SHAPE ASPHERICAL COMPACTA—APPLICATIONS OF A THEOREM OF KAN AND THURSTON TO COHOMOLOGICAL DIMENSION AND SHAPE THEORIES

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ABSTRACT. Dydak and Yokoi introduced the notion of shape aspherical compactum. In this paper, we use this notion to obtain a generalization of Kan and Thurston theorem for compacta and pro-homology. As an application, we obtain a characterization of cohomological dimension with coefficients in \( \mathbb{Z} \) and \( \mathbb{Z}/p \) (\( p \) prime) in terms of acyclic maps from a shape aspherical compactum, which improves the theorems of Edwards and Dranishnikov. Furthermore, we obtain the shape version of the theorem and as a consequence we show that every compactum has the stable shape type of a shape aspherical compactum.

1. Introduction

First recall

**Theorem 1.1** (Kan and Thurston [KT]). For each path-connected space \( X \), there exist a space \( TX \) and a map \( t : TX \to X \), natural for maps on \( X \), with the following properties:

- \( t_* : H_\ast(TX; t^*A) \to H_\ast(X; A) \) and \( t^* : H^\ast(X; A) \to H^\ast(TX; t^*A) \) are isomorphisms of singular homologies and cohomologies with local coefficients;

- and

- \( (KT2) \): \( t_* : \pi_1(TX) \to \pi_1(X) \) is onto, and \( \pi_i(TX) \cong 0 \) for \( i \neq 1 \).

Mauner gave a simpler proof to the theorem and obtained the following variation:

**Theorem 1.2** (Mauner [Ma]). For each finite connected simplicial complex \( K \), there exist a finite simplicial complex \( TK \) of the same dimension, and a map \( t_K : TK \to K \), natural for simplicial maps on \( K \), with properties \((KT1)\) and \((KT2)\).

Throughout the paper, a compactum means a compact metric space, and a continuum means a connected compactum.

The paper consists of three parts. In the first part, we generalize those results as follows:
Theorem A. For each continuum $X$ (resp., continuum with $\dim X < \infty$), there exist an approximately aspherical compactum $Y$ (resp., approximately aspherical compactum $Y$ with $\dim Y = \dim X$) and a surjective map $\varphi : Y \to X$ with the following properties:

(S1): $\varphi$ induces isomorphisms of Čech homologies and cohomologies;
(S2): $\varphi_* : \text{pro-} \pi^S_1(Y) \to \text{pro-} \pi^S_1(X)$ is an epimorphism; and
(S3): For each connected closed subset $A$ of $X$, $\varphi^{-1}(A)$ is an approximately aspherical compactum, and $\varphi|\varphi^{-1}(A) : \varphi^{-1}(A) \to A$ satisfies properties (S1) and (S2).

Here, for each compactum $X$, $\dim X$ denotes the covering dimension of $X$. A compactum $X$ is said to be approximately aspherical if every map of $X$ into a polyhedron factors up to homotopy through a finite aspherical CW complex. Note that our definition is slightly stronger than the original definition of shape asphericity of Dydak and Yokoi [DY] by requiring the finiteness of the factoring CW complex. Asphericity of compacta in the study of cell-like maps was first considered by Daverman [Da] and continued by Daverman and Dranishnikov [DD].

As an application of Theorem A, in the second part of the paper we obtain a characterization of cohomological dimension with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/p$ for any prime $p$, which improves the well-known characterizations by Edwards and Dranishnikov in the theorems below.

For each compactum $X$ and abelian group $G$, the cohomological dimension $\text{cdim}_G X \leq n$ if $X\tau K(G,n)$, where for any ANR $P$, $X\tau P$ denotes the property that every map of any closed subset of $X$ into $P$ extends over $X$.

Theorem 1.3 (Edwards [E, W]). For each compactum $X$, $\text{cdim}_\mathbb{Z} X \leq n$ if and only if there exists a cell-like map $f : Y \to X$ from a compactum $Y$ of $\dim Y \leq n$.

Theorem 1.4 (Dranishnikov [Dr]). For each compactum $X$ and for each prime $p$, $\text{cdim}_{\mathbb{Z}/p} X \leq n$ if and only if there exists a surjective map $f : Y \to X$ from a compactum $Y$ of $\dim Y \leq n$ such that each fibre is acyclic modulo $p$.

Koyama [K] and Koyama and Yokoi [KY] extended those results to approximable dimensions with arbitrary coefficient groups. Note the approximable dimension with a finitely generated coefficient group coincides with the cohomological dimension.

We obtain the following:

Theorem B. For each continuum $X$ and for each prime $p$, $\text{cdim}_{\mathbb{Z}} X \leq n$ (resp., $\text{cdim}_{\mathbb{Z}/p} X \leq n$) if and only if there exist an approximately aspherical compactum $Y$ with $\dim Y \leq n$ and a surjective map $f : Y \to X$ such that each fibre is acyclic (resp., acyclic modulo $p$).

In the third and final part of the paper we give applications to shape theory. Let $sd X$ denote the shape dimension of $X$ (see [MaS, p. 95]).

Theorem C. For each continuum $X$ of $sd X < \infty$, there exist an approximately aspherical compactum $Y$ of $\dim Y = sd X$ and a shape morphism $\varphi : Y \to X$ with properties (S1) and (S2).

Theorem D. 1. Every continuum has the weak stable shape type of an approximately aspherical compactum.
2. Every continuum $X$ of $\text{sd} X < \infty$ has the stable shape type of an approximately aspherical compactum $Y$ of $\dim Y = \text{sd} X$.

See [MiS1] for the definitions in stable shape theory.

2. CHARACTERIZATIONS OF APPROXIMATELY ASPHERICAL COMPACTA

**Theorem 2.1.** For every compactum $X$, the following are equivalent:

i) $X$ is an approximately aspherical compactum;

ii) $X$ admits an expansion of $X$, $p = (p_i) : X \to X = \langle X_i, p_{ii+1}, \mathbb{N} \rangle$, such that each $X_i$ is a finite aspherical polyhedron (here, the expansion is in the sense of [MaS] p. 19); and

iii) Every polyhedral expansion of $X$, $p = (p_i) : X \to X = \langle X_i, p_{ii+1}, \mathbb{N} \rangle$ has the property that every $i$ admits $i' \geq i$ such that $p_{i'}$ factors through a finite aspherical polyhedron.

**Proof.** ii) $\Rightarrow$ i) is obvious. We wish to verify i) $\Rightarrow$ iii) $\Rightarrow$ ii). For i) $\Rightarrow$ iii), let $p = (p_i) : X \to X = \langle X_i, p_{ii+1}, \mathbb{N} \rangle$ be a polyhedral expansion of $X$. For each $i$, there exist a finite aspherical polyhedron $P$ and homotopy maps $g : X \to P$ and $h : P \to X_i$ such that $p_i = hg$. Then for some $i'' \geq i$ there exists a homotopy map $g'' : X_{i''} \to P$ such that $g = g'p_{i''}$. So, $p_{i''}p_{i''} = p_i = hg'g_{i''}$, and hence there exists $i' \geq i''$ such that $p_{i'} = hg'p_{i''}$ as desired. For iii) $\Rightarrow$ ii), let $p = (p_i) : X \to X = \langle X_i, p_{ii+1}, \mathbb{N} \rangle$ be any polyhedral expansion of $X$. Then by iii), there is an increasing sequence $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq \cdots$ and finite aspherical polyhedra $Y_k$, $k = 1, 2, \ldots$, such that $p_{i_ki_{k+1}} = h_kg_k$ for some $g_k : X_{i_{k+1}} \to Y_k$ and $h_k : Y_k \to X_{i_k}$. For each $k = 1, 2, \ldots$, let $q_k = g_kp_{i_{k+1}} : X \to Y_k$ and $q_{kk+1} = g_kh_{k+1} : Y_{k+1} \to Y_k$. Then it is a routine to check that $q = (q_k) : X \to Y = \langle Y_k, q_{kk+1}, \mathbb{N} \rangle$ forms an expansion of $X$.

**Remark 2.2.** Analogous characterization for Dydak and Yokoi’s definition holds without the finiteness conditions on the aspherical polyhedra in i) and iii).

3. PROOF OF THEOREM A

Before we prove the theorem, we observe the following properties for the map $t_K : TK \to K$ in Maunder’s Theorem, which were obtained in the original proof [Ma].

- (M1): For each connected subcomplex $M$ of $K$, $\dim t_K^{-1}(M) = \dim M$, and $t_K|t_K^{-1}(M) : t_K^{-1}(M) \to M$ satisfies properties (KT1) and (KT2) and is natural in the following sense: For any simplicial map $f : K \to K'$, if $L$ and $L'$ are subcomplexes of $K$ and $K'$, respectively, such that $f(L) \subseteq L'$, then the following diagram commutes:

\[
\begin{array}{ccc}
L & \xrightarrow{f|L} & L' \\
\uparrow & & \uparrow \\
t_K^{-1}(L) & \xrightarrow{tf|t_K^{-1}(L)} & t_K^{-1}(L')
\end{array}
\]

where $Tf : TK \to TK'$ is the induced simplicial map;

- (M2): Each fibre of $t_K$ is either a point or an acyclic and aspherical subcomplex of $TK$; and

- (M3): $t_K$ is onto.
Let $X$ be a continuum, and let $U_i$, $i = 1, 2, \ldots$, be a sequence of finite open coverings of $X$ which form a base for the topology on $X$. For each $i$, let $K_i$ be the nerve of $U_i$ with realization $X_i$, let $p_{i+1}: K_{i+1} \to K_i$ be a connecting simplicial map and let $p_i: X \to X_i$ be a canonical map. Then the map $p = (p_i): X \to X = (X_i, p_{i+1}, N)$ forms an inverse limit. By Theorem 1.2 and the above observation, for each $i$, there exist a complex $TK_i$ and a map $\varphi_i = t_{K_i}: TK_i \to K_i$ with properties (KT1), (KT2), (M1), (M2) and (M3) and a simplicial map $q_{i+1} = Tp_{i+1}: TK_{i+1} \to TK_i$ which makes the following diagram commute:

\[
\begin{array}{ccc}
K_i & \xrightarrow{p_{i+1}} & K_{i+1} \\
\varphi_i \downarrow & & \uparrow \varphi_{i+1} \\
TK_i & \xrightarrow{Tp_{i+1}} & TK_{i+1}
\end{array}
\]

For each $i$, let $Y_i$ be the realization of $TK_i$, and let $Y$ be the limit of the inverse sequence $Y = (Y_i, q_{i+1}, N)$ with the projections $q_i: Y \to Y_i$. Then the level morphism $\varphi = (\varphi_i): Y \to X$ induces the limit map $\varphi: Y \to X$, which is surjective. By properties (KT1) and (KT2) for each $\varphi_i: Y_i \to X_i$, $\varphi$ satisfies properties (S1) and (S2). To verify property (S3), let $A$ be a closed subset of $X$. For each $i$, let $L_i$ be the nerve of the open covering $U_i\cap A = \{U \cap A : U \in U_i\}$ of $A$. Then $L_i$ is a subcomplex of $K_i$. So, for each $i$, if $A_i$ is the realization of $L_i$, then $A_i$ is a subpolyhedron of $X_i$, and by property (M1) we have the following commutative diagram:

\[
\begin{array}{ccc}
A_i & \xrightarrow{p_{i+1}|A_{i+1}} & A_{i+1} \\
\varphi_i|\varphi_i^{-1}(A_i) \downarrow & & \uparrow \varphi_{i+1}|\varphi_{i+1}^{-1}(A_{i+1}) \\
\varphi_i^{-1}(A_i) & \xrightarrow{q_{i+1}|\varphi_i^{-1}(A_{i+1})} & \varphi_{i+1}^{-1}(A_{i+1})
\end{array}
\]

and each $\varphi_i|\varphi_i^{-1}(A_i): \varphi_i^{-1}(A_i) \to A_i$ satisfies properties (KT1) and (KT2). Note that the restricted maps $p|A = (p_i|A): A \to A = (A_i, p_{i+1}|A_i, N)$ and $q|\varphi^{-1}(A) = (q_i|\varphi^{-1}(A)): \varphi^{-1}(A) \to \varphi^{-1}(A) = (\varphi^{-1}(A), q_{i+1}|\varphi_i^{-1}(A_{i+1}), N)$ form the inverse limits of $A$ and $\varphi^{-1}(A)$, respectively. So the map $\varphi|\varphi^{-1}(A): \varphi^{-1}(A) \to A$ which is the limit map of the level morphism $\varphi|\varphi^{-1}(A) = (\varphi_i|\varphi_i^{-1}(A_i)): \varphi^{-1}(A) \to A$ has properties (S1) and (S2). Since each $\varphi_i^{-1}(A_i)$ is aspherical, $\varphi^{-1}(A)$ is approximately aspherical by Proposition 2.1. Hence property (S3) is fulfilled.

Now suppose $\dim X = n < \infty$. Then we can take the base $U_i$, $i = 1, 2, \ldots$, so that the nerves of $U_i$ have dimension at most $n$. So, for each $i$, $\dim Y_i = \dim TK_i = \dim K_i \leq n$, and hence $\dim Y \leq n$. On the other hand, the commutative diagram for each closed subset $A$ of $X$

\[
\begin{array}{ccc}
\hat{H}^q(\varphi^{-1}(A); \mathbb{Z}) & \xrightarrow{\varphi^{-1}(A)^*} & \hat{H}^q(A; \mathbb{Z}) \\
\downarrow j_A & & \downarrow j_A \\
\hat{H}^q(Y; \mathbb{Z}) & \xrightarrow{c^*} & \hat{H}^q(X; \mathbb{Z})
\end{array}
\]

and property (S3) imply $\dim_{\mathbb{Z}} X \leq \dim_{\mathbb{Z}} Y$, and by Alexandroff theorem, $\dim X = \dim_{\mathbb{Z}} X$ and $\dim_{\mathbb{Z}} Y = \dim Y$. Hence $\dim Y = n$, as required.
4. Proof of Theorem B

Assume there is a surjective map \( f : Y \to X \) from an approximately aspherical compactum \( Y \) with \( \dim Y \leq n \) such that \( H^*(f^{-1}(x); \mathbb{Z}) = 0 \) for all \( x \in X \). Using Vietoris-Begle theorem, we can obtain \( \text{cdim}_Y X \leq \text{cdim}_Y Y = \dim Y \leq n \). Hence \( \text{cdim}_Y Y \leq n \).

Conversely, suppose \( \text{cdim}_Y Y \leq n \). Then Edwards theorem (Theorem 1.3) implies that there exists a cell-like map \( g : X' \to X \) from a compactum \( X' \) with \( \dim X' \leq n \). By taking each component of \( X' \), without loss we can assume \( X' \) is connected. Theorem A implies that there exists a surjective map \( h : Y \to X' \) from a shape aspherical map \( Y \) of \( \dim Y = \dim X' \) such that for each closed subset \( B \) of \( X' \), the restricted map \( h|^{-1}(B) : h^{-1}(B) \to B \) induces an isomorphism \( (h|h^{-1}(B))^* : H^q(B; \mathbb{Z}) \to H^q(h^{-1}(B); \mathbb{Z}) \) for each \( q \). So, if we let \( f = gh : Y \to X \), then \( H^q(f^{-1}(x); \mathbb{Z}) \cong H^q(g^{-1}(x); \mathbb{Z}) = 0 \) for each \( q \). The case for \( \mathbb{Z}/p \) is proved similarly, using Dranishnikov theorem (Theorem 1.4).

5. Proofs of Theorems C and D

Proof of Theorem C. If \( \text{sd} X \leq n < \infty \), then there is a polyhedral expansion \( \mathbf{p} = (p_i) : X \to X = (X_i, p_{i+1}, N) \) of \( X \) such that \( X_i \) are finite polyhedra with \( \dim X_i \leq n \). Then choose a triangulation \( K_1 \) of \( X_1 \), and using the simplicial approximation theorem, we can inductively choose triangulations \( K_i \) of \( X_i \) and simplicial maps \( a_{i+1} : K_{i+1} \to K_i \) that represent the corresponding homotopy classes \( p_{i+1} : X_{i+1} \to X_i \). As in the proof of Theorem A, for each \( i \), there exist a simplicial complex \( TK_i \) with \( \dim TK_i = \dim K_i \) and maps \( \varphi_i : TK_i \to K_i \) with properties (KT1) and (KT2) and \( q_{i+1} : TK_{i+1} \to TK_i \). Let \( Y \) be the limit of the inverse sequence \( Y = (Y_i, q_{i+1}, N) \) where \( Y_i \) are the realizations of \( TK_i \), and let \( q_i : Y \to Y_i \) be the projection maps. Then \( q = (q_i) : Y \to Y \) induces a polyhedral expansion of \( Y \), so the maps \( \varphi_i : Y_i \to X_i \) form a level morphism \( \varphi = (\varphi_i) : Y \to X \) which represents a shape morphism \( \varphi : Y \to X \) with properties (S1) and (S2). Since \( \dim Y_i = \dim X_i \leq n \), then \( \dim Y \leq n \).

Thus \( \dim Y \leq \text{sd} X \). On the other hand, by [1], property (S3) and Alexandroff Theorem, \( \text{sd} X \leq \text{cdim}_Y X \leq \text{cdim}_Y Y = \dim Y \). Hence \( \text{sd} X = \dim Y \).

Corollary 5.1. 1. For each continuum \( X \) (resp., continuum \( X \) with \( \dim X < \infty \)), there exists an approximately aspherical compactum \( Y \) (resp., approximately aspherical compactum \( Y \) with \( \dim Y = \dim X \)) and a surjective map \( \varphi : Y \to X \) such that the induced map \( \text{SP}^\infty(\varphi) : \text{SP}^\infty Y \to \text{SP}^\infty X \) is a weak shape equivalence.

2. For each continuum \( X \) with \( \text{sd} X < \infty \), there exists an approximately aspherical compactum \( Y \) with \( \dim Y = \text{sd} X \) and a shape morphism \( \varphi : Y \to X \) such that the induced map \( \text{SP}^\infty(\varphi) : \text{SP}^\infty Y \to \text{SP}^\infty X \) is a weak shape equivalence.

Proof. This easily follows from Theorems A and C and [DT].

Proof of Theorem D. Let \( X \) be a continuum. Then by Theorem A, there exists a map \( \varphi : Y \to X \) from an approximately aspherical compactum \( Y \) onto \( X \) such that \( \varphi_* : \text{pro}-H_q(Y; \mathbb{Z}) \to \text{pro}-H_q(X; \mathbb{Z}) \) is an isomorphism for each \( q \), which implies by [MS2, Corollary 7.8] that \( \varphi_* : \text{pro}-\pi^1_q(Y) \to \text{pro}-\pi^1_q(X) \) is an isomorphism for each \( q \) as required. If \( \text{sd} X < \infty \), then Theorem C implies that there exists a shape morphism \( \varphi : Y \to X \) from an approximately aspherical compactum \( Y \) of
dim$Y = \text{sd } X$ such that $\varphi_* : \text{pro-} H_q(Y; \mathbb{Z}) \to \text{pro-} H_q(X; \mathbb{Z})$ is an isomorphism for each $q$, so $\varphi_* : \text{pro-} \pi^X_q(Y) \to \text{pro-} \pi^X_q(X)$ is an isomorphism for each $q$. Now by [MiS1, Theorem 6.1], $\varphi$ is an equivalence in the stable shape category.

References


[KT] D. M. Kan and W. P. Thurston, Every connected space has the homology of a $K(\pi,1)$, Topology 15 (1976), 253 – 258. MR 54:12110


