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SUFFICIENT CONDITIONS FOR A LINEAR FUNCTIONAL TO BE MULTIPLICATIVE

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ABSTRACT. A commutative Banach algebra $\mathcal{A}$ is said to have the $P(k,n)$ property if the following holds: Let M be a closed subspace of finite codimension n such that, for every $x \in M$, the Gelfand transform $\hat{x}$ has at least k distinct zeros in $\Delta(\mathcal{A})$, the maximal ideal space of $\mathcal{A}$. Then there exists a subset Z of $\Delta(\mathcal{A})$ of cardinality k such that $\hat{M}$ vanishes on Z, the set of common zeros of M. In this paper we show that if $X \subset \mathbf{C}$ is compact and nowhere dense, then $R(X)$, the uniform closure of the space of rational functions with poles off X, has the $P(k,n)$ property for all $k,n \in \mathbf{N}$. We also investigate the $P(k,n)$ property for the algebra of real continuous functions on a compact Hausdorff space.

1. Introduction

The well known Gleason-Kahane-Zelazko theorem ([6, 10]) states that if $\mathcal{A}$ is any commutative unital Banach algebra and M is a closed subspace with codim(M)=1 such that M does not contain invertible elements, then M is a maximal ideal in $\mathcal{A}$. In other words if $\varphi \in \mathcal{A}^*$ is such that $\varphi(a) \in \sigma(a)$ for every $a \in \mathcal{A}$, then φ is multiplicative. This theorem has been generalized to higher codimensions (see [1, 12, 13, 7]). To state these generalizations we first need a definition.

Let $\mathcal{A}$ be a commutative complex Banach algebra with identity. We say that $\mathcal{A}$ satisfies the $P(k,n)$ property if the following holds: Let M be a closed subspace of $\mathcal{A}$ of finite codimension n. Suppose that for every $x \in M$, the Gelfand transform $\hat{x}$ has at least k distinct zeros in $\Delta(\mathcal{A})$, the maximal ideal space of $\mathcal{A}$. Then there exists a subset Z of $\Delta(\mathcal{A})$ of cardinality k such that $\hat{x}$ vanishes on Z for every $x \in M$. We call Z a set of common zeros of M. In fact the hypothesis states that if every $x \in M$ is contained in k maximal ideals $I_1^x,\ I_2^x, \cdots, I_k^x$ (depending on x), then there exist k maximal ideals $I_1, \cdots, I_k$ such that every $x \in M$ is contained in $I_1,\ I_2, \cdots, I_k$.

Note that if for $k > 1$, the $P(k,n)$ holds for $C(X)$ the space of continuous complex valued functions on a compact Hausdorff space X, then every point of X is a G_δ set. We now discuss some background history of the subject.

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In [12] Warner and Whitely conjectured that if X is a compact Hausdorff space such that each point of X is a G_δ set, then $C(X)$ satisfies the $P(k,n)$ property. This conjecture was settled by K. Jarosz [7]. Garimella and Rao [5] showed that $C^n[a,b]$ and $L^1(\mathbf{R})$ satisfy $P(k,n)$. Chen and Cohen [2] have proved that a selfadjoint regular n-point spectral Banach algebra satisfies $P(k,n)$. Rao [11] removed the spectrality condition in the Chen and Cohen theorem, thus making the result applicable to $C^n[a,b]$. He generalized the above result to any selfadjoint, regular commutative Banach algebra with identity. We would also like to investigate the $P(k,n)$ property. Extensions of the Gleason-Kahane-Zelazko theorem in other directions can be found in [9].

In §2 we generalize some results in [7] and [5] and show that if X is a compact and nowhere dense subset of the plane, then $R(X)$, the uniform closure of the algebra of rational functions with poles off X, has the $P(k,n)$ property for all $k,n \in \mathbf{N}$.

Note that the GKZ theorem does not hold for real Banach algebras. For example let $\mathcal{A} = \mathrm{Re}C[0,1]$ and $\varphi(f) = \frac{1}{2}(f(0)+f(1))$ for all f in $\mathcal{A}$. By the intermediate value theorem $\varphi(f) \in \sigma(f)$ for all $f \in \mathcal{A}$ but φ is not multiplicative. This example, in turn, shows that the proof of the GKZ theorem depends heavily on complex analysis techniques. But the GKZ theorem is true for $\mathrm{Re}C(X)$, X compact Hausdorff, if and only if X is assumed to be totally disconnected (see Farnum and Whitely [4]). We may use the definition of the $P(k,n)$ property without any difficulty for real Banach algebras and in §3 we show that if X is a compact Hausdorff and totally disconnected space, then $\mathrm{Re}C(X)$ has the $P(1,n)$ property for all $n > 1$ and also has the $P(k,n)$ property for all $k,n \in \mathbf{N}$, if we further assume that the points of X are G_δ. We also show that the $P(1,n)$ property holds for all unital commutative real Banach algebras with a totally disconnected maximal ideal space.

2. Algebras of rational functions

To prove the $P(1,n)$ property for $C(X)$ where X is a compact Hausdorff space, Jarosz proves a lemma [7, Lemma 1] stating that if X is a compact subset of the real line and $p_1, \cdots, p_n$ are polynomials such that each linear combination of them has a zero in X, then there is a common zero for $p_1, \cdots, p_n$ in X. Garimella and Rao [5] generalized this result for those closed subsets X of $\mathbf{C}$ that have an empty interior and called this result the *polynomial lemma*. The assumption that X has an empty interior is essential as the following example shows:

Let $p_1 = (iz+1)^2$ and $p_2 = (z+i)^2$ on $\overline{\mathbf{D}}$ the closed unit disk. We show that every linear combination of p_1 and p_2 has a zero in the disk. To see this note that the Mőbius transformation $\frac{iz+1}{z+i}$ sends $\overline{\mathbf{D}} \setminus \{-i\}$ to the lower closed half plane and so the function $h = \frac{(iz+1)^2}{(z+i)^2}$ sends it to the whole plane. Consequently, every linear combination of p_1 and p_2 has a zero in the disk while there is no common zero for p_1 and p_2 in the disk.

This example also shows that the assumption of being finite codimensional is essential in the generalizations of GKZ. See also Jarosz [7, §2].

In the proof of Jarosz and Garimella and Rao for the polynomial lemma, the method of several variables is used. Here we prove a generalization of this lemma that uses only elementary complex analysis and needs no assumption of closedness. We only assume that the closure of X in $\mathbf{C}$ has an empty interior, i.e. X is nowhere dense in $\mathbf{C}$. But first we need a lemma.

Lemma 2.1. *Let X be a nowhere dense subset of $\mathbf{C}$ and f be a holomorphic function on a neighborhood of X. Then $f(X)$ has an empty interior in $\mathbf{C}$. Consequently, there are infinitely many complex numbers outside $f(X)$.*

Proof. Denote by $\overline{X}$ the closure of X in $\mathbf{C}$. By definition, for each $x \in X$ there is a neighborhood of x such that f is holomorphic there. In this proof we restrict ourselves to those neighborhoods V such that f is holomorphic on a neighborhood of $\overline{V}$. Note that f need not be defined on $\overline{X}$.

Let x be in X. Two cases are possible:

(i) $f'(x) \neq 0$. In this case there exists a bounded open set V containing x such that f is one-to-one on $\overline{V}$. Thus $\overline{V}$ is compact and $f_1 = f|_{\overline{V}}$ is invertible and if $f_1(\overline{X} \cap \overline{V}) = U$ or equivalently ${f_1}^{-1}(U) = \overline{V} \cap \overline{X}$, then U has an empty interior in $\mathbf{C}$ since otherwise ${f_1}^{-1}$, being holomorphic and nonconstant on U, sends open sets to open sets, and so $\overline{V} \cap \overline{X}$ and hence $\overline{X}$ would contain an open set, which contradicts the hypothesis. Therefore $f(\overline{V} \cap \overline{X}) = f_1(\overline{V} \cap \overline{X})$ has an empty interior in $\mathbf{C}$.

(ii) $f'(x) = 0$. In this case again two cases may occur. First, there exists a neighborhood V of x such that $f' = 0$ on $\overline{V}$. Then f is constant on V and hence $f(\overline{V} \cap \overline{X})$ is a singleton and so is closed and has an empty interior. The second possibility is that there exists a neighborhood V such that f' has a single zero x in V. In this case we may suppose that V is an open ball with center x. Choose closed annuli V_n such that $\overline{V} \setminus \{x\} = \bigcup_{n=1}^{\infty} V_n$. By ($i$), for all $n = 1, 2, 3, \cdots$, $f(V_n \cap \overline{X})$ has an empty interior in $\mathbf{C}$ and therefore $f(\overline{V} \cap \overline{X}) = \bigcup_{n=1}^{\infty} f(V_n \cap \overline{X}) \cup \{f(x)\}$ has an empty interior in $\mathbf{C}$ by the Baire category theorem.

Therefore for each $x \in X$, there exists a suitable neighborhood V of x such that $f(\overline{V} \cap \overline{X})$ has an empty interior in $\mathbf{C}$. But X can be covered with a countable union of such suitable neighborhoods V. Since for such a neighborhood V, $f(\overline{V} \cap \overline{X})$ is closed and has an empty interior in $\mathbf{C}$, their countable union also has an empty interior by the Baire category theorem. Hence, $f(X)$, being a subset of this countable union, has an empty interior in $\mathbf{C}$. $\square$

We also need the following well-known facts from linear algebra.

Lemma 2.2. (i) *If a finite dimensional complex or real linear space is a countable union of subspaces, then one of these subspaces must contain the others.*

(ii) *If a complex or real linear space is a finite union of subspaces, then one of these subspaces must contain the others.* $\square$

Theorem 2.3. *Suppose X is a nowhere dense subset of the plane and $h_0, \cdots, h_n$ are functions analytic on a neighborhood of X such that each linear combination of them has a zero in X. If one of these h_j, $0 \leq j \leq n$, has countably many zeros in X, then there exists a common zero for $h_0, \cdots, h_n$ in X.*

Proof. Suppose h_0 has countably many zeros in X. Set $Z = \{x \in X : h_0(x) = 0\}$. For a linear combination h of $h_0, \cdots, h_n$ consider the function $f = h/h_0$ on $X' = X \setminus Z$. Since X' is also nowhere dense in $\mathbf{C}$, by Lemma 2.1 there exists a complex number $\beta \neq 0$ such that $f(x) \neq \beta$ for all $x \in X'$. But the equation $h(x) = \beta h_0(x)$ has a solution in X and this solution is not in X', so that it is in Z. Let M denote the complex linear space of linear combinations of $h_0, \cdots, h_n$, i.e. $M = [h_0, \cdots, h_n]$. We have shown that every element in M has a zero in Z. For a point $z \in Z$ let $M_z = \{f \in M : f(z) = 0\}$, so that $M = \bigcup_{z \in Z} M_z$. But Z is a countable set and by Lemma 2.2 (i), one of these M_z must contain the others, i.e. $h_0, \cdots, h_n$ has a common zero in X. □

Corollary 2.4. *Under the same hypothesis, if each linear combination of* $h_0, \cdots, h_n$ *has at least* k *zeros in* X*, then there exist* k *common zeros for* $h_0, \cdots, h_n$ *in* X*.*

Proof. There exists a common zero for $h_0, \cdots, h_n$. Remove it from X and continue if necessary. □

Corollary 2.5. *Suppose* A *is a linear space of functions on a nowhere dense subset* X *of the plane each of which is analytic on a neighborhood of* X*, and has at least* k *zeros in* X*. Furthermore assume that one element of* A *has finitely many zeros in* X*. Then there exist* k *common zeros for* A *in* X*.*

For example, A can be a space of polynomials or rational functions.

Proof. Let p in A have only finitely many zeros in X. The set $Z = \{x \in X : \ p(x) = 0\}$ is finite and has cardinality $\geq k$. For an arbitrary element $q \in A$, let $h = q/p$, so by Lemma 2.1, $h(X \setminus Z)$ has empty interior in $\mathbf{C}$. Thus there exists $\beta \neq 0$ such that $h(x) \neq \beta$ for all $x \in X \setminus Z$. But by the hypothesis, $q - \beta p$ is in A and has k zeros in X and none of the zeros are in $X \setminus Z$; hence all are in Z. Therefore q has at least k zeros in Z. For a subset K of Z of cardinality k, set

$$A_K = \{f \in A : \ f(x) = 0 \text{ for all } x \in K\}.$$

Thus $A = \bigcup_K A_K$ where K runs over all subsets of Z of cardinality k. But Z is a finite set and has only finitely many subsets of cardinality k. By Lemma 2.2 (ii) one of these A_K must contain the others, i.e. A has k common zeros in X. □

Now the $P(k, n)$ property for the algebra of rational functions over a compact nowhere dense subset of the plane follows.

Theorem 2.6. *Let* $R(X)$ *be the uniform closure of the algebra of rational functions* $\mathrm{Rat}(X)$ *with poles off a compact nowhere dense subset* X *of the plane. If* M *is a finite codimensional subspace of* $R(X)$ *such that every element of it has at least* k *zeros in* X*, then there are* k *common zeros for* M *in* X*. That is, the* $P(k, n)$ *property holds for* $R(X)$*.*

Proof. Note that the maximal ideal space of $R(X)$ is X. Let M be a finite codimensional subspace of $R(X)$ such that each element of M has at least k zeros in X. Let $M' = M \cap \mathrm{Rat}(X)$. Since M is finite codimensional in $R(X)$, M' would be dense in M. Also each element of M' has at least k zeros in X and hence by Corollary 2.5, M' has k common zeros in X. Now M, being the uniform closure of M', has k common zeros in X. □

It follows that the $P(k, n)$ property holds for $C(X)$ whenever $X \subset \mathbf{C}$ is compact and nowhere dense with a connected complement (especially whenever $X \subset \mathbf{R}$) since rational functions are dense in $C(X)$ in this case.

The above proof is also applicable to $C^n[a, b]$ and to $C^\infty[a, b]$, although the latter has no Banach algebra norm, since polynomials are dense in these algebras.

3. What happens for Re $C(X)$?

As stated in the introduction, the GKZ theorem does not hold in general for real Banach algebras and it is true for Re$C(X)$, X compact, if and only if X is totally disconnected [4]. In fact, if U is a nontrivial connected component of X, then we have an abundance of examples of continuous linear functionals φ with $\varphi(f) \in \text{Im(f)}$ for all f in Re$C(X)$ such that φ is not multiplicative.

Let x and y be two distinct points of U and $\varphi(f) = \frac{f(x)+f(y)}{2}$. Then for all f in Re$C(X)$, $\varphi(f) \in \text{Im(f)}$ by the intermediate value theorem and obviously φ is not multiplicative. If we consider U as a subset of the dual space of Re$C(X)$ (the points of U will then be the point mass measures at points of U), a deeper observation reveals that each element φ of $\overline{\text{co}}(U)$, the closure of the convex hull of U in the dual space of Re$C(X)$, has the property $\varphi(f) \in \text{Im(f)}$ for all f in Re$C(X)$.

Now we show that if X is a totally disconnected compact Hausdorff space, then $P(1,n)$ is true for Re$C(X)$ for all $n \in \mathbf{N}$.

Theorem 3.1. *If X is a totally disconnected compact Hausdorff space, then* Re $C(X)$ *has the $P(1,n)$ property for all $n \in \mathbf{N}$.*

Proof. Let M be a finite codimensional subspace of Re$C(X)$ consisting only of non-invertible elements. Since X is totally disconnected, X can be considered as a closed subspace of a discrete product $\{0,1\}^J$ where J is some index set. For a finite subset J_0 of J denote

$$M(J_0) = \{f \in M : \ f \text{ depends only on the coordinates from } J_0\}$$

and

$$X(J_0) = \{x \in X : \ \ f(x) = 0 \text{ for all } f \in M(J_0)\}.$$

Clearly $M(J_0)$ can be considered as a subspace of $\mathbf{R}^{2^{J_0}}$ and since each element of $M(J_0)$ is non-invertible; then $X(J_0)$ would be closed (and hence compact) and nonvoid by Lemma 2.2 (i).

Now for a family $J_1, \cdots, J_m$ of finite subsets of J let $J_0 = \bigcup_{i=1}^m J_i$. Thus $X(J_0) \subseteq \bigcap_{i=1}^m X(J_i)$ and since $X(J_0)$ is not empty, then it follows that the class

$$\{X(J_0) : \ \ J_0 \text{ is a finite subset of } J\}$$

has the finite intersection property and therefore has a nonempty intersection Z which is compact and is a zero set for the subspace

$$M' = \{f \in M : \ f \text{ depends only on a finite number of coordinates from } J\}.$$

But the linear subspace

$$Y = \{f \in \text{Re}C(X) : \ f \text{ depends only on a finite number of coordinates from } J\}$$

is a dense subspace of Re$C(X)$ and M is finite codimensional in Re$C(X)$, so that $M' = Y \cap M$ is dense in M and therefore each element of M is zero on Z; that is, M has at least a common zero in X. □

Now we are ready to prove the $P(k,n)$ property for Re$C(X)$ if we further assume that the points of X are G_δ. In general, all the points of the space $\{0,1\}^J$ are not G_δ. For example, if $J = [0,1]$, then the point $\bar{\mathbf{0}}$ in $\{0,1\}^J$, which is zero in all of its coordinates, is not a G_δ set.

Theorem 3.2. *If X is a totally disconnected compact Hausdorff space such that each point of X is a G_δ, then the $P(k,n)$ property holds for* $\operatorname{Re} C(X)$ *for all* $k, n \in \mathbf{N}$.

Proof. Let M be a subspace of $\mathrm{Re}C(X)$ of codimension n such that each element of M has at least k zeros in X. By Theorem 3.1, there is at least one common zero for M in X. Let $Z = \{x_1, \cdots, x_m\}$ denote the set of common zeros for M in X. Since the multiplicative functionals on a commutative Banach algebra are linearly independent, it follows that $m \leq n$. If $k \leq m$ we are done. If $k > m$, put

$$M^\perp = [\delta_1, \cdots, \delta_m, \mu_1, \cdots, \mu_{m'}]$$

where $m + m' = n$ and δ_j denotes the point mass measure at x_j for $1 \leq j \leq m$. Let

$$M' = \{f \in \mathrm{Re}C(X) : \ \mu_j(f) = \int f d\mu_j = 0, \text{for all } \ j, \ 1 \leq j \leq m'\}.$$

We may suppose, by adding suitable linear combination of $\delta_1, \cdots, \delta_m$ to each μ_j, that $|\mu_j|(Z) = 0$ for $1 \leq j \leq m'$.

Since there is no common zero for M', by Theorem 3.1 there is a function $f \in M'$ such that $f \neq 0$ everywhere on X. By using the duality notions in the subalgebra

$$\{f \in \mathrm{Re}C(X) : \ \ f(Z) = 0\},$$

of $\mathrm{Re}C(X)$, we may choose f_j such that f_j is zero on Z and $\int f_i d\mu_j = \delta_{ij}$, $1 \leq i, j \leq m'$. Let $\epsilon > 0$ be such that $g = |f| - \epsilon \sum_j |f_j|$ is a strictly positive function. Set $s = \sup_{x \in X} g(x)$. Since each μ_j is regular and zero on Z, there exists an open neighborhood V of Z such that $|\mu_j|(V) < \epsilon/s$.

Since Z is a G_δ set, we may choose h in $\mathrm{Re}C(X)$ such that $0 \leq h \leq 1$ on X, h is 1 only on Z and is zero outside V. Let $u = gh(\mathrm{sgn} f)$ (since $f \neq 0$ everywhere on X, $\mathrm{sgn} f$ is continuous). Then

$$|\int u d\mu_j| \leq |\mu_j|(V) \sup_{x \in X} g(x) < (\frac{\epsilon}{s})s = \epsilon.$$

If we consider the function $w = f - u + \sum_j \mu_j(u) f_j$, then a simple computation shows that w is zero on Z and $\int w d\mu_j = 0$, $1 \leq j \leq m'$. Thus w belongs to M. Furthermore, $|w| \geq |f| - |u| - \epsilon \sum_j |f_j| = (1-h)g > 0$, outside Z. Hence $w \in M$ has exactly m zeros which is absurd. □

Corollary 3.3. *If X is a totally disconnected compact Hausdorff space with points G_δ and M is a subspace of codimension n in* $\operatorname{Re} C(X)$ *such that each element of M has at least n zeros in X, then M is an ideal.* □

In [11, Theorem (0.4)], Rao proved that a complex unital commutative Banach algebra $\mathcal{A}$ has the $P(k,n)$ property if and only if its corresponding semisimple algebra $\frac{\mathcal{A}}{\mathcal{B}}$ has this property where $\mathcal{B}$ denotes the Jacobson radical of $\mathcal{A}$. The proof there is also applicable to real Banach algebras as well. Therefore we conclude the following:

Theorem 3.4. *Let $\mathcal{A}$ be a unital real commutative Banach algebra with a totally disconnected maximal ideal space X. If M is a finite codimensional subspace of $\mathcal{A}$ such that the Gelfand transform of each element of M has a zero in X, then M is contained in a maximal ideal.*

Proof. We can assume that $\mathcal{A}$ is semisimple. Let Λ be the Gelfand map from $\mathcal{A}$ into $\mathrm{Re}C(X)$. Then $N = \overline{\Lambda(M)}$ is finite codimensional in $\mathrm{Re}C(X)$ and each element of N is noninvertible in $\mathrm{Re}C(X)$. So by Theorem 3.1 there exists a common zero for N in X and this is a common zero for M. Consequently M is contained in a maximal ideal. □

4. Some further results

In 1991, K. Jarosz [9] posed the following problems:

Problem 1. Does any commutative complex unital Banach algebra have the $P(1,n)$ property for $n \geq 2$?

Problem 2. In particular does the disk algebra have the $P(1,n)$ property for $n \geq 2$?

Problem 3. If some element f of a commutative complex Banach algebra $\mathcal{A}$ does not belong to any regular maximal ideal of $\mathcal{A}$, does $\mathcal{A}$ have the $P(1,n)$ property for $n \geq 1$?

Problem 4. If the points of the maximal ideal space of such an algebra $\mathcal{A}$ are G_δ, does $\mathcal{A}$ have the $P(k,n)$ property for all positive integers k and n?

However, in 1989, Rao [11] showed that $P(2,3)$ does not hold for $C^1(B)$, the algebra of all continuously differentiable functions on B where B denotes the closed unit ball of $\mathbf{R}^3$. So the fourth problem of Jarosz is settled negatively. We give this example and modify it to give a negative answer to the third problem. However, the first two problems are still open.

Example 4.1 ([11])**.** Let $\mathcal{A} = C^1(B)$. It is clear that the maximal ideal space of $\mathcal{A}$ is B. Let P and Q be arbitrary points of B with Q in the interior. Consider the three continuous linear functionals ν_1, ν_2 and ν_3 defined by $\nu_1(f) = f(P) + f_x(Q)$, $\nu_2(f) = f(Q)$, and $\nu_3(f) = f_x(Q) + if_y(Q)$ for all $f \in \mathcal{A}$, where x, y, z denote the coordinates of a point in $\mathbf{R}^3$. We show that each element of the subspace

$$M = {}^{\perp}[\nu_1, \nu_2, \nu_3] = \{f \in \mathcal{A} : \ \nu_1(f) = \nu_2(f) = \nu_3(f) = 0\}$$

has at least two distinct zeros in B while M has only one common zero Q.

Let $f \in M$. If $f(P) = f(Q) = 0$, we are done. If $f(P) \neq 0$, then also by the definition of M, $f_x(Q)$ and $f_y(Q)$ are not equal to zero. Write $f = g + ih$ where g and h are real valued functions in $\mathcal{A}$. Consequently $g_x(Q) = h_y(Q)$ and $g_y(Q) = -h_x(Q)$. Since $f_x(Q) \neq 0$, the previous two quantities are not equal to zero, i.e. $g_x(Q) = h_y(Q) \neq 0$. Therefore, by the implicit function theorem, there are two 2-dimensional C^1 manifolds passing through Q such that $g = 0$ on one of them and $h = 0$ on the other. So on their intersection, which contains at least a C^1 curve passing through Q, f is identically zero.

To show that Problem 3 of Jarosz has also a negative answer, we modify the above example.

Example 4.2. We use the same notations as in Example 4.1**.** Put

$$\mathcal{A}_0 = \{f \in \mathcal{A} : \ f(Q) = 0\}.$$

It is not difficult to see that $\mathcal{A}_0$ is a non-unital commutative Banach algebra with the maximal ideal space $B' = B \setminus \{Q\}$ and there are functions in $\mathcal{A}_0$ which are nonzero on B', i.e. do not belong to any regular maximal ideal of $\mathcal{A}_0$. Now put

$$M_0 =^{\perp} [\nu_1, \nu_3] = \{f \in \mathcal{A}_0 : \ \ \nu_1(f) = \nu_3(f) = 0\}$$

in $\mathcal{A}_0$. Then M_0 is of codimension 2 in $\mathcal{A}_0$ and as in the above we see that every element of M_0 has at least one zero in B' but there is no common zero for M_0 in B'. Therefore $P(1,2)$ does not hold for $\mathcal{A}_0$ and the answer to the third problem would also be negative.

Finally we treat the most important type of non-unital commutative Banach algebras $C_0(X)$, the algebra of all continuous complex-valued functions on a locally compact Hausdorff space X vanishing at infinity.

If X is not σ-compact, then every element of $C_0(X)$ has infinitely many zeros in X. So, it does not make any sense to speak about the $P(k,n)$ property in this case. But if X is assumed to be σ-compact the $P(1,n)$ property follows for $C_0(X)$ from Theorem 2 of [8] for all $n \in \mathbf{N}$. If we further assume that all the points of X are G_δ, the $P(k,n)$ property follows for $C_0(X)$ from [7] for all $k,n \in \mathbf{N}$ by only considering the natural embedding of $C_0(X)$ into $C(X_\infty)$, where X_∞ denotes the one point compactification of X.

Here we present a simple proof of $P(1,1)$ for $C_0(X)$ that is independent of [8]:

Theorem 4.3. *Let X be a locally compact Hausdorff space that is σ-compact. Then $C_0(X)$, the space of all complex valued functions on X vanishing at infinity, satisfies $P(1,1)$.*

Proof. First we show that there exists a positive function f_0 in $C_0(X)$. Write $X = \bigcup_{n=1}^{\infty} K_n$, where K_n is compact for each $n \geq 1$. By the Urysohn lemma we find a sequence $g_n \in C_c(X)$ such that $0 \leq g_n \leq 1$ and $g_n = 1$ on K_n. Set $f_0 = \sum_{i=1}^{\infty} \frac{g_i}{2^i}$. Then $f_0 \in C_0(X)$ and $f_0 > 0$ on X.

Now, let M be a closed subspace of $C_0(X)$ of codimension 1 such that every $f \in M$ has a zero in X. Then $M^{\perp} = [\mu_0]$ for some regular Borel measure μ_0 on X. Hence

$$M = \{f \in C_0(X) : \ \int f d\mu_0 = 0\}.$$

Note that $f_0 \notin M$ because every element of M vanishes at some point of X. Let $\alpha = \int f_0 d\mu_0$. Clearly $\alpha \neq 0$. Replacing μ_0 by $\alpha^{-1}\mu_0$ we obtain a new $\mu_0 \perp M$ such that $\int f_0 d\mu_0 = 1$. Now given $f \in C_0(X)$ we write $g = f - (\int f d\mu_0) f_0$. Then $\int g d\mu_0 = \int f d\mu_0 - \int f d\mu_0 = 0$. Therefore $g \in M$. Hence there exists $x_0 \in X$ such that $g(x_0) = 0$, i.e. $\int f d\mu_0 = f(x_0)/f_0(x_0)$. Now if $f \geq 0$, then $\int f d\mu_0 \geq 0$; hence $\mu_0 \geq 0$.

We have shown that $\int f d\mu_0 \in \mathrm{Im}(f/f_0)$ for every $f \in C_0(X)$. Replacing f by $f f_0$ we get $\int f f_0 d\mu_0 \in \mathrm{Im}(f)$, $f \in C_0(X)$. From this we conclude that $f_0 d\mu_0 = d\delta_v$, the point mass at some $v \in X$ [3, Lemma 2.5]. Write $E = X \setminus \{v\}$. Then $\int_E f_0 d\mu_0 = 0$; hence $f_0 = 0$ a.e. μ_0 on E. Because f_0 is strictly positive, we conclude that $\mu_0(E) = 0$, that is, $\mu_0 = c\delta_v$ for some constant $c > 0$, so that,

$$M = \{f \in C_0(X) : \ f(v) = 0\}.$$

The proof is now complete. □

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