying
\[ r(A_n B_n) < \frac{1}{n}r(A_n)r(B_n). \]
Then
\[ r \left( \left\| \frac{A_n}{\|A_n\|} \frac{B_n}{\|B_n\|} \right\| \right) < \frac{1}{n} r \left( \frac{A_n}{\|A_n\|} \right) r \left( \frac{B_n}{\|B_n\|} \right). \]

Since the unit sphere of the space of linear transformations is compact, there are transformations $E$ and $F$ and an increasing sequence of positive integers $\{n_i\}$ such that $\left\{ \frac{A_n}{\|A_n\|} \right\} \to E$ and $\left\{ \frac{B_n}{\|B_n\|} \right\} \to F$. It follows that $r(EF) = 0$ (spectral radius is continuous in finite-dimensions). Since $EF$ is nilpotent it is a zero divisor and hence $EF = 0$. If $EF = 0$, then, since $\|E\| = \|F\| = 1$, $E$ and $F$ are non-zero zero divisors, which contradicts the hypothesis.

**Definition.** We say that multiplication is bounded below on a subset $S$ of a Banach algebra if there is an $\varepsilon > 0$ such that $\|AB\| \geq \varepsilon \|A\| \|B\|$ for all $A$ and $B$ in $S$.

**Corollary 2.** Let $S$ be a semigroup of linear transformations on a finite-dimensional space, on which multiplication is bounded below. If there exists a $k$ such that
\[ r(AB) \leq kr(A)r(B) \quad \text{for all } A \text{ and } B \text{ in } S, \]
then
\[ r(AB) \geq \frac{1}{k}r(A)r(B) \quad \text{for all } A \text{ and } B \text{ in } S. \]

**Proof.** Let $T$ be the closure of
\[ \{ tS : t \geq 0, S \in S \}. \]
Then the inequality of the hypothesis extends to the semigroup $T$ so the corollary will follow from Theorem 2 if we show that $T$ has no non-zero zero divisors. Note that
\[ \|AB\| \geq \varepsilon \|A\| \|B\| \]
for all $A$ and $B$ in $S$ implies the same for all $A$ and $B$ in $T$, so $T$ does not have non-zero zero divisors.

**Corollary 3.** If multiplication is bounded below on a semigroup $S$ of linear transformations on a finite-dimensional space, and if spectral radius is submultiplicative on $S$, then spectral radius is multiplicative on $S$.

**Proof.** $\frac{1}{k} = 1$.

There is a result analogous to Theorem 2 for the reverse inequality.

**Theorem 3.** Let $S$ be a semigroup of linear transformations on a finite-dimensional space which is closed under multiplication by positive real numbers and is also closed in the topological sense. If $S$ has no non-zero zero divisors and there is an $\epsilon > 0$ such that $r(AB) \geq \epsilon r(A)r(B)$ for all $A$ and $B$ in $S$, then $r(AB) \leq \frac{1}{k} r(A)r(B)$ for all $A$ and $B$ in $S$.

**Proof.** It suffices to show that there is some $k$ such that $r(AB) \leq kr(A)r(B)$ for all $A$ and $B$ (by Theorem 1). If not, then for every $k$ there are $A_k$ and $B_k$ in $S$ satisfying

$$r(A_k B_k) > kr(A_k)r(B_k)$$

with $r(A_k)$ and $r(B_k) \neq 0$. For each $k$, let $C_k = \frac{A_k}{r(A_k)}$ and $D_k = \frac{B_k}{r(B_k)}$.

If $\{C_k\}$ was not bounded in norm, $\left\{\frac{C_k}{\|C_k\|}\right\}$ would have a subsequence which converged to a linear transformation of norm 1 and spectral radius 0, contradicting the lack of zero divisors in $S$. Thus $\{C_k\}$ and $\{D_k\}$ are bounded sequences, and so therefore is $\{C_k D_k\}$. But $r(C_k D_k) > k$ for each $k$, which is a contradiction.

Theorem 3 has corollaries similar to those of Theorem 2.

4. **Semigroups of compact operators**

The results of section 3 do not appear to extend to semigroups of compact operators. However, a theorem of [1] can be extended to give a result in the case of irreducible semigroups (i.e., semigroups with no non-trivial invariant (closed) subspaces).

**Theorem 4.** If $S$ is an irreducible semigroup of compact operators on Hilbert space and if there exists $k > 0$ such that $r(AB) \leq kr(A)r(B)$ for all $A$ and $B$ in $S$, then $r(AB) \geq \frac{1}{k} r(A)r(B)$ for all $A$ and $B$ in $S$.

**Proof.** The proof is a very small modification of the proof of the case $k = 1$ given in Theorem 2.1 of [1]. As in [1], we assume that $S$ is closed under multiplication by positive numbers and is also closed in the topological sense. Also as in [1], we first consider the case where $A^2 = A$ and $B^2 = B$ and $r(A) = r(B) = 1$. Then

$$r(AB) = r(A^2 B^2) = r(BA AB) \leq k(r(AB))^2.$$  

So, since $r(AB) \neq 0$ (as in [1]), it follows that

$$\frac{1}{k} \leq r(AB).$$

For the general case where $r(A) = r(B) = 1$, choose, as in [1], increasing sequences of positive integers and scalars $a$ and $b$ of modulus 1 such that $\{a A^n\} \to P$ and $\{b B^n\} \to Q$ with $P$ and $Q$ non-zero idempotents.
Following [1] but inserting “$k$” gives
\[
  r(A^{n_j} B^{n_j}) = r(AB^{n_j-1} A^{n_j-1}) \\
  \leq kr(AB)r(B^{n_j-1} A^{n_j-1}) \\
  \leq k^2 r(AB) r(B^{n_j-1}) r(A^{n_j-1}) \\
  = k^2 r(AB).
\]

Taking limits yields
\[
  r(PQ) \leq k^2 r(AB).
\]

Since \( \frac{1}{k^2} \leq r(PQ) \) by the first part of this proof, it follows that
\[
  \frac{1}{k^2} \leq r(AB).
\]

This holds for all \( A \) and \( B \) in \( S \) of spectral radius 1. Thus
\[
  \frac{1}{k^2} r(A)r(B) \leq r(AB) \leq kr(A)r(B)
\]
for all \( A \) and \( B \) in \( S \). Theorem 1 then gives \( \frac{1}{k^2} r(A)r(B) \leq r(AB) \).

5. An example

Let \( S \) be the semigroup consisting of all positive integral multiples of
\[
  A = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}.
\]

Direct multiplication shows that \( S \) is a semigroup, and the inequalities
\[
  \frac{1}{4} r(E)r(F) \leq r(EF) \leq 4r(E)r(F)
\]
hold and are sharp for \( S \).

References


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