REAL GROUPS TRANSITIVE ON COMPLEX FLAG MANIFOLDS

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Abstract. Let $Z = G/Q$ be a complex flag manifold. The compact real form $G_u$ of $G$ is transitive on $Z$. If $G_0$ is a noncompact real form, such transitivity is rare but occasionally happens. Here we work out a complete list of Lie subgroups of $G$ transitive on $Z$ and pick out the cases that are noncompact real forms of $G$.

0. The problem

Let $Z = G/Q$ be a complex flag manifold where $G$ is a complex connected semisimple Lie group and $Q$ is a parabolic subgroup. Let $G_0$ be a real form of $G$. If $G_0$ is the compact real form, then it is transitive on $Z$. On a number of occasions the question has come up as to whether any noncompact real form of $G$ can be transitive on $Z$. Here I'll record the answer. The rough answer is "yes, but just a few." The precise answer, Corollaries 1.7 and 2.3 below, follows from a more general classification, Theorems 1.6 and 2.2. This more general classification uses a technique of D. Montgomery [M], together with some results of [W1] that depend in an essential way on a classification [O1] of A. L. Onishchik.

After this paper was written I learned of Onishchik's book [O2]. There is some overlap for compact groups, but there are no inclusions.

1. The solution for irreducible flags

We formulate the problem in terms of transitive subgroups. Let $G_u$ be the compact real form of $G$, so $Z = G_u/(G_u \cap Q)$ and $G_u \cap Q$ is the compact real form of the reductive part of $Q$. Let $A \subseteq G$ be a closed subgroup that is transitive on $Z$. The identity component $A^0$ of $A$ is transitive on $Z$, because $Z$ is connected, so a maximal compact subgroup $B^0 \subset A^0$ already is transitive on $Z$, according to Montgomery [M]. We may replace $A$ by a conjugate and assume $B = A \cap G_u$. So
now we have several expressions:

\[ Z = G/Q = G_u/(G_u \cap Q) = A/(A \cap Q) = B/(B \cap Q) \]

\[ = A^0/(A^0 \cap Q) = B^0/(B^0 \cap Q). \]

According to [1], there are just a few possibilities for a homogeneous almost–\( \text{hermitian manifold} \) \( Z \) to have distinct expressions such as \( G_u/L_u \) and \( B^0/(B^0 \cap L_u) \), where \( G_u \) is the identity component of the group of all almost–\( \text{hermitian isometries} \), \( G_u \) is simple, \( L_u \) is the centralizer of a torus subgroup of \( G_u \), and \( B^0 \subset G_u \) with \( B^0 \) connected. They are:

(1.2) \( Z = S^2/\mathbb{C}(\mathbb{C}) = SU(2n)/U(2n-1) = Sp(n)/(Sp(n-1) \cdot U(1)) \), complex projective space.

(1.3) \( Z = SO(2r+2)/U(r+1) = SO(2r+1)/U(r) \), unitary structures on \( \mathbb{R}^{2r+2} \).

(1.4) \( Z = SO(7)/(SO(5) \cdot SO(2)) = G_2/U(2) \), 5–dimensional complex quadric.

(1.5) \( Z = SO(8)/(SO(6) \cdot SO(2)) = \{Spin(7)/Z_2\}/U(3) \), 6–dimensional complex quadric.

This applies in our situation because \( L_u = G_u \cap Q \) is the centralizer of a torus subgroup of \( G_u \), and \( Z \) has a \( G_u \)-invariant hermitian metric.

Now return to the expression \( Z = G/Q \). \( G \) (and thus \( G_u \)) is simple. Let \( A \subset G \) be a closed subgroup that is transitive on \( Z \) and let \( B \) be its maximal compact subgroup. We may assume \( B = A \cap G_u \). Then \( B \subset G_u \), \( B^0 \) is transitive on \( Z \), and the expression \( Z = G_u/L_u = B^0/(B^0 \cap L_u) \) is given above. In each case the group \( B^0 \) is simple, so \( A^0 \) has Levi decomposition \( A^0 = A_{ss}A_{rad} \) into semisimple part and solvable radical, where \( B^0 \) is a maximal compact subgroup of \( A_{ss}^0 \). We run through the 4 possibilities listed above.

For (1.2), \( G = SL(2n; \mathbb{C}) \) and \( B^0 = Sp(n) \). The semisimple Lie groups with maximal compact subgroup \( Sp(n) \) are \( Sp(n), Sp(n; \mathbb{C}), \) the quaternionic linear group \( SL(n; \mathbb{H}) \), and, for \( n = 4 \), the real group \( F_4 \). But \( F_4 \) does not have a representation of degree 8, in other words \( F_4 \not\subset G \), so now \( A_{ss}^0 \) is one of \( Sp(n), Sp(n; \mathbb{C}) \) and \( SL(n; \mathbb{H}) \). Each of them is irreducible on \( \mathbb{C}^{2n} \), so the unipotent radical of the algebraic hull of \( A^0 \) acts trivially on \( \mathbb{C}^{2n} \) and the center of the reductive part of \( A^0 \) acts by scalars. As \( G \) acts effectively and by transformations of determinant 1 on \( \mathbb{C}^{2n} \) now \( A_{ss}^0 = A^0 \), so \( A^0 \) is one of \( Sp(n), Sp(n; \mathbb{C}) \) and \( SL(n; \mathbb{H}) \). If \( g \in G \) normalizes \( A^0 \), then some element \( g' \in gA^0 \) centralizes \( A^0 \), because \( A^0 \) has no rational outer automorphism. As \( A^0 \) is irreducible on \( \mathbb{C}^{2n} \) now \( g' \) is scalar (and thus acts trivially on \( Z \)). Thus \( A = A^0 F \) where \( F \) can be any subgroup of the center \( \{e^{2\pi ik/2n}I \mid 0 \leq k < 2n \} \) of \( G \).

For (1.3), \( G = SO(2r+2; \mathbb{C}) \) and \( B^0 = SO(2r+1) \). The semisimple Lie groups with maximal compact subgroup \( SO(2r+1) \) are \( SO(2r+1), SO(2r+1; \mathbb{C}), SO(1,2r+1), \) and \( SL(2r+1; \mathbb{R}) \). But \( A_{ss}^0 = SL(2r+1; \mathbb{R}) \) would give \( SL(2r+1; \mathbb{C}) \subset SO(2r+2; \mathbb{C}) \), so the respective dimensions would satisfy \( 4r^2 + 4r \leq 2r^2 + 3r + 1 \), forcing \( r = 0 \) and \( Z = \{ \text{point} \} \). Thus \( A_{ss}^0 \neq SL(2r+1; \mathbb{R}) \). Now \( A_{ss}^0 \) is one of \( SO(2r+1), SO(2r+1; \mathbb{C}) \), and \( SO(1,2r+1) \). The last one acts irreducibly on \( \mathbb{C}^{2r+2} \), and there \( A_{ss}^0 = A^0 \) as above. For the first two, recall that \( SO(2r+1) \) is absolutely irreducible on the tangent space \( so(2r+2)/so(2r+1) \) of the sphere \( S^{2r+1} \), so \( A_{rad}^0 \) has Lie algebra reduced to 0, and again \( A_{ss}^0 = A^0 \). Now \( A^0 \) is one of \( SO(2r+1), SO(2r+1; \mathbb{C}) \), and \( SO(1,2r+1) \). If \( g \in G \) normalizes \( A^0 \), then some

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1 The author thanks the referee for a comment that improved and clarified his treatment of this \( SL(2r+1; \mathbb{R}) \) case.
element \( g' \in gA^0 \) centralizes \( A^0 \), because \( A^0 \) has no rational outer automorphism. Thus either \( A = A^0 \) or \( A/A^0 \) has order 2 where \( A \) is one of \( O(2r + 1), O(2r + 1; \mathbb{C}) \), and \( SO(1, 2r + 1) \cdot \{ \pm I \} \).

For (1.4), \( G = SO(7; \mathbb{C}) \) and \( B^0 = G_2 \). The semisimple Lie groups with maximal compact subgroup \( G_2 \) are \( G_2 \) and its complexification \( G_{2, \mathbb{C}} \). They are irreducible on \( \mathbb{C}^7 \) and have no rational outer automorphisms, so, as before, \( A^0 \) is either \( G_2 \) or \( G_{2, \mathbb{C}} \), and if \( g \in G \) normalizes \( A^0 \), then some element \( g' \in gA^0 \) centralizes \( A^0 \). This forces \( g' \) to be central in \( SO(7; \mathbb{C}) \), so \( g' = 1 \) and \( A = A^0 \). Thus \( A \) is either \( G_2 \) or \( G_{2, \mathbb{C}} \).

Finally, (1.5) is obtained from the case \( r = 3 \) of (1.3) by applying the triality automorphism, so it does not give us anything more.

In summary,

**Theorem 1.6.** Consider a complex flag manifold \( Z = G/Q \). Suppose that \( Z \) is irreducible, i.e., that \( G \) is simple. Then the closed subgroups \( A \subset G \) transitive on \( Z \), \( G \neq A \neq G \), are precisely those given as follows:

1. \( Z = SU(2n)/U(2n - 1) = P^{2n-1}(\mathbb{C}) \) complex projective \((2n - 1)-\)space; \( G = SL(2n; \mathbb{C}) \) and \( A = A^0F \) where \( A^0 \) is one of \( Sp(n) \), \( Sp(n; \mathbb{C}) \) and \( SL(n; \mathbb{H}) \), and \( F \) is any subgroup of the center \( \{ e^{2\pi i k/2n}I \mid 0 \leq k < 2n \} \) of \( G \). Here \( F \) acts trivially on \( Z \), so \( A \) and \( A^0 \) have the same action on \( Z \).

2. \( Z = SO(2r + 2)/U(r + 1) \), unitary structures on \( \mathbb{R}^{2r+2} \); \( G = SO(2r + 2; \mathbb{C}) \) and \( A = A^0F \) where \( A^0 \) is one of \( SO(2r + 1) \), \( SO(2r + 1; \mathbb{C}) \), and \( SO(1, 2r + 1) \), and where \( F \) is any subgroup of the center \( \{ \pm I \} \) of \( G \). Here \( F \) acts trivially on \( Z \), so \( A \) and \( A^0 \) have the same action on \( Z \).

3. \( Z = SO(7)/(SO(5) \cdot SO(2)) \), 5-dimensional complex quadric; \( G = SO(7; \mathbb{C}) \) and \( A \) is either the compact connected group \( G_2 \) or its complexification \( G_{2, \mathbb{C}} \).

Picking out the cases where \( A \) is a real form of \( G \) we have

**Corollary 1.7.** Consider a complex flag manifold \( Z = G/Q \). Suppose that \( Z \) is irreducible, i.e., that \( G \) is simple. Then the (connected) noncompact real forms \( G_0 \subset G \) transitive on \( Z \) are precisely those given as follows:

1. \( Z = SU(2n)/U(2n - 1) = P^{2n-1}(\mathbb{C}) \) complex projective \((2n - 1)-\)space; \( G = SL(2n; \mathbb{C}) \) and \( G_0 \) is the quaternion linear group \( SL(n; \mathbb{H}) \), which has maximal compact subgroup \( Sp(n) \).

2. \( Z = SO(2r + 2)/U(r + 1) \), unitary structures on \( \mathbb{R}^{2r+2} \); \( G = SO(2r + 2; \mathbb{C}) \) and \( G_0 \) is the Lorentz group \( SO(1, 2r + 1) \), which has maximal compact subgroup \( SO(2r + 1) \).

2. The solution for flag manifolds in general

We complete the solution of the problem by reducing it to the case where \( Z \) is irreducible.

**Proposition 2.1.** Decompose \( G = \prod G_i \), the local direct product of complex connected simple Lie groups. Thus \( Z = \prod Z_i \), the product of irreducible flag manifolds \( Z_i = G_i/Q_i \) where \( Q_i = Q \cap G_i \). Then \( A^0 = \prod A^0_i \) with \( A^0_i = A^0 \cap G_i \) and \( B^0 = \prod B^0_i \) with \( B^0_i = B^0 \cap G_i \). The groups \( A^0_i \) and \( B^0_i \) are connected, simple, and transitive on \( Z_i \).

**Proof.** The solvable radical of \( A^0 \) is contained in a Borel subgroup of \( G \), and thus has a fixed point on \( Z \). It is normal in the transitive group \( A^0 \) so it fixes every point. Thus \( A^0 \) is semisimple. Similarly \( B^0 \) is semisimple.
Let $\pi_i : G \to G_i$ denote the projection. The compact connected group $\pi_i(B^0)$ is transitive on $Z_i$. So it must be the compact real form $G_{u,i} = G_i \cap G_u$ of $G_i$ or one of the compact connected transitive groups described in (1.2), (1.3) or (1.4). (Recall that (1.5) is in fact a special case of (1.3).) In all cases, $\pi_i(B^0)$ is nontrivial and simple. Now $\pi_i$ annihilates all but one of the simple factors of $B^0$. Obviously no simple factor of $B^0$ is annihilated by every $\pi_i$. So now $B^0 = \prod B^0_\alpha$ where the $B^0_\alpha$ are simple and where the index set $I$ for $G = \prod_i G_i$ is a disjoint union of subsets $I_\alpha$ with $B^0_\alpha \subset \prod_{i \in I_\alpha} G_i$. The proof of Proposition 2.1 is reduced to the case where $B^0$ (and thus also $A^0$) is simple, and there it is reduced to the proof that $G_u$ is simple.

We may now assume $B^0$ simple. Suppose that $G_u$ is not simple. Projecting to $G_1 \times G_2$ we may assume $G = G_1 \times G_2$. View the isomorphisms $\pi_i : B^0 \cong \pi_i(B^0)$ as identifications. Denote $E_i = \pi_i(B^0)$, the complexification of the image of $B^0$ in $G_i$. Denote $E_{u,i} = \pi_i(B^0)$, the compact real form of $E_i$. Denote $P_i = E_i \cap Q_i$, the parabolic subgroup of $E_i$ that is its isotropy subgroup in $Z_i$, so $Z_i = E_i/P_i$. Now $B^0_C = \{(e,e) \mid e \in E_1\}$, $B^0_C \cap Q = \{(p,p) \mid p \in (P_1 \cap P_2)\}$, and $Z = B^0_C/(B^0_C \cap Q) \cong E_1/(P_1 \cap P_2)$. In particular $P_1 \cap P_2$ is a parabolic subgroup of $E_1$. Compute complex dimensions: $\dim E_1 - \dim (P_1 \cap P_2) = \dim B^0 - \dim (B^0 \cap Q) = \dim Z = \dim Z_1 + \dim Z_2 = (\dim E_1 - \dim P_1) + (\dim E_1 - \dim P_2)$. On the Lie algebra level this says $\dim \mathfrak{t}_1 = \dim \mathfrak{p}_1 + \dim \mathfrak{p}_2 - \dim (\mathfrak{p}_1 \cap \mathfrak{p}_2)$, in other words $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{t}_1$. As $\mathfrak{p}_1 \cap \mathfrak{p}_2$ is a parabolic subalgebra of $\mathfrak{t}_1$ we have a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{s} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$. In the root order such that $\mathfrak{s}$ is the sum of $\mathfrak{h}$ and the negative root spaces, no parabolic containing $\mathfrak{s}$ can contain the root space for the maximal root. This contradicts $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{t}_1$. The contradiction proves $G_u$ simple and completes the proof. □

Combining Proposition 2.1 with Theorem 1.6 we have

**Theorem 2.2.** Let $Z = G/Q$, the complex flag manifold, where $G$ is a complex connected semisimple Lie group acting with finite kernel on $Z$. Then the closed subgroups $A \subset G$ transitive on $Z$ are precisely those given as follows. Decompose $G = \prod G_i$ with $G_i$ simple, so $Z = \prod Z_i$ with $Z_i = G_i/(Q \cap G_i)$. Then $A = A^0F$ where $A^0 = \prod A_i$ with $A_i = (A \cap G_i)^0$, and $A_i$ is equal to $G_i$, or to its compact real form $G_{u,i}$, or to one of the three types listed in Theorem 1.6 and $F$ is any subgroup of the center of $G$. Here $F$ acts trivially on $Z$, so $A$ and $A^0$ have the same action on $Z$.

Picking out the cases where $A$ is a real form of $G$ we have, as in Corollary 1.7

**Corollary 2.3.** Let $Z = G/Q$, the complex flag manifold, where $G$ is a complex connected semisimple Lie group acting with finite kernel on $Z$. Then the (connected) real forms $G_0 \subset G$ transitive on $Z$ are precisely those given as follows. Decompose $G = \prod G_i$ with $G_i$ simple, so $Z = \prod Z_i$ with $Z_i = G_i/(Q \cap G_i)$. Then $A = \prod A_i$ where $A_i = A_i \cap G_i$ either is the compact real form $G_{u,i}$ of $G_i$ or is one of the two types listed in Corollary 1.7.

**References**


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