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$0^\sharp$ AND ELEMENTARY END EXTENSIONS OF V_κ

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ABSTRACT. In this paper we prove that if κ is a cardinal in $L[0^\sharp]$, then there is an inner model M such that $M \models (V_\kappa, \in)$ has no elementary end extension. In particular if $0^\sharp$ exists, then weak compactness is never downwards absolute. We complement the result with a lemma stating that any cardinal greater than $\aleph_1$ of uncountable cofinality in $L[0^\sharp]$ is Mahlo in every strict inner model of $L[0^\sharp]$.

1. Introduction

In this paper we consider the question of existence of elementary end extensions of models of the form $(V_\kappa, \in)$.

Definition 1.1. 1. Let $(\mathbb{E}_M, \prec_M)$ denote the structure of all non-trivial elementary end extensions of M, with $A \prec_M B$ iff B is an elementary end extension of A.

2. Let $(\mathbb{E}_M^{\mathrm{wf}}, \prec_M)$ denote the structure of all non-trivial *well founded* elementary end extensions of M, with $A \prec_M B$ iff B is an elementary end extension of A.

Several results regarding the existence of elements in $\mathbb{E}_M$ were proved by Keisler, Silver and Morley.

Theorem 1.1 (Keisler, Morley). *Let M be a model of ZFC, $cof(On^M) = \omega$. Then $\mathbb{E}_M \neq \emptyset$.*

Theorem 1.2 (Keisler, Silver). *Let $M = (V_\kappa, \in)$ be a model of ZFC, where κ is weakly compact cardinal. Then, for every $S \subseteq M$, $\mathbb{E}^{wf}_{(V_\kappa,\in,S)} \neq \emptyset$.*

Villaveces [5], [6] has proved several other results regarding the existence of elementary end extensions of V_κ.

Theorem 1.3 (Villaveces). *The theory "ZFC + GCH + $\exists\lambda(\lambda$ measurable$) + \forall\kappa[\kappa$ inaccessible not weakly compact $\to\exists$ transitive $M_\kappa \models ZFC$ such that $o(M) = \kappa$ and $\mathbb{E}_M^{wf} = \emptyset]$" is consistent relative to the theory "ZFC + $\exists\lambda(\lambda$ measurable$)$+the weakly compact cardinals are cofinal in On".*

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He also proved that the property $\mathbb{E}^{\mathrm{wf}}_{V_\kappa} \neq \emptyset$ is not preserved in certain generic extensions by destroying a weakly compact cardinal. In this paper we consider the problem of downwards absoluteness of the existence of well founded elementary end extensions of V_κ. We prove the following:

Theorem 1.4. *If $0^\sharp$ exists, then for every cardinal κ there is an inner model M such that*

$$M \models \mathbb{E}_{V_\kappa} = \emptyset. \tag{1.1}$$

In particular, weak compactness is never downwards absolute, once we have $0^\sharp$ in the universe. On the other hand we will prove that any cardinal with uncountable cofinality is Mahlo in any strict inner model of $L[0^\sharp]$. I would like to thank the referee for pointing out an inaccuracy in the formulation of Lemma 3.1 and for asking the question at the end of the paper.

2. Main theorem

In this section we prove Theorem 1.4. Let κ be a cardinal. Since we assume that $0^\sharp$ exists we can construct our model inside the inner model $L[0^\sharp]$. Note that since κ is a cardinal in V it remains a cardinal in $L[0^\sharp]$, and hence it is weakly compact in L. Our model will be a generic extension of L, such that we will be able to construct a generic object inside $L[0^\sharp]$. The basic idea will be to construct a generic Suslin tree and then to code it. For the construction of the Suslin tree we will follow Kunen's construction [2], while the coding will use Levy collapse of certain L cardinals. Then we will obtain the generic filter inside $L[0^\sharp]$.

The following theorem by Kunen gives us the forcing for generating the Suslin tree.

Theorem 2.1. *Let κ be a weakly compact cardinal and P_κ be the forcing for adding a Cohen subset to κ. Then $P_\kappa \simeq R_\kappa * T_\kappa$, where R_κ is a forcing that adds a Suslin tree T_κ to κ, and T_κ is the forcing defined by the tree.*

Let $\mathbf{P}$ be the reverse Easton iteration for adding a Cohen subset to each inaccessible, defined by:

Definition 2.1.

$$\mathbf{P} = (P_\alpha, Q_\alpha | \alpha \in On),$$

where:

$P_0 = \emptyset$.
If α is not inaccessible, then $P_\alpha \Vdash Q_\alpha = \emptyset$.
If α is inaccessible, then Q_α is a P_α name for a partial order adding a Cohen subset to α, i.e. $P_\alpha \Vdash Q_\alpha = (2^{<\alpha}, \subseteq)$.

Direct limits are taken at inaccessible limits of inaccessibles and inverse limits otherwise.

Solovay (see M. Stanley [4]) proved that the reverse Easton support iteration for adding Cohen subsets to every L inaccessible has a generic filter in $L[0^\sharp]$, and therefore our iteration up to κ has a generic filter as well.

Let $\mathbf{G} = \langle G_\alpha | \alpha \leq \kappa \rangle$ be $\mathbf{P}$ generic. By Kunen's theorem we can interpret G_κ as a pair $G_\kappa = \langle T_\kappa, b_\kappa \rangle$ where T_κ is a κ Suslin tree and b_κ is a branch through T_κ.

Next we define the forcing used to code the tree T_κ. Let $\mathbf{S}$ be the Easton supported product of collapsing of α^{+3} to α^{+2} defined inside L:

$$\mathbf{S} = \prod \{S_\alpha : \alpha \text{ is inaccesible}\}$$

where $S_\alpha = Coll(\alpha^{+2}, \alpha^{+3})$.

Proposition 2.1. *There is a* $\mathbf{P} \times \mathbf{S}$ *generic over* L*, inside* $L[0^\sharp]$.

Proof. The method of proof of this lemma is almost identical to M. Stanley's proof of Solovay's theorem which states that there exists a $\mathbf{P}$ generic filter over L inside $L[0^\sharp]$. We shall build the generic filter by induction on the Silver indiscernibles. The main point will be taking care that at limits the generic filter will be the direct limit of the previously built generic filters.

Let $\langle i_\alpha : \alpha < \kappa \rangle$ be an increasing enumeration of the indiscernibles below κ. For any indiscernible λ the forcing can be factored as

$$\mathbf{P} \times \mathbf{S} = \left(\mathbf{P}^{\lambda+1} * \mathbf{P}_{\lambda+1}\right) \times \left(\mathbf{S}^{\lambda} \times \mathbf{S}_\lambda\right) \tag{2.1}$$

where $\mathbf{P}^\lambda$ is the iteration up to λ, and $\mathbf{P}_\lambda$ is the iteration from λ upwards. For each α we shall define $\left(G^{i_\alpha}, H^{i_\alpha}\right)$, and then define $\left(G^{i_\alpha+1}, H^{i_\alpha+1}\right)$ such that $G^{i_\alpha+1} \times H^{i_\alpha+1}$ is $\left(\mathbf{P}^{i_\alpha} * Q_{i_\alpha}\right) \times \mathbf{S}^{i_\alpha+1}$ generic over $L[G^{i_\alpha} \times H^{i_\alpha}]$.

<u>i_0 or $i_{\alpha+1}$.</u> We have that in L for every indiscernible λ both $\mathbf{P}'_{\lambda+1}$ and $\mathbf{S}_\lambda$ are λ^{++} closed, where

$$\mathbf{P}'_{\lambda+1} = \left\{\tau : \tau \text{ is a name and } \Vdash_{\mathbf{P}^{\lambda+1}} \tau \in \tilde{\mathbf{P}}_{\lambda+1}\right\} \tag{2.2}$$

is the term forcing for $\mathbf{P}_{\lambda+1}$. Hence $\mathbf{P}_{\lambda+1} \times \mathbf{S}_\lambda$ is λ^+-distributive over $L^{\mathbf{P}^{\lambda+1} \times \mathbf{S}^\lambda}$, since $\mathbf{P}^{\lambda+1} \times \mathbf{S}^\lambda$ is obviously λ^+-c.c.

By the same argument $\mathbf{P}^{i_{\alpha+1}}_{i_\alpha+1} \times \mathbf{S}^{i_{\alpha+1}}_{i_\alpha}$ is also i_α^+ distributive. Let

$$M = L^{\mathbf{P}^{i_\alpha+1} \times \mathbf{S}^{i_\alpha+1}}. \tag{2.3}$$

Note that each L name for a dense subset of $\mathbf{P}^{i_{\alpha+1}}_{i_\alpha+1} \times \mathbf{S}^{i_{\alpha+1}}_{i_\alpha}$ in M belongs to the Skolem hull of the ordinals up to i_α and finitely many indiscernibles above $i_{\alpha+1}$, say $\{i_{\alpha+1}, \dots, i_{\alpha+n}\}$. Hence in $L[0^\sharp]$ we can represent the dense subsets of $\mathbf{P}^{i_{\alpha+1}}_{i_\alpha+1} \times \mathbf{S}^{i_{\alpha+1}}_{i_\alpha+1}$ in M by a countable union of families of dense subsets each of size i_α. Now using the i_α^+ distributivity we can meet each of these dense subsets. To ensure downwards compatibility we also demand that $\left(G^{i_\alpha+1}, H^{i_\alpha+1}\right)$ extends $\left(G^{i_\alpha}, H^{i_\alpha}\right)$. Finally use the same distributivity argument to define a generic filter $G(i_{\alpha+1})$ for $Q_{i_{\alpha+1}}$ over $L^{\left(\mathbf{P}^{i_{\alpha+1}} \times \mathbf{S}^{i_{\alpha+1}}\right)}$. Again in order to ensure extension we demand that $G(i_{\alpha+1})$ extends $G(i_\alpha)$ by putting a condition forcing it into the generic. Since $\mathbf{S}$ is not active at these stages and using the fact that $\mathbf{P}$ is a reverse Easton iteration this is possible.

<u>i_α for α limit.</u> We have built generic objects $\left\langle G^i \times H^i : i < \alpha \right\rangle$ for the product up to α. Now we would like to build a generic filter for $\mathbf{P}^{i_\alpha} \times \mathbf{S}^{i^\alpha}$. Note that since i_α is Mahlo in L we take direct limit. Moreover $\mathbf{P}^{i_\alpha} \times \mathbf{S}^{i_\alpha}$ is i_α–c.c. Define $G^{i_\alpha}, H^{i_\alpha}$ by

$$p \in G^{i_\alpha} \text{ iff } \forall \gamma < i_\alpha p{\restriction}\gamma \in G^{i_\gamma}, \tag{2.4}$$

$$s \in H^{i_\alpha} \text{ iff } \forall \gamma < i_\alpha s{\restriction}\gamma \in H^{i_\gamma}. \tag{2.5}$$

We prove that $G^{i_\alpha} \times H^{i_\alpha}$ is $\mathbf{P}^{i_\alpha} \times \mathbf{S}^{i_\alpha}$ generic over L. Suppose that $D \subseteq \mathbf{P}^{i_\alpha} \times \mathbf{S}^{i_\alpha}$ is dense open. D belongs to the Skolem hull of finitely many ordinals below i_α $\mathbf{a} = \langle \gamma_1, \dots, \gamma_n \rangle$ and finitely many indiscernibles above α, say $\mathbf{i}_n = \langle i_{\alpha+1}, \dots, i_{\alpha+n} \rangle$. Let $\sup(\mathbf{a}) < i_\beta < i_\alpha$. Define an elementary embedding $j : L \rightarrow L$ by

$$j(i_\gamma) = \begin{cases} i_\gamma & \text{if } \gamma < \beta, \\ i_{\alpha+\delta} & \text{if } \gamma = \beta + \delta, 0 \leq \delta. \end{cases} \tag{2.6}$$

Obviously $D \in \mathrm{rng} j$, and $j^{-1}(D)$ is dense open in $\mathbf{P}^\beta \times \mathbf{S}^\beta$. Let $(p', q') \in j^{-1}(D) \cap (\mathbf{P}^\beta \times \mathbf{S}^\beta)$. Since both p', q' are trivial on an end segment we obtain that

$$j((p', q')) = (p, q)^\wedge \left\langle \emptyset^{Q_\gamma \times S_\gamma} : \beta \leq \gamma < \alpha \right\rangle . \tag{2.7}$$

Hence by our choice of $\left(G^{i_\alpha}, H^{i_\alpha}\right)$ we obtain that $j((p', q')) \in \left(G^{i_\alpha}, H^{i_\alpha}\right)$.

Finally we prove that we can find a generic object $G(i_\alpha)$ for Q_{i_α} over $L^{\left(G^{i_\alpha} \times H^{i_\alpha}\right)}$. Define

$$G(i_\alpha) = \bigcup_{\beta < \alpha} G(i_\beta). \tag{2.8}$$

Let D be a dense subset of Q_{i_α} in $L^{G^{i_\alpha} \times H^{i_\alpha}}$. Let $\tilde{\mathbf{D}}$ be a name for D in $\mathbf{P}^{i_\alpha} \times \mathbf{S}^{i_\alpha}$. Again $\tilde{\mathbf{D}}$ is in the Skolem hull of some $i_\beta < i_\alpha$ and finitely many indiscernibles $\mathbf{i}_n = (i_{\alpha+1}, \dots, i_{\alpha+n})$. Define $j : L \rightarrow L$ as above. As we have proved, if $(p, q) \in G^{i_\beta} \times H^{i_\beta}$, then $j(p, q) \in G^{i_\alpha} \times H^{i_\alpha}$. Hence the embedding j has a canonical extension to an embedding $\hat{j} : L[G^{i_\beta} \times H^{i_\beta}] \rightarrow L[G^{i_\alpha} \times H^{i_\alpha}]$ defined by

$$\hat{j}(\tau(G^{i_\beta} \times H^{i_\beta})) = j(\tau)(G^{i_\alpha} \times H^{i_\alpha}). \tag{2.9}$$

Since $\tilde{\mathbf{D}}$ is in $\mathrm{rng} j$ we have $D \in \mathrm{rng} \hat{j}$. The proof ends as follows:

Let

$$p' \in G(i_\beta) \cap \hat{j}^{-1}(D). \tag{2.10}$$

p' exists by the induction hypothesis. $G(i_\beta)$ is Q_{i_β} generic, and $\hat{j}^{-1}(D)$ is dense in Q_{i_β} by elementarity, and hence $\hat{j}(p') \in D$. Since $p' \in L_{i_\beta}[G^{i_\beta} \times H^{i_\beta}]$ we have $j(p') = p'$. So

$$p' \in G(i_\beta) \cap D \subseteq G(i_\alpha) \cap D. \tag{2.11}$$

□

Let $\mathbf{G} \times \mathbf{H}$ be $\mathbf{P} \times \mathbf{S}$ generic over L. Suppose that $\mathbf{H} = \langle h_\alpha | \alpha < \kappa \rangle$ is the $\mathbf{S}$ generic filter. Let $\langle \cdot, \cdot \rangle$ be a definable pairing function in L, such that for every β, γ, $\langle \beta, \gamma \rangle$ is an L inaccessible. Since the pairing is definable and κ is an indiscernible it is closed under the pairing function.

Let T be the tree part of $G(\kappa)$. Our final model will be $N = L[T, \langle h_\alpha | \alpha \in C_T \rangle]$ where

$$C_T = \{\alpha | \exists \beta, \gamma \, (\alpha = \langle \beta, \gamma \rangle \wedge \beta <_T \gamma)\}.$$

To finish the proof of the theorem we have to prove:

Proposition 2.2.

$$N \models \text{“}V_\kappa \textit{ has no elementary end extension”}. \tag{2.12}$$

Proof. The proof will be done by a sequence of claims.

Claim 2.1. $N \models T$ is Suslin.

Proof. The claim follows from the fact that the forcing $\mathbf{S}$ is κ-Knaster in $L[T]$. Hence $\mathbf{S} \times T$ is κ-c.c. in $L[T]$, so especially T is κ-c.c. in $N' = L[T, \langle h_\alpha | \alpha < \kappa\rangle]$. However, $N \subseteq N'$ and $\kappa^N = \kappa^{N'}$; thus N contains no large anti-chains of T as well. □

Claim 2.2. For every inaccessible α

$$N \models \alpha^{+++L} < \alpha^{+++} \iff \alpha \in C_T. \tag{2.13}$$

Proof. Since for every $\alpha \in C_T$ the claim obviously holds, it will be enough to prove that other cardinals are not collapsed inside $L[\mathbf{G}, \langle h_\alpha | \alpha \in C_T\rangle]$. For each $\mu \notin C_T$ we can even work inside $L[\mathbf{G}, \langle h_\alpha | \alpha \neq \mu\rangle]$. However since both forcing notions $\mathbf{P}$ and

$$\mathbf{S}^{-\mu} = \prod \{S_\alpha : \alpha \neq \mu \text{ and } \alpha \text{ is inaccesible}\}$$

factor nicely, it is obvious that the only L-cardinals collapsed are the triple successors of cardinals in C_T. □

Notice that by the inaccessibility of κ all the collapsing functions are inside V_κ^N.

Now we finish the proof of Proposition 2.2. In $(V_\kappa^N, \in)$ the tree T is definable by the first order formula

$$\beta <_T \gamma \iff \exists \alpha \left(\alpha \text{ is inaccessible } \wedge \alpha = \langle \beta, \gamma\rangle \wedge \alpha^{+++L} < \alpha^{+++}\right).$$

$(V_\kappa^N, \in) \models T$ is a κ tree, i.e., for every ordinal α $\{x \in T | \text{hight}_T(x) = \alpha\}$ is a set, and for every ordinal α there is an element of T of hight α. Assume that (M, E) is an end extension of $(V_\kappa^N, \in)$. Let a be a new ordinal in M. In M there is a tree T' that end extends the tree T, since T was definable. By elementarity

$$M \models \text{there is a branch } b \text{ in } T' \text{ of length } a.$$

Now it follows that

$$N \models \{x \in b | rk(x) < \kappa\} \text{ is a branch through } T.$$

Hence any end extension of $(V_\kappa^N, \in)$ will provide a branch through T in N. This is a contradiction since $N \models T$ is Suslin. □

3. Mahloness in inner models

In view of the previous result it is natural to ask whether we can get an inner model $M \subsetneq L[0^\sharp]$ such that for every inaccessible cardinal $\alpha \in M$, $(V_\alpha, \in)$ has no well founded elementary end extension. This turns out to be impossible by the following lemma:

Lemma 3.1. *Let $\kappa > \aleph_1$ be a cardinal in $L[0^\sharp]$, $cf(\kappa) > \aleph_0$. Then κ is weakly Mahlo in any strictly inner model $M \subsetneq L[0^\sharp]$. Moreover if κ is a limit cardinal, then κ is strongly Mahlo in every $M \subsetneq L[0^\sharp]$.*

Proof. The basic idea is to use the covering theorem to prove that certain cardinals are not collapsed in any strict inner model of $L[0^\sharp]$. Then we use the covering theorem again to prove that actually there must be a stationary set of inaccessibles below κ. Let $M \subsetneq L[0^\sharp]$ be an inner model. Let $I = \{i_\alpha | \alpha \in On\}$ be an increasing enumeration of Silver's indiscernibles. Then for every α such that $\omega < \text{cf}(\alpha)$ we have that $M \models i_\alpha^{+L}$ is a cardinal. The proof of this uses an idea of Beller [1]. Assume $M \models i_\alpha^{+L}$ is not a cardinal. Then $|i_\alpha^{+L}|^M = |i_\alpha|^M$. Also by the covering theorem $M \models \text{cf}(i_\alpha^{+L}) = \text{cf}(i_\alpha) = |i_\alpha|$. Therefore in M there is an f $f : i_\alpha \to i_\alpha^{+L}$ which maps in an order preserving way a cofinal subset of i_α into a cofinal subset of i_α^{+L}. Since $L[0^\sharp] \models \mathrm{cf}(i_\alpha^{+L}) = \omega$ choose a cofinal sequence (in i_α^{+L}) $\{\beta_n : n < \omega\}$ inside $L[0^\sharp]$. Now let γ_n be the least γ such that $f(\gamma) > \beta_n$. We obtain that $\{\gamma_n : n < \omega\}$ is cofinal in i_α so $L[0^\sharp] \models \mathrm{cf}(i_\alpha) = \omega$. This contradicts the fact that i_α has uncountable cofinality. Hence every limit of indiscernibles of uncountable cofinality is a limit cardinal. By the covering theorem it must be a regular cardinal, so it is weakly inaccessible. Especially any uncountable cardinal is weakly inaccessible.

Suppose now that i_α is not Mahlo in M and i_α is a limit of indiscernibles of uncountable cofinality. Then there is a club $C \subseteq i_\alpha$ consisting of singular cardinals in M. By the covering theorem (between L and M) each element of C is singular in L. Hence $C \cap I = \emptyset$. Hence $L[0^\sharp] \models \mathrm{cf}(i_\alpha) = \omega$ (since it has two disjoint clubs through i_α). Therefore if $\mathrm{cf}(i_\alpha) > \omega$ and i_α is a limit of indiscernibles of uncountable cofinality it must be Mahlo in any strict inner model.

If κ is also a limit cardinal in $L[0^\sharp]$ it is strong limit by GCH. Hence it is strong limit in any inner model, so it is strongly Mahlo in M. □

Therefore if κ is limit in $L[0^\sharp]$ and $\mathrm{cf}(\kappa) > \omega$, then in every inner model there is an inaccessible $\alpha < \kappa$ such that $\mathbb{E}^{\mathrm{wf}}_{(V_\alpha,\varepsilon)} \neq \emptyset$.

A natural question is whether one can have no weakly compacts in a strictly inner model of $L[0^\sharp]$. We comment that if there is a κ such that $L[0^\sharp] \models \kappa \to (\omega)^{<\omega}$, then by a result of Silver [3], for any inner model M, $M \models \kappa \to (\omega)^{<\omega}$; hence there are many ineffable cardinals in M. Similarly if there is a subtle cardinal κ, in $L[0^\sharp]$, then obviously κ is subtle in every inner model (the definition is Π_1). Hence there are many large cardinals below it in any inner model (e.g., totally indescribables).

However the following question remains open:

Question ($ZFC + V = L[0^\sharp]$)**.** Let M be an inner model. Is it consistent that M has no weakly compact cardinals? Is it consistent that for no κ $M \models \kappa \to (\omega)^{<\omega}$?

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