BOREL COMPLEXITY OF THE SPACE
OF PROBABILITY MEASURES

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Abstract. Using a technique developed by Louveau and Saint Raymond, we
find the complexity of the space of probability measures in the Borel hierarchy:
if \( X \) is any non-Polish Borel subspace of a Polish space, then \( P(X) \), the
space of probability Borel measures on \( X \) with the weak topology, is always
true \( \Pi^0_\xi \), where \( \xi \) is the least ordinal such that \( X \) is \( \Pi^0_\xi \).

1. Introduction

For \( X \) a separable metric space, let \( P(X) \) be the space of probability Borel
measures on \( X \) with the usual topology of weak convergence, so that \( P(X) \)
is also a separable metrizable space. Many results relating the descriptive complexities of
\( X \) and \( P(X) \) are classical and standard. For example (see [1], [5]), \( X \) is compact-
metrizable (resp. Polish) iff \( P(X) \) is compact-metrizable (resp. Polish), \( X \) is Borel
(resp. projective) iff \( P(X) \) is Borel (resp. projective), etc. Beyond the domain of
Borel sets we have: \( X \) is analytic (resp. co-analytic) iff \( P(X) \) is analytic (resp. co-
analytic), which follows from results of Kechris [2], and this also extends to every
level of the projective hierarchy under additional set-theoretic hypotheses. Shreve’s
theorem ([6]) establishes the same equivalence for each level of the \( C \)-hierarchy of
Selivanovski.

In this note we prove a similar result for the Borel hierarchy. It is known ([2])
that, for \( \alpha \geq 2 \), \( P(X) \) is \( \Pi^0_\alpha \) iff \( X \) is \( \Sigma^0_\alpha \); hence if \( X \) is \( \Sigma^0_\alpha \), then \( P(X) \) is \( \Pi^0_{\alpha+1} \).
This suggests the question: Is this the best bound? More generally, for \( \Gamma \) and \( \Gamma' \)
any two intrinsic Borel pointclasses, we ask: If \( X \) is in \( \Gamma \), is \( P(X) \) in \( \Gamma' \)? It is also
easy to find, for each \( \alpha \geq 3 \), an example of a \( P(X) \) which is true \( \Pi^0_\alpha \); just take \( X \)
to be any space which is true \( \Pi^0_\alpha \); since \( X \) is always embedded in \( P(X) \) as a closed
subset, \( P(X) \) cannot be \( \Sigma^0_\alpha \) lest \( X \) be \( \Sigma^0_\alpha \). This suggests the question: For which
\( \alpha \geq 3 \) can we find \( X \) such that \( P(X) \) is true \( \Sigma^0_\alpha \) (or true \( \Delta^0_\alpha \))?

Theorem 3.1 below answers these questions, and determines uniquely the true
Borel class of \( P(X) \) from the true Borel class of \( X \); it says: \( P(X) \) is Polish if \( X \) is
Polish, and for every \( \alpha \geq 3 \), \( P(X) \) is true \( \Pi^0_\alpha \) if \( X \) is true \( \Pi^0_\alpha \), \( P(X) \) is true \( \Pi^0_{\alpha+1} \).
if $X$ is true $\Sigma_\alpha^0$, and $P(X)$ is true $\Pi_\alpha^0$ if $X$ is true $\Delta_\alpha^0$. (These cases are mutually exclusive and exhaustive for Borel $X$.) In particular, if $X$ is Borel, then either $P(X)$ is Polish or $P(X)$ is true $\Pi_\alpha^0$ for some (unique) $\alpha \geq 3$, so that if $\alpha \geq 3$, there is no $X$ such that $P(X)$ is true $\Sigma_\alpha^0$ or true $\Delta_\alpha^0$.

2. Terminology

We use the notation of [1] and [4] for the Borel pointclasses: the additive, multiplicative, and ambiguous classes of level $\alpha$ are denoted, respectively, by $\Sigma_\alpha^0$, $\Pi_\alpha^0$, and $\Delta_\alpha^0$, with $\Sigma_1^0$ denoting the pointclass of open sets. $\Gamma$ is a Borel pointclass if $\Gamma$ is one of $\Sigma_\alpha^0$, $\Pi_\alpha^0$, and $\Delta_\alpha^0$. If $X$ is a Polish space, we use the notation $\Sigma_\alpha^0 | X$ to denote the $\Sigma_\alpha^0$ subsets of $X$, and similarly for $\Pi_\alpha^0$ and $\Delta_\alpha^0$. Note that for $\alpha \geq 2$ the pointclass $\Pi_\alpha^0$ is intrinsic $^\dagger$ and for $\alpha \geq 3$ the pointclasses $\Sigma_\alpha^0$ and $\Delta_\alpha^0$ are also intrinsic. For an intrinsic pointclass $\Gamma$, we can speak (unambiguously) of a separable metrizable space $X$ being $\Gamma$, without mentioning any Polish space in which $X$ is embedded. The notion of “the true Borel class” of a Borel set is defined in the usual way.

$\omega^*$ denotes the Baire space, i.e., the space of irrationals. If $X$, $Y$ are Polish spaces, $B \subseteq Y$, and $\mathcal{C}$ is a collection of subsets of $X$, we say that $B$ is $\mathcal{C}$-hard if for all $C \in \mathcal{C}$ there is a continuous $f: X \to Y$ such that $C = f^{-1}(B)$.

If $X$ is a metrizable space, $P(X)$ denotes the space of probability measures on $X$ with the weak topology $^\ddagger$.

For a separable metrizable space $Y$, and a Borel $X \subseteq Y$, we can (topologically) identify (see [1]) the space $P(X)$ with the subspace $P(X/Y)$ of $P(Y)$, where

$$P(X/Y) \overset{\text{def}}{=} \{ \mu \in P(Y) \mid \mu(Y \setminus X) = 0 \}.$$  

3. The Borel complexity of $P(X)$

**Theorem 3.1.** If $X$ is any non-Polish Borel subspace of a Polish space, then $P(X)$ is true $\Pi_\xi^0$, where $\xi$ is the least such that $X$ is $\Pi_\xi^0$.

**Proof.** We will use the following two lemmas:

**Lemma 3.2** (Louveau and Saint Raymond). If $Y$ is a Polish space, $\alpha$ is a countable ordinal $\geq 2$, $A \subseteq Y$ such that $A$ is Borel, and $A \not\in \Pi_\alpha^0$, then $A$ is $\Sigma_\alpha^0 | \omega^*$-hard, i.e. for all $B \subseteq \omega^*$, if $B$ is $\Sigma_\alpha^0$, then $B = f^{-1}(A)$ for some continuous $f: \omega^* \to Y$.

**Proof.** This is an immediate consequence of [3] Theorem 3, p. 455].

**Lemma 3.3** (The $\delta$-propagation lemma). Let $Y$, $Z$ be Polish spaces, $\mathcal{C}$ a collection of subsets of $Z$, and $X$ a subset of $Y$ which is $\mathcal{C}$-hard, i.e. for every $A$ in $\mathcal{C}$ there is a continuous map $f$ from $Z$ to $Y$ such that $A = f^{-1}(X)$. Then $P(X/Y)$ is $\mathcal{C}_\delta$-hard, where $\mathcal{C}_\delta$ denotes the countable intersections of sets in $\mathcal{C}$.  

$^\dagger$ $\Gamma$ is an intrinsic pointclass if for all Polish $X$, $Y$ and $A \subseteq X$, $B \subseteq Y$, if $A$ is a $\Gamma$-set in $X$ and $A$ is homeomorphic to $B$, then $B$ is a $\Gamma$-set in $Y$.

$^\ddagger$ The collection of all countably additive non-negative real-valued functions $\mu$ defined on the Borel sets of $X$ such that $\mu(X) = 1$.

$^3$ The weakest topology such that, for every bounded continuous real-valued function $f$ on $X$, the real-valued map $\mu \to \int fd\mu$ defined on $P(X)$ is continuous.
Proof. Let \( X, Y, Z, \) and \( C \) be as above. To show that \( P(X/Y) \) is \( C_{\delta} \)-hard, let \( F \) be an arbitrary subset of \( Z \) in \( C_{\delta} \). Then there is sequence \( (E_n)_{n=0}^\infty \) of subsets of \( Z \) such that \( (\forall n)(E_n \in C) \), and

\[
F = \bigcap_{n=0}^\infty E_n.
\]

For each \( n \in \omega \), choose a continuous function \( f_n: Z \to Y \) such that

\[
E_n = f_n^{-1}(X).
\]

Define \( \Psi: Z \to P(Y) \) by setting for each \( x \in Z \) and each Borel \( E \subseteq Y \):

\[
\Psi(x)(E) = \sum_{k=0}^\infty \frac{\chi_E(f_k(x))}{2^{k+1}},
\]

where, for any set \( A \), \( \chi_A \) denotes the characteristic function of \( A \). In other words, for any \( x \in Z \), \( \Psi(x) \) is the measure

\[
\Psi(x) = \sum_{k=0}^\infty \frac{1}{2^{k+1}} \delta_{f_k(x)} ,
\]

where, for any \( y \in Y \), \( \delta_y \) denotes the probability measure on \( Y \) known as the “unit mass” at \( y \). \( \Psi \) is well-defined and continuous. It is now easy to verify that \( \Psi \) reduces \( F \) to \( P(X/Y) \).

Now let \( X \) be non-Polish Borel, and \( Y \) be any metrizable compactification of \( X \); let \( \alpha \) be the least such that \( X \in \Sigma^0_\alpha \). Then \( \alpha \geq 3 \), as \( X \) is not \( \Pi^0_3 \) in \( Y \). Also, \( P(X) \) is \( \Pi^0_\alpha \). It remains to show that \( P(X) \) is not \( \Sigma^0_\alpha \) (\( \Sigma^0_\alpha \) is an intrinsic pointclass since \( \alpha \geq 3 \)). Put \( Z = \omega^\omega \), and

\[
\mathcal{E} = \bigcup_{\beta < \alpha} \Sigma^0_\beta | \omega^\omega.
\]

By Lemma 57.2 \( X \) is \( \Sigma^0_\beta | \omega^\omega \)-hard for every \( \beta < \alpha \), and hence is \( \mathcal{E} \)-hard. The hypotheses of the \( \delta \)-propagation lemma now hold. Therefore \( P(X/Y) \) is \( C_{\delta} \)-hard; but \( \mathcal{E} = \Pi^0_\alpha | \omega^\omega \). So \( P(X) \) must be true \( \Pi^0_\alpha \).

It follows from the above theorem that the Borel complexity of the space of probability measures on \( \mathbb{Q} \) (the rationals) is \( \mathcal{F}_{\sigma\delta} \) but not \( \mathcal{G}_{\delta\sigma} \).

References


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