Borel Complexity of the Space of Probability Measures

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Abstract. Using a technique developed by Louveau and Saint Raymond, we find the complexity of the space of probability measures in the Borel hierarchy: if $X$ is any non-Polish Borel subspace of a Polish space, then $P(X)$, the space of probability Borel measures on $X$ with the weak topology, is always true $\Pi^0_\xi$, where $\xi$ is the least ordinal such that $X$ is $\Pi^0_\xi$.

1. Introduction

For $X$ a separable metric space, let $P(X)$ be the space of probability Borel measures on $X$ with the usual topology of weak convergence, so that $P(X)$ is also a separable metrizable space. Many results relating the descriptive complexities of $X$ and $P(X)$ are classical and standard. For example (see [1], [5]), $X$ is compact-metrizable (resp. Polish) iff $P(X)$ is compact-metrizable (resp. Polish), $X$ is Borel (resp. projective) iff $P(X)$ is Borel (resp. projective), etc. Beyond the domain of Borel sets we have: $X$ is analytic (resp. co-analytic) iff $P(X)$ is analytic (resp. co-analytic), which follows from results of Kechris [2], and this also extends to every level of the projective hierarchy under additional set-theoretic hypotheses. Shreve’s theorem (6) establishes the same equivalence for each level of the $C$-hierarchy of Selivanovski.

In this note we prove a similar result for the Borel hierarchy. It is known (2) that, for $\alpha \geq 2$, $P(X)$ is $\Pi^0_\alpha$ iff $X$ is $\Sigma^0_\alpha$; hence if $X$ is $\Sigma^0_\alpha$, then $P(X)$ is $\Pi^0_{\alpha+1}$. This suggests the question: Is this the best bound? More generally, for $\Gamma$ and $\Gamma'$ any two intrinsic Borel pointclasses, we ask: If $X$ is in $\Gamma$, is $P(X)$ in $\Gamma'$? It is also easy to find, for each $\alpha \geq 3$, an example of a $P(X)$ which is true $\Pi^0_\alpha$: just take $X$ to be any space which is true $\Pi^0_\alpha$; since $X$ is always embedded in $P(X)$ as a closed subset, $P(X)$ cannot be $\Sigma^0_\alpha$ lest $X$ be $\Sigma^0_\alpha$. This suggests the question: For which $\alpha \geq 3$ can we find $X$ such that $P(X)$ is true $\Sigma^0_\alpha$ (or true $\Delta^0_\alpha$)?

Theorem 3.1 below answers these questions, and determines uniquely the true Borel class of $P(X)$ from the true Borel class of $X$; it says: $P(X)$ is Polish if $X$ is Polish, and for every $\alpha \geq 3$, $P(X)$ is true $\Pi^0_\alpha$ if $X$ is true $\Pi^0_\alpha$, $P(X)$ is true $\Pi^0_{\alpha+1}$.
if $X$ is true $\Sigma^0_\alpha$, and $P(X)$ is true $\Pi^0_\alpha$ if $X$ is true $\Delta^0_\alpha$. (These cases are mutually exclusive and exhaustive for Borel $X$.)

In particular, if $X$ is Borel, then either $P(X)$ is Polish or $P(X)$ is true $\Pi^0_\alpha$ for some (unique) $\alpha \geq 3$, so that if $\alpha \geq 3$, there is no $X$ such that $P(X)$ is true $\Sigma^0_\alpha$ or true $\Delta^0_\alpha$.

2. Terminology

We use the notation of $[1]$ and $[3]$ for the Borel pointclasses: the additive, multiplicative, and ambiguous classes of level $\alpha$ are denoted, respectively, by $\Sigma^0_\alpha$, $\Pi^0_\alpha$, and $\Delta^0_\alpha$, with $\Sigma^0_1$ denoting the pointclass of open sets. $\Gamma$ is a Borel pointclass if $\Gamma$ is one of $\Sigma^0_\alpha$, $\Pi^0_\alpha$, and $\Delta^0_\alpha$. If $X$ is a Polish space, we use the notation $\Sigma^0_\alpha X$ to denote the $\Sigma^0_\alpha$ subsets of $X$, and similarly for $\Pi^0_\alpha X$ and $\Delta^0_\alpha X$. Note that for $\alpha \geq 2$ the pointclass $\Pi^0_\alpha$ is intrinsic and for $\alpha \geq 3$ the pointclasses $\Sigma^0_\alpha$ and $\Delta^0_\alpha$ are also intrinsic. For an intrinsic pointclass $\Gamma$, we can speak (unambiguously) of a separable metrizable space $X$ being $\Gamma$, without mentioning any Polish space in which $X$ is embedded. The notion of “the true Borel class” of a Borel set is defined in the usual way.

$\omega^\omega$ denotes the Baire space, i.e., the space of irrationals. If $X$, $Y$ are Polish spaces, $B \subseteq Y$, and $\mathcal{C}$ is a collection of subsets of $X$, we say that $B$ is $\mathcal{C}$-hard if for all $C \in \mathcal{C}$ there is a continuous $f: X \to Y$ such that $C = f^{-1}(B)$.

If $X$ is a metrizable space, $P(X)$ denotes the space of probability measures on $X$ with the weak topology.

For a separable metrizable space $Y$, and a Borel $X \subseteq Y$, we can (topologically) identify (see $[1]$) the space $P(X)$ with the subspace $P(X/Y)$ of $P(Y)$, where

$$P(X/Y) \overset{\text{def}}{=} \{ \mu \in P(Y) \mid \mu(Y \setminus X) = 0 \}.$$

3. The Borel complexity of $P(X)$

**Theorem 3.1.** If $X$ is any non-Polish Borel subspace of a Polish space, then $P(X)$ is true $\Pi^0_\xi$, where $\xi$ is the least such that $X$ is $\Pi^0_\xi$.

**Proof.** We will use the following two lemmas:

**Lemma 3.2** (Louveau and Saint Raymond). If $Y$ is a Polish space, $\alpha$ is a countable ordinal $\geq 2$, $A \subseteq Y$ such that $A$ is Borel, and $A \notin \Pi^0_\alpha$, then $A$ is $\Sigma^0_\alpha | \omega^\omega$-hard, i.e., for all $B \subseteq \omega^\omega$, if $B$ is $\Sigma^0_\alpha$, then $B = f^{-1}(A)$ for some continuous $f: \omega^\omega \to Y$.

**Proof.** This is an immediate consequence of $[3]$ Theorem 3, p. 455.

**Lemma 3.3** (The $\delta$-propagation lemma). Let $Y$, $Z$ be Polish spaces, $\mathcal{C}$ a collection of subsets of $Z$, and $X$ a subset of $Y$ which is $\mathcal{C}$-hard, i.e., for every $A$ in $\mathcal{C}$ there is a continuous map $f$ from $Z$ to $Y$ such that $A = f^{-1}(X)$. Then $P(X/Y)$ is $\mathcal{C}_\delta$-hard, where $\mathcal{C}_\delta$ denotes the countable intersections of sets in $\mathcal{C}$.

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1. $\Gamma$ is an intrinsic pointclass if for all Polish $X$, $Y$ and $A \subseteq X$, $B \subseteq Y$, if $A$ is a $\Gamma$-set in $X$ and $A$ is homeomorphic to $B$, then $B$ is a $\Gamma$-set in $Y$.

2. The collection of all countably additive non-negative real-valued functions $\mu$ defined on the Borel sets of $X$ such that $\mu(X) = 1$.

3. The weakest topology such that, for every bounded continuous real-valued function $f$ on $X$, the real-valued map $\mu \to \int f d\mu$ defined on $P(X)$ is continuous.
Proof. Let \( X, Y, Z, \) and \( C \) be as above. To show that \( P(X=Y) \) is \( \mathcal{C}_\delta \)-hard, let \( F \) be an arbitrary subset of \( Z \) in \( \mathcal{C}_\delta \). Then there is sequence \((E_n)_{n=0}^\infty \) of subsets of \( Z \) such that \( (\forall n)(E_n \in \mathcal{C}) \), and
\[
F = \bigcap_{n=0}^\infty E_n.
\]
For each \( n \in \omega \), choose a continuous function \( f_n : Z \to Y \) such that
\[
E_n = f_n^{-1}(X).
\]
Define \( \Psi : Z \to P(Y) \) by setting for each \( x \in Z \) and each Borel \( E \subseteq Y \):
\[
\Psi(x)(E) = \sum_{k=0}^\infty \frac{X E(f_k(x))}{2^{k+1}}.
\]
where, for any set \( A \), \( \chi_A \) denotes the characteristic function of \( A \). In other words, for any \( x \in Z \), \( \Psi(x) \) is the measure
\[
\Psi(x) = \sum_{k=0}^\infty \frac{1}{2^{k+1}} \delta_{f_k(x)},
\]
where, for any \( y \in Y \), \( \delta_y \) denotes the probability measure on \( Y \) known as the “unit mass” at \( y \). \( \Psi \) is well-defined and continuous. It is now easy to verify that \( \Psi \) reduces \( F \) to \( P(X/Y) \).

Now let \( X \) be non-Polish Borel, and \( Y \) be any metrizable compactification of \( X \); let \( \alpha \) be the least such that \( X \) is \( \Pi^0_\alpha \). Then \( \alpha \geq 3 \), as \( X \) is not \( \Pi^0_2 \) in \( Y \). Also, \( P(X) \) is \( \Pi^0_\alpha \). It remains to show that \( P(X) \) is not \( \Sigma^0_\alpha \) (\( \Sigma^0_\alpha \) is an intrinsic pointclass since \( \alpha \geq 3 \)). Put \( Z = \omega^\omega \), and
\[
\mathcal{E} = \bigcup_{\beta < \alpha} \Sigma^0_\beta \setminus \omega^\omega.
\]
By Lemma 3.2 \( X \) is \( \Sigma^0_\beta \setminus \omega^\omega \)-hard for every \( \beta < \alpha \), and hence is \( \mathcal{E} \)-hard. The hypotheses of the \( \delta \)-propagation lemma now hold. Therefore \( P(X/Y) \) is \( \mathcal{C}_\delta \)-hard; but \( \mathcal{C}_\delta = \Pi^0_\alpha \setminus \omega^\omega \). So \( P(X) \) must be true \( \Pi^0_\alpha \).

It follows from the above theorem that the Borel complexity of the space of probability measures on \( \mathbb{Q} \) (the rationals) is \( F_{\sigma \delta} \) but not \( G_{\delta \sigma} \).

References


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