THE FEFFERMAN-STEIN TYPE INEQUALITY 
FOR THE KAKEYA MAXIMAL OPERATOR

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Abstract. Let $K_{\delta}$, $0 < \delta \ll 1$, be the Kakeya maximal operator defined as the supremum of averages over tubes of the eccentricity $\delta$. We shall prove the so-called Fefferman-Stein type inequality for $K_{\delta}$, 

$$
\|K_{\delta}f\|_{L^p(\mathbb{R}^d;w)} \leq C_{d,p}\left(\frac{1}{\delta}\right)^{d/p-1}\left(\log\left(\frac{1}{\delta}\right)\right)^{\alpha(d)}\|f\|_{L^p(\mathbb{R}^d,K_{\delta}w)} ,
$$

in the range $(1 < p \leq (d^2-2)/(2d-3))$, $d \geq 3$, with some constants $C_{d,p}$ and $\alpha(d)$ independent of $f$ and the weight $w$.

1. Introduction 

The purpose of this note is to investigate the so-called Fefferman-Stein type inequality for the Kakeya maximal operator. Throughout this note $0 < \delta \ll 1$ will be a small parameter. For $f$ a locally integrable function on $\mathbb{R}^d$, $d \geq 2$, define 

$$(K_{h,\delta}f)(x) = \sup_{T} \frac{1}{|T|} \int_{T} |f(y)|dy ,$$

where the supremum is taken over all tubes $T$ containing $x \in \mathbb{R}^d$ with the length $h$ and the radius of the cross section $h\delta$. We define the Kakeya maximal operator $K_{\delta}$ by 

$$(K_{\delta}f)(x) = \sup_{h>0} (K_{h,\delta}f)(x) .$$

We call a non-negative Borel measurable function $w$ a weight if it is a locally integrable function on $\mathbb{R}^d$. By $w(A)$ we mean the $w(x)dx$ measure of a set $A$.

It is verified for $d = 2$ that in the range $1 < p \leq d$ the Fefferman-Stein type inequality 

$$(1.1) \quad \left( \int_{\mathbb{R}^d} (K_{\delta}f)(x)^p w(x)dx \right)^{1/p} \leq C_{d,p}\epsilon\left(\frac{1}{\delta}\right)^{d/p-1+\epsilon}\left(\int_{\mathbb{R}^d} |f(x)|^p(K_{\delta}w)(x)dx\right)^{1/p}$$

holds for all $\epsilon > 0$ (M"{u}ller and F. Solia, [MS]). But in higher dimensions this fact has been verified only in the range $1 < p \leq (d+1)/2$ (A. M. Vargas, [Va]). The main difficulty of this problem lies in making the exponent $p$ as high as possible.

Bourgain proved that an unweighted version of $(1.1)$ (putting $w \equiv 1$) holds in the range $1 < p \leq p_d$, where $(d+1)/2 < p_d < (d+2)/2$ is some exponent given by

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a recursive formula starting from $p_3 = 7/3$ \[Bo1\]. Wolff improved this result \[Wo\]. He proved that an unweighted version of (1.1) holds in the range $1 < p \leq (d + 2)/2$. Recently, in higher dimensions Bourgain improved it further to $1 < p \leq (1/2 + c)d$ ($c > 0$ independent of $d$) \[Bo2\].

A different approach to this problem (an unweighted version) was given by Igari. He investigated the most difficult case $p = d$. He proved that an unweighted version of (1.1) holds for a special basis \[Ig\]. He restricted the bases for taking the supremum to only tubes $T$ of which the axis intersects a fixed line. The author proved the weighted version of this restricted result \[Ta2\]. In this note we shall improve Vargas’s result by using this restricted estimates.

The main theorem of this note is the following.

**Theorem 1.** Let $d \geq 3$. There exist constants $C_{d,p}$ and $\alpha(d)$ independent of $\delta, f,$ and $w$ such that

$$
\|K_\delta f\|_{L^p(\mathbb{R}^d,w)} \leq C_{d,p}(\frac{1}{\delta})^{d/p-1}(\log(\frac{1}{\delta}))^{\alpha(d)}\|f\|_{L^p(\mathbb{R}^d,K_\delta w)}
$$

holds in the range $1 < p \leq (d^2 - 2)/(2d - 3)$.

By using sieve arguments and three-points interpolation lemma our result can be reduced to the discrete analogue as stated in the following theorem. (See \[MS\], \[Va\], and also \[Ta2\].)

Let $Q = (-1/2,1/2)^d$ and $\hat{Q} = (-2,2)^d$. We divide $\hat{Q}$ into $\delta$-tubes, $Q_i, \text{centered at } i \in I$, where $I$ is the set of lattice points with the $\delta$-separation.

**Theorem 2.** Let $d \geq 3$. For a measurable set $A \subset Q$ and $0 < \lambda \leq 1$ define

$$I = \{i \in I \mid (K_1 R_A)(i) > \lambda \}.$$ 

Then

$$
\sum_{i \in I} w(Q_i) \leq C_d(\frac{1}{\delta})^{d-p(d)}(\frac{1}{\lambda})^{p(d)}(\log(\frac{1}{\delta}))^{\beta(d)}(K_\delta w)(A),
$$

where

$$p(d) = \frac{(d^2 - 2)}{(2d - 3)}$$

and

$$\beta(d) = \frac{(d+1)(d-2)}{(2d-3)}.$$ 

In the following $C$‘s will denote constants which may be different in each occasion but depend only on the dimension $d$.

2. **Proof of Theorem**

2.1. **Preliminaries.** We summarize some known results for later use.

Given any line $L$ in $\mathbb{R}^d$ define

$$(K_1^L f)(x) = \sup_T \frac{1}{|T|} \int_T |f(y)|dy,$$

where the supremum is taken over all $\delta$-tubes $T$ which contain $x$ and of which the axis intersects $L$. Here, $\delta$-tube is the tube with the length 1 and the radius of the cross section $\delta$. 

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Lemma 3 (Theorem 2 in [Ta2]). Let $d \geq 3$. Let $\lambda > 0$ and $L$ be any line in $\mathbb{R}^d$. Then
\[
\{x \in \mathbb{R}^d : (K^L_{1,\delta} f)(x) > \lambda\} \leq C\left(\frac{1}{\lambda}\right)^d (\log(\frac{1}{\delta}))^{d+1} \|f\|_{L^d(\mathbb{R}^d,d\omega)}.
\]

Let $B_{\leq 1/\delta}$ be the class of all rectangles in $\mathbb{R}^d$ which satisfy
\[
1 \leq \text{(the length of longest sides)}/\text{(the length of shortest sides)} \leq \frac{1}{\delta}.
\]
The corresponding maximal operator associated to this base $B_{\leq 1/\delta}$ will be denoted by $K_{\leq 1/\delta}$.

Lemma 4 (Theorem 3 in [Ta1]). Let $d \geq 2$. There exist constants $C_1$ and $C_2$ depending only on $d$ such that
\[
C_1(K_{\delta} f)(x) \leq (K_{\leq 1/\delta} f)(x) \leq C_2(K_{\delta} f)(x)
\]
holds for every $x \in \mathbb{R}^d$.

2.2. Main argument. Write $W = K_{\delta} w$. Fix $A \subset Q$ and $0 < \lambda \leq 1$. Recall $I = \{i \in \mathcal{I} : (K_{1,\delta} \chi_A)(i) > \lambda\}$. Then for every $i \in I$ we can select a $\delta$-tube $T_i$, which contains $i$, such that
\[
|A \cap T_i| > C\delta^{d-1}\lambda.
\]
Then, it follows from (2.1) and the Schwarz inequality that
\[
(C\delta^{d-1}\lambda \sum_{i \in I} w(Q_i))^2
\leq (\sum_{i \in I} w(Q_i)|A \cap T_i|)^2 = (\int_A \sum_{i \in I} w(Q_i)\chi_{T_i})^2
\leq \left\{ \int_A (\sum_{i \in I} w(Q_i)\chi_{T_i})^2 W^{-1}\right\} W(A)
\leq \left\{ \sum_{i \in I} w(Q_i) \sum_{j \in I} w(Q_j) W^{-1}(T_i \cap T_j) \right\} W(A)
\leq \max_{i \in I} (\sum_{j \in I} w(Q_j) W^{-1}(T_i \cap T_j)) \cdot (\sum_{i \in I} w(Q_i)) W(A).
\]
Hence
\[
N^{-1} \sum_{i \in I} w(Q_i) \leq C\left(\frac{1}{\delta}\right)^{2(d-1)} \left(\frac{1}{\lambda}\right)^2 W(A),
\]
which corresponds to low multiplicity of Wolff (see the proof of Lemma 3.1 in [W0]), where
\[
N = \max_{i \in I} (\sum_{j \in I} w(Q_j) W^{-1}(T_i \cap T_j)).
\]
By the fact that $p(d) > 2$ and $\frac{1}{d} \geq 1$, we may assume that
\[
C\delta^{2(d-1)} \leq N.
\]
The following proposition, corresponding to high multiplicity of Wolff, will be proven later.
Proposition 5. With previous setup we have
\[
N \{ \sum_{i \in I} w(Q_i) \}^{(d-2)/(d-1)} \leq C \left( \frac{1}{\delta} \right)^{-d} \{ (\log \left( \frac{1}{\delta} \right) \}^{d+1} \left( \frac{1}{\lambda} \right)^d W(A) \}^{(d-2)/(d-1)}.
\]

Multiplying both sides of (2.2) and (2.4) together, we obtain the desired inequality (1.2).

2.3. Proof of Proposition 5. Take some \( i_0 \in I \) so that
\[
N = \sum_{j \in I} w(Q_j) W^{-1}(T_{i_0} \cap T_j).
\]
Let
\[
I_0 = \{ j \in I : T_{i_0} \cap T_j \neq \emptyset \}
\]
and
\[
s_0 = \inf_{y \in T_{i_0}} W(y).
\]

By the geometric observation of Córdoba [Co] one sees that
\[
|T_{i_0} \cap T_j| \leq C \frac{\delta^d}{\delta + \text{dist}(T_{i_0}, J)}.
\]
From (2.5)–(2.8) we have
\[
N \leq C(s_0)^{-1} \sum_{j \in I_0} w(Q_j) \frac{\delta^d}{\delta + \text{dist}(T_{i_0}, J)}.
\]

Define the subset of \( I_0 \) as
\[
\sigma_k = \{ j \in I_0 : (k-1)\delta \leq \text{dist}(T_{i_0}, j) < k\delta \}, \quad k = 1, 2, \ldots,
\]
and rewrite
\[
(s_0)^{-1} \sum_{j \in I_0} w(Q_j) \frac{\delta^d}{\delta + \text{dist}(T_{i_0}, J)} = (s_0)^{-1} \sum_k \sum_{j \in \sigma_k} w(Q_j) \frac{\delta^d}{\delta + \text{dist}(T_{i_0}, J)}.
\]
Then
\[
N \leq C(s_0)^{-1} \delta^{d-1} \sum_k \sum_{j \in \sigma_k} \frac{w(Q_j)}{k}.
\]

It follows for some \( k_0 \) to be specified later that
\[
\delta^{d-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} \frac{w(Q_j)}{k} = \delta^{d-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} w(Q_j) \left( \sum_{l=k}^{k_0} \frac{1}{l(l+1)} + \frac{1}{k_0+1} \right)
\]
\[
= \delta^{d-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} w(Q_j) \left( \sum_{l=k}^{k_0} \frac{1}{l(l+1)} \right) + \delta^{d-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} w(Q_j)
\]
\[
= I + II.
\]
By reversing the order of summation we have

\[ I = \delta^{d-1} \sum_{l=1}^{k_0} \left( \sum_{k=1}^{l} \sum_{j \in \sigma_k} w(Q_j) / (l(l+1)) \right) \]

\[ \leq C \delta^{2d-2} \sum_{l=1}^{k_0} l^{d-3} \left( \sum_{k=1}^{l} \sum_{j \in \sigma_k} w(Q_j) / ((l\delta)^{d-1}) \right). \]

By using Lemma 4 we see that

\[ (s_0)^{-1} \left( \sum_{k=1}^{l} \sum_{j \in \sigma_k} w(Q_j) / ((l\delta)^{d-1}) \right) \leq C(s_0)^{-1} \frac{f_{R_l} w}{|R_l|} \leq C, \]

where

\[ R_l = \{ x \in \mathbb{R}^d : \text{dist}(T_{i_0}, x) \leq l\delta \}. \]

Hence

\[ (s_0)^{-1} I \leq C \delta^{2d-2} \sum_{l=1}^{k_0} l^{d-3} \leq C \delta^d (k_0 \delta)^{d-2}. \]

Combining these inequalities we obtain

\[ N \leq C \delta^d \{(k_0 \delta)^{d-2} + (k_0 \delta)^{-1} (s_0)^{-1} \sum_{j \in I_0} w(Q_j) \}. \]

Now, we can choose some \( k_0 \) so that

\[ (k_0 \delta)^{d-1} \sim (s_0)^{-1} \sum_{j \in I_0} w(Q_j) \]

by (2.3) and (2.10). Then the two terms in the right-hand side of (2.11) balance and hence

\[ N \leq C \delta^d \left\{ (s_0)^{-1} \sum_{j \in I_0} w(Q_j) \right\}^{(d-2)/(d-1)}. \]

Applying Lemma 3 with \( L = \) the axis of \( T_{i_0} \) and \( f = \chi_A \), we clearly obtain

\[ \sum_{j \in I_0} w(Q_j) \leq C \left( \frac{1}{\lambda} \right)^d (\log \frac{1}{\lambda})^{d+1} W(A). \]

Thus, from (2.12) and (2.13) we have

\[ N (s_0)^{(d-2)/(d-1)} \leq C \delta^d \left\{ \left( \frac{1}{\lambda} \right)^d (\log \frac{1}{\lambda})^{d+1} W(A) \right\}^{(d-2)/(d-1)}. \]

Finally, again by Lemma 3 we observe that

\[ \sum_{i \in I} w(Q_i) \leq C \frac{w(Q)}{|Q|} \leq C' s_0. \]

Thus, from (2.14) and (2.15) we obtain

\[ N \left\{ \sum_{j \in I} w(Q_j) \right\}^{(d-2)/(d-1)} \leq C \delta^d \left\{ \left( \frac{1}{\lambda} \right)^d (\log \frac{1}{\lambda})^{d+1} W(A) \right\}^{(d-2)/(d-1)}. \]
References


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