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THE KANENOBU-MIYAZAWA CONJECTURE AND THE VASSILIEV-GUSAROV SKEIN MODULES BASED ON MIXED CROSSINGS

JÓZEF H. PRZYTYCKI AND KOUKI TANIYAMA

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Abstract. We show that a Brunnian link of n components and the n component trivial link share the same first $n-1$ coefficients of the Jones-Conway (Homflypt) polynomial (answering the question of Kanenobu and Miyazawa). We prove also the similar result for the Kauffman polynomial of Brunnian links. We place our solution in the context of Vassiliev-Gusarov skein modules based on mixed singular crossings.

1. Kanenobu-Miyazawa conjecture

A Brunnian link of n components is an oriented link such that the removal of any of its components leaves us with a trivial link of $n-1$ components (for $n=2$ we assume also that the linking number is equal to 0).

Let $P_L(v,z) \in Z[v^{\pm 1}, z^{\pm 1}]$ be the Jones-Conway (Homflypt) polynomial of oriented links in S^3. That is, $P_{T_n}(v,z) = (\frac{v^{-1}-v}{z})^{n-1}$ for a trivial link of n components, T_n, and $v^{-1}P_{L_+} - vP_{L_-} = zP_{L_0}$ for a Conway skein triple L_+, L_- and L_0.

Let $\tilde{P}_L(v,z) = z^{n-1}P_L(v,z)$ for a link L of n components. $\tilde{P}_L(v,z)$ satisfies: $\tilde{P}_{T_n}(v,z) = (v^{-1}-v)^{n-1}$ and

$$v^{-1}\tilde{P}_{L_+} - v\tilde{P}_{L_-} = \begin{cases} z^2\tilde{P}_{L_0} & \text{for a mixed crossing,} \\ \tilde{P}_{L_0} & \text{for a self-crossing.} \end{cases}$$

In particular $\tilde{P}_L(v,z) \in Z[v^{\pm 1}, z]$ and z is not an invertible element in the ring.

Theorem 1.1 (Kanenobu-Miyazawa conjecture)**.** *If L is a Brunnian link of n components, then*

$$\tilde{P}_L - \tilde{P}_{T_n} \equiv 0 \ mod(z^{2n-2}).$$

Proof. Consider a diagram D_L of a nontrivial Brunnian link L of n components ($n \geq 2$) which consists of a trivial link T_{n-1} (no crossings) with components $L_1, ..., L_{n-1}$, and a component L_n which intersects all other components. Let S_i denote the subset of crossings of L_i and L_n at which L_i is below L_n. We use the notation $L^{S_1,...,S_{n-1}}_{\epsilon_1,...,\epsilon_{n-1}}$ where $\epsilon_i = -1$ or 1, to denote a link obtained from L by changing all

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the crossings in S_i of the diagram D_L iff $\epsilon_i = -1$. In particular $L_{1,\ \ldots,\ 1}^{S_1,\ldots,S_{n-1}} = L$. On the other hand if not all $\epsilon_i = 1$, then $L_{\epsilon_1,\ldots,\epsilon_{n-1}}^{S_1,\ldots,S_{n-1}}$ is a trivial link of n components, because when $\epsilon_i = -1$, then L_i is above the rest of the link. Therefore

$$\tilde{P}_L - \tilde{P}_{T_n} = \sum_{\epsilon_1,\ldots,\epsilon_{n-1}} \epsilon_1 \cdots \epsilon_{n-1} \tilde{P}_{L_{\epsilon_1,\ldots,\epsilon_{n-1}}^{S_1,\ldots,S_{n-1}}}.$$

Now, because $lk(L_i, L_j) = 0$ for Brunnian links, we can write the right-hand sum of the equation (using skein relation between mixed crossings) as:

$$z^{2n-2} \sum_{\epsilon(S_1),\ldots,\epsilon(S_{n-1})} (-1)^* v^* \tilde{P}_{L_{\epsilon(S_1),\ldots,\epsilon(S_{n-1})}^{S_1,\ldots,S_{n-1}}}.$$

We should explain the notation and the formula will be clear (and our proof of Theorem 1.1 completed). We order crossings of S_i, say $q_1, q_2, \ldots., q_{k_i}$, and $\epsilon(S_i)$ denote the number between 1 and k_i (i.e. $1 \leq \epsilon(S_i) \leq k_i$). The meaning of $L_{\epsilon(S_1),\ldots,\epsilon(S_{n-1})}^{S_1,\ldots,S_{n-1}}$ is that we change crossings q_j in S_i if $j < \epsilon(S_i)$, we smooth the crossing $q_{\epsilon(S_i)}$ and we do not change other crossings of S_i. The exponents of -1 and v depend on sign of changed/smoothed crossings but are inessential for us. The important thing to observe is that we always operate on mixed crossings, and every link of the sum is a knot obtained by n smoothings of mixed crossings (and some changings) so finally we have the common power z^{2n-2}. □

We have the similar result for the Kauffman polynomial. We delay a proof of it because the result follows as a corollary of a more general theory.

Theorem 1.2. *If L is a Brunnian link of n components, then*

$$\tilde{F}_L - \tilde{F}_{T_n} \equiv 0 \ mod(z^{2n-2})$$

where $\tilde{F}_L = z^{n-1} F_L$ and F_L is the Dubrovnik version of the Kauffman polynomial. That is, it satisfies: $F_{T_k} = (\frac{a-a^{-1}+z}{z})^{k-1}$ and $a^{w(L_+)} F_{L_+} - a^{w(L_-)} F_{L_-} = z(a^{w(L_0)} F_{L_0} - a^{w(L_\infty)} F_{L_\infty})$ where $w(L)$ for a link diagram L is the writhe (or Tait) number of L, that is, the sum of signs of all crossings of L.

2. Vassiliev-Gusarov skein modules based on mixed crossings

Our proof is motivated by the notion of n-triviality of links, as introduced by Yamamoto [Y], Ohyama [O] and Gusarov [G]. This notion is related to Vassiliev-Gusarov skein modules (and the dual space of Vassiliev invariants), and Jones-Conway and Kauffman polynomials. The difference with respect to the standard treatment (see [P]) is that we consider only mixed singular crossings. We can develop our theory in any 3-manifolds, but we will limit ourselves here to S^3.

(1) Skein modules. Let $\mathcal{L}$ be the set of oriented links up to ambient isotopy, R a commutatative ring with identity, and $R\mathcal{L}$ the free module over $\mathcal{L}$. Let $\mathcal{L}^{sgm}$ be the set of "mixed" singular links, that is, a set of singular links such that, for any k, smoothing k singular crossings reduces the number of components by k. If we divide $R\mathcal{L}^{sgm}$ by the singular crossing splitting relations ($L_{sgm} = L_+ - L_-$), we get $R\mathcal{L}$ back. Let C_i^{mix} be a submodule of $R\mathcal{L}$ generated by links with i "mixed" singular crossings. We define the "mixed" Vassiliev-Gusarov skein module as the quotient $W_k^{mix} = R\mathcal{L}/C_{k+1}^{mix}$.

(2) "Mixed" n-trivial links.[1] A link L of m components ($m \geq n+1$) is called a *"mixed" n-trivial link* if there is a diagram of L with n groups of crossings $S_1, ..., S_n$ such that $L^{S_1,\ldots,S_n}_{\epsilon_1,\ldots,\epsilon_n}$ is a trivial link of m components with the possible exception of $L = L^{S_1,\ldots,S_n}_{1,\ \ldots,\ 1}$. We require furthermore the "mixed" crossing condition, that is, if $p_i \in S_i$ then the smoothing of the crossings $p_1, ..., p_n$ reduces the number of components of L by n. It follows from the considerations in the first section that:

Lemma 2.1. *A Brunnian link of n components is "mixed" $n-1$ trivial.*

(3) "Mixed" n-trivial links and "mixed" skein modules.

Theorem 2.2. *If L is a "mixed" n-trivial link of m components ($m \geq n+1$), then*

$$L - T_m \in C_n^{mix}$$

or equivalently $L = T_m$ in the "mixed" Vassiliev-Gusarov skein module W_n^{mix}.

Proof. Following our proof of Theorem 1.1, but using singular links instead of smoothings, we can write (in $R\mathcal{L}$):

$$\begin{aligned} L - T_m &= \sum_{\epsilon_1,\ldots,\epsilon_n} L^{S_1,\ldots,S_n}_{\epsilon_1,\ldots,\epsilon_n} \\ &= \sum_{\delta(S_1),\ldots,\delta(S_n)} L^{S_1,\ldots,S_n}_{\delta(S_1),\ldots,\delta(S_n)}. \end{aligned}$$

Below we explain our notation.

We order the crossings of S_i, say $q_1, q_2, ...q_{k_i}$, and $\delta(S_i)$ denote a number between 1 and k_i (i.e. $1 \leq \delta(S_i) \leq k_i$). The meaning of $L^{S_1,\ldots,S_{n-1}}_{\delta(S_1),\ldots,\delta(S_n)}$ is that we change the crossings q_j in S_i if $j < \delta(S_i)$, we put a singular crossing in place of the crossing $q_{\delta(S_i)}$ and we do not change other crossings of S_i. Theorem 2.2 follows from the formula. □

(4) Jones-Conway and Kauffman polynomials for "mixed" n-trivial links. Applying Theorem 2.2 we obtain:

Theorem 2.3. *Let L be a "mixed" n-trivial link of m components. For $n = 1$ assume additionally that $lk(L) = 0$. Then:*

(P) $\tilde{P}_L - \tilde{P}_{T_m} \equiv 0$ *mod* (z^{2n}).
(F) $\tilde{F}_L - \tilde{F}_{T_m} \equiv 0$ *mod* (z^{2n}) *where* $\tilde{F}_L = z^{m-1}F_L$ *and* F_L *is the Dubrovnik version of the Kauffman polynomial.*

As a corollary we obtain the following generalization of the Kanenobu- Miyazawa conjecture.

After [S] we define an *n-color Brunnian link L* as a link whose components are divided into n nonempty groups $L_1, L_2, ..., L_n$ in such a way that $L - L_i$ is a trivial link and for $n = 2$, $lk(L) = 0$.

Corollary 2.4. *Let L be an n-color Brunnian link of m components. Then*

(T) *L is a "mixed" $n-1$ trivial link.*
(P) $\tilde{P}_L - \tilde{P}_{T_m} \equiv 0$ *mod* (z^{2n-2}).
(F) $\tilde{F}_L - \tilde{F}_{T_m} \equiv 0$ *mod* (z^{2n-2}).

[1] In choice of "n" we follow [O]. In [G] or [P] it would be "$n-1$".

(5) "Mixed" n-similarity. The notion of an n-trivial link is generalized in [T, G] to the concept of an n-similar links. We adjust the concept to the "mixed" crossings case.

Definition 2.5. A link L is *"mixed" n-similar* to a link L_1 if there is a diagram of L with n groups of crossings $S_1, ..., S_n$ such that $L^{S_1,\ldots,S_n}_{\epsilon_1,\ldots,\epsilon_n}$ represents the link L_1 with the possible exception of $L = L^{S_1,\ldots,S_n}_{1,\ \ldots,\ 1}$.

Theorems 2.2 and 2.3 generalize to the case of a "mixed" n-similarity.

Theorem 2.6. *Let L be a link "mixed" n-similar to L_1. Then:*
(T) $L - L_1 \in C_n^{mix}$.
Let additionally $lk(L) = lk(L_1)$ for $n = 1$. Then:
(P) $\tilde{P}_L - \tilde{P}_{L_1} \equiv 0\ mod\ (z^{2n})$.
(F) $\tilde{F}_L - \tilde{F}_{L_1} \equiv 0\ mod\ (z^{2n})$.

"Mixed" n-similar links can be constructed from Brunnian links:

Example 2.7. Let L be a Brunnian link of n components $K_1, ..., K_n$ and $B_1, ..., B_n$ be links in a solid torus. Then the link L_1 obtained from L by decorating every K_i by B_i is "mixed" $(n-1)$-similar to the link L_2 obtained by decorating a trivial link of n components by $K_1, ..., K_n$ (informally L_2 is a disjoint sum of K_i's).

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Department of Mathematics, University of Maryland, College Park, Maryland 20742

(On leave from) Department of Mathematics, George Washington University, Washington, DC 20052
E-mail address: przytyck@gwu.edu

Department of Mathematics, Tokyo Woman's Christian University, Zempukuji, Suginamiku, Tokyo, 167-8585, Japan
E-mail address: taniyama@twcu.ac.jp