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VIETORIS CONTINUOUS SELECTIONS AND DISCONNECTEDNESS-LIKE PROPERTIES

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Abstract. Suppose that X is a Hausdorff space such that its Vietoris hyperspace $(\mathcal{F}(X), \tau_V)$ has a continuous selection. Do disconnectedness-like properties of X depend on the variety of continuous selections for $(\mathcal{F}(X), \tau_V)$ and vice versa? In general, the answer is "yes" and, in some particular situations, we were also able to set proper characterizations.

1. Introduction

Let X be a Hausdorff space, and let $\mathcal{F}(X)$ be the family of the non-empty closed subsets of X. Also, let $\mathcal{D} \subset \mathcal{F}(X)$. A map $f : \mathcal{D} \to X$ is a *selection* for $\mathcal{D}$ if $f(S) \in S$ for every $S \in \mathcal{D}$. A map $f : \mathcal{D} \to X$ is a *continuous* selection for $\mathcal{D}$ if it is a selection which is continuous with respect to the relative Vietoris topology τ_V on $\mathcal{D}$. Let us recall that the *Vietoris topology* τ_V on $\mathcal{F}(X)$ is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \cap V \neq \emptyset,\ V \in \mathcal{V}, \text{ and } S \subset \bigcup \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of X.

In what follows, we use $\mathcal{S}e\ell(X)$ to denote the set of all continuous selections for $\mathcal{F}(X)$, and $\dim(X)$ to denote the *covering dimension* of X. Also, we denote by $\operatorname{ind}(X)$ the *small inductive dimension* of X. Finally, we shall say that X is *zero-dimensional* if it has a base of clopen sets (i.e., if $\operatorname{ind}(X) = 0$), and that X is *strongly zero-dimensional* if $\dim(X) = 0$. Note that every strongly zero-dimensional space is zero-dimensional but the converse fails ([12], [13], see also [11]).

The dimension-type of restrictions play an important role in the selection theory for hyperspaces. For instance, $\mathcal{S}e\ell(X) \neq \emptyset$ for every strongly zero-dimensional completely metrizable space X (see [3], [5]). On the other hand, we have the following two results in the opposite direction.

Theorem 1.1 ([8]). *If X is a compact Hausdorff space with $\mathcal{S}e\ell(X) \neq \emptyset$, then it is a linear ordered topological space. In particular,* $\dim(X) \leq 1$.

Theorem 1.2 ([10]). *If X is a compact Hausdorff space with $\mathcal{S}e\ell(X) \neq \emptyset$, then it has finitely many connected components if and only if $\mathcal{S}e\ell(X)$ is finite. In particular,* $\dim(X) = 1$ *provided X is infinite and $\mathcal{S}e\ell(X)$ is finite.*

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In the present paper we are interested in relations between the set $\mathcal{S}e\ell(X)$ and zero-dimensionality of X. As Theorems 1.1 and 1.2 suggest, we may expect "sufficiently many" continuous selections for $\mathcal{F}(X)$ provided $\dim(X) = 0$ and $\mathcal{S}e\ell(X) \neq \emptyset$. In fact, this is the first result of the paper. The following theorem will be proved in the next section.

Theorem 1.3. *If X is a zero-dimensional Hausdorff space such that $\mathcal{S}e\ell(X) \neq \emptyset$, then the set $\{f(X) : f \in \mathcal{S}e\ell(X)\}$ is dense in X.*

The proof of Theorem 1.3 does not involve such complicated arguments. However, in general, the conclusion "$\{f(X) : f \in \mathcal{S}e\ell(X)\}$ is dense in X" cannot be strengthened to "$\{f(X) : f \in \mathcal{S}e\ell(X)\} = X$" (see Example 2.1). On the other hand, this becomes possible provided X is a first countable space which allows us to obtain also the converse.

Theorem 1.4. *Let X be a first countable Hausdorff space such that $\mathcal{S}e\ell(X) \neq \emptyset$. Then it is zero-dimensional if and only if for every point $x \in X$ there exists $f_x \in \mathcal{S}e\ell(X)$ such that $f_x^{-1}(x) = \{S \in \mathcal{F}(X) : x \in S\}$.*

In view of Theorem 1.4, it would be interesting to know if the converse of Theorem 1.3 holds as well. Here we have only the following partial result.

Theorem 1.5. *If X is a Hausdorff space such that $\{f(X) : f \in \mathcal{S}e\ell(X)\}$ is dense in X, then it is totally disconnected.*

Since every totally disconnected locally compact space is zero-dimensional, by Theorems 1.3 and 1.5, we get the following consequence.

Corollary 1.6. *Let X be a locally compact Hausdorff space such that $\mathcal{S}e\ell(X) \neq \emptyset$. Then $\operatorname{ind}(X) = 0$ if and only if $\{f(X) : f \in \mathcal{S}e\ell(X)\}$ is dense in X.*

A word should be said about the proofs of Theorems 1.4 and 1.5. A preparation for that is done in Sections 3 and 4. A proof of Theorem 1.5 is obtained in Section 5. Since the proof of Theorem 1.4 involves that of Theorem 1.5, it will be finally accomplished in Section 6 of the paper.

2. Selections and clopen sets

Throughout this section, and in the sequel, X is always a Hausdorff space. We first prove Theorem 1.3. In fact, this theorem is a consequence of the following lemma.

Lemma 2.1. *Let X be a space such $\mathcal{S}e\ell(X) \neq \emptyset$, and let G be a non-empty clopen subset of X. Then there exists a selection $g \in \mathcal{S}e\ell(X)$ with $g(X) \in G$.*

Proof. Note that the sets

$$\mathcal{G}_0 = \{S \in \mathcal{F}(X) : S \cap G = \emptyset\} \quad \text{and} \quad \mathcal{G}_1 = \{S \in \mathcal{F}(X) : S \cap G \neq \emptyset\}$$

constitute a disjoint and τ_V-open cover of $\mathcal{F}(X)$. Define another subset $\mathcal{G}_2$ of $\mathcal{F}(X)$ by $\mathcal{G}_2 = \{S \in \mathcal{F}(X) : S \subset G\}$. Take a selection $f \in \mathcal{S}e\ell(X)$. Note that each $f_i = f|\mathcal{G}_i$, $i = 0, 1, 2$, is a continuous selection for $\mathcal{G}_i$. We now consider the map $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ defined by $\varphi(S) = S \cap G$ for every $S \in \mathcal{G}_1$. This map is continuous with respect to the relative Vietoris topologies on $\mathcal{G}_1$ and $\mathcal{G}_2$. Finally, define a map $g : \mathcal{F}(X) \to X$ by $g|\mathcal{G}_0 = f_0$ and $g|\mathcal{G}_1 = f_2 \circ \varphi$. Clearly, g is a selection for $\mathcal{F}(X)$. That $g \in \mathcal{S}e\ell(X)$ follows from the fact that φ is continuous and $\mathcal{G}_0$ and $\mathcal{G}_1$

are τ_V-open in $\mathcal{F}(X)$. This g is the required one because $X \in \mathcal{G}_1$ and, therefore, $g(X) = f_2(\varphi(X)) = f_2(X \cap G) = f(G) \in G$. □

Example 2.2. There exists a space X_p with only one non-isolated point $p \in X_p$ such that $\mathcal{S}e\ell(X_p) \neq \emptyset$ and $f(X_p) \neq p$ for every $f \in \mathcal{S}e\ell(X_p)$.

Proof. Let $p = \omega_1$ be the first uncountable ordinal, and let $X_p = \{\alpha + 1 : \alpha < \omega_1\} \cup \{p\}$. We endow X_p with the relative topology as a subspace of the ordinal space $\omega_1 + 1$. Then, X_p is a space with only one non-isolated point p such that every countable subset of $X_p \backslash \{p\}$ is closed in X_p. Hence, by a result of [1], $f(X_p) \neq p$ for every $f \in \mathcal{S}e\ell(X_p)$. On the other hand, X_p is a linear ordered space and $f(S) = \min S$, $S \in \mathcal{F}(X_p)$, defines a continuous selection for $\mathcal{F}(X_p)$. □

In contrast to Example 2.2, we have the following lemma demonstrating the relationship of the problem with the convergent structure of X_p in p.

In what follows, we use ω to denote the first infinite ordinal. As a space, every ordinal has the usual order topology.

Lemma 2.3. *Let X be a space, $p \in X$, and let Y_q be the quotient space on the disjoint union $X \sqcup (\omega + 1)$ obtained by identifying the points p and ω to a single point $q \in Y_q$. If $\mathcal{S}e\ell(Y_q) \neq \emptyset$, then there exists $g \in \mathcal{S}e\ell(Y_q)$ with $g(Y_q) = q$.*

Proof. Let

$$\mathcal{V}_0 = \{F \in \mathcal{F}(Y_q) : 0 \notin F\} \quad \text{and} \quad \mathcal{V}_1 = \{F \in \mathcal{F}(Y_q) : 0 \in F\}.$$

Since $0 \in Y_q$ is an isolated point, this defines a disjoint τ_V-open cover $\{\mathcal{V}_0, \mathcal{V}_1\}$ of $\mathcal{F}(Y_q)$. Consider the sets

$$\mathcal{V}_1^0 = \{F \in \mathcal{V}_1 : \omega \backslash F \neq \emptyset\} \quad \text{and} \quad \mathcal{V}_1^1 = \{F \in \mathcal{V}_1 : \omega \subset F\}.$$

Next, for every $F \in \mathcal{V}_1^0$, define $n(F) = \min\{n \in \omega \cap F : n + 1 \notin F\}$. Now, take $f \in \mathcal{S}e\ell(Y_q)$ and then define another selection $g : \mathcal{F}(Y_q) \to Y_q$ for $\mathcal{F}(Y_q)$ by $g|\mathcal{V}_0 = f|\mathcal{V}_0$ while $g(F) = n(F)$ if $F \in \mathcal{V}_1^0$ and $g(F) = q$ otherwise. Since $Y_q \in \mathcal{V}_1^1$, we have $g(Y_q) = q$. Thus to finish the proof it only remains to show that g is continuous. In fact, it suffices to show that $g|\mathcal{V}_1$ is continuous because $g|\mathcal{V}_0 = f|\mathcal{V}_0$. Take an $F \in \mathcal{V}_1$. We distinguish the following two cases. If $F \in \mathcal{V}_1^0$, then $g(F) = n(F) < \omega$ and $\{n < \omega : n \leq n(F)\} \subset F$. Set $\mathcal{U} = \{\{n\} : n \leq n(F)\} \cup \{Y_q \backslash \{n(F) + 1\}\}$. Then $\langle \mathcal{U} \rangle \subset \mathcal{V}_1^0$ is a τ_V-neighbourhood of F such that $n(S) = n(F)$ for every $S \in \langle \mathcal{U} \rangle$. Hence, $g(\langle \mathcal{U} \rangle) = \{g(F)\}$. Suppose now that $F \in \mathcal{V}_1^1$, and let V be a neighbourhood of $q = g(F)$. Take an $m < \omega$ such that $\{n < \omega : n \geq m\} \subset V$. Then, let $\mathcal{U} = \{\{n\} : n \leq m\} \cup \{Y_q\}$. In this way, we get a τ_V-neighbourhood $\langle \mathcal{U} \rangle$ of F such that $g(\langle \mathcal{U} \rangle) \subset V$. Indeed, take an $S \in \langle \mathcal{U} \rangle$. In case $\omega \subset S$, by definition, we have $g(S) = q \in V$. Otherwise, $\{n \in \omega \cap S : n \leq m\} \subset S$ implies that $n(S) \geq m$. Therefore, $g(S) = n(S) \in V$. □

3. Selections and order-like relations

In this section we collect some known facts we need for the proof of Theorem 1.5. Suppose that X is a space with $\mathcal{S}e\ell(X) \neq \emptyset$. Following [7], to every selection $f \in \mathcal{S}e\ell(X)$ we associate an order-like relation $\preceq_f$ on X defined for $x, y \in X$ by

$$x \preceq_f y \qquad \text{if and only if} \qquad f(\{x, y\}) = x.$$

In what follows, we shall refer to "$\preceq_f$" as an *f-order* on X. Also, let us agree to write $x \prec_f y$ provided $x \preceq_f y$ and $x \neq y$.

Finally, for every $f \in \mathcal{S}e\ell(X)$ and $x \in X$, we consider the following special subsets of X:

$$(-\infty, x]_f = \{z \in X : z \preceq_f x\} \quad \text{and} \quad [x, +\infty)_f = \{z \in X : x \preceq_f z\},$$

$$(-\infty, x)_f = \{z \in X : z \prec_f x\} \quad \text{and} \quad (x, +\infty)_f = \{z \in X : x \prec_f z\}.$$

The following observation is an immediate consequence of the continuity of f (see [7, Lemma 7.2]) and we left the corresponding arguments to the interested reader.

Lemma 3.1. *Let X be a space, and let $f \in \mathcal{S}e\ell(X)$. Then, for every $x \in X$, the sets $(-\infty, x)_f$ and $(x, +\infty)_f$ are open in X. In particular, $(-\infty, x]_f$ and $[x, +\infty)_f$ are closed in X.*

It should be mentioned that, in contrast to usual linear orders on X, the f-orders are not engaged to be *transitive*. However, in case of connected spaces, this is so. The following result is a partial case of [7, Lemmas 7.2 and 7.3] (see also [9, Lemma 10]).

Lemma 3.2. *Let X be a connected space such that $\mathcal{S}e\ell(X) \neq \emptyset$. Then, $|\mathcal{S}e\ell(X)| \leq 2$ and, for every $f \in \mathcal{S}e\ell(X)$, the following hold.*

1. *The f-order on X is transitive.*
2. *$f(X) = \min_{\preceq_f} X$.*
3. *$g(X) = \max_{\preceq_f} X$ provided $g \in \mathcal{S}e\ell(X) \setminus \{f\}$.*

In case of connected subsets of X we also have the following property of f-orders.

Lemma 3.3. *Let X be a space, $f \in \mathcal{S}e\ell(X)$, and let $A \subset X$ be connected. Also, let $x, y \in A$ be such that $x \preceq_f y$. Then,*

$$(x, y)_f = \{z \in X : x \prec_f z \prec_f y\} \subset A.$$

Proof. Suppose that there exists $z \in (x, y)_f \setminus A$. Then, according to Lemma 3.1, $U = (-\infty, z]_f \cap A$ is a clopen (in A) subset of A because $U = (-\infty, z)_f \cap A$. Note that $x \in U$ while $y \in A \setminus U$. Since A is connected, this is impossible. □

4. Selections and components

Let X be a space. For every $x \in X$, we shall use $\mathcal{C}[x]$ to denote the *component* of the point x and $\mathcal{C}^*[x]$ the corresponding *quasi-component*. Let us recall that

$$\mathcal{C}[x] = \bigcup \{C \subset X : x \in C \text{ and } C \text{ is connected}\}$$

and, respectively,

$$\mathcal{C}^*[x] = \bigcap \{C \subset X : x \in C \text{ and } C \text{ is clopen}\}.$$

In this section we establish the following result which may have some independent interest.

Theorem 4.1. *Let X be a space with $\mathcal{S}e\ell(X) \neq \emptyset$. Then, $\mathcal{C}^*[x] = \mathcal{C}[x]$ for every point $x \in X$.*

To prepare for the proof of Theorem 4.1, we need the following lemma.

Lemma 4.2. *Let X be such that $\mathcal{S}e\ell(X) \neq \emptyset$, and let $f \in \mathcal{S}e\ell(X)$. Then for every $x \in X$ and $y, z \in \mathcal{C}^*[x]$, with $y \preceq_f z$, we have*

$$[y, z]_f = \{t \in X : y \preceq_f t \preceq_f z\} \subset \mathcal{C}^*[x].$$

Proof. Suppose that there are points $y, z \in \mathcal{C}^*[x]$ and $t \in X \backslash \mathcal{C}^*[x]$ such that $y \prec_f t \prec_f z$. Since $t \notin \mathcal{C}^*[x]$, there exists a clopen subset $V \subset X$ such that $\mathcal{C}^*[x] \subset V$ and $t \notin V$. Then, by Lemma 3.1, $U = (-\infty, t]_f \cap V$ defines a clopen (in X) neighbourhood of y because $U = (-\infty, t)_f \cap V$. Note that $z \notin U$. However, this is impossible because $z \in \mathcal{C}^*[x] = \mathcal{C}^*[y] \subset U$. A contradiction. ☐

Proof of Theorem 4.1. Let $x \in X$. It suffices to show that $\mathcal{C}^*[x]$ is connected. Towards this end, note that, by Lemma 4.2,

$$\mathcal{C}^*[x] = \bigcup \{[y, z]_f : y, z \in \mathcal{C}^*[x],\ y \preceq_f z, \text{ and } y \preceq_f x \preceq_f z\}.$$

Hence, it will be sufficient to show that for every $y, z \in \mathcal{C}^*[x]$, with $y \preceq_f z$, the set $[y, z]_f$ is connected. Suppose to the contrary that $[y, z]_f$ is not connected for some points $y, z \in \mathcal{C}^*[x]$ with $y \preceq_f z$. Then, there exists a clopen (in $[y, z]_f$) neighbourhood $W \subset [y, z]_f$ of z such that $[y, z]_f \backslash W \neq \emptyset$. Take a point $t \in [y, z]_f \backslash W$ and then set $T = W \cap [t, z]_f$. Thus, we get a clopen (in $[t, z]_f$) neighbourhood T of z such that $t \notin T$. Then, the set $G = T \cup [z, +\infty)_f$ is clopen in X. Indeed, according to Lemma 3.1, G is closed in X as a union of two closed subsets. To show that it is also open in X, note that there exists an open subset $E \subset (t, +\infty)_f$ such that $E \cap [t, z]_f = T$ because $t \notin T \subset [t, z]_f \subset [t, +\infty)_f$. Hence, by Lemma 3.1, the set G is open in X because $G = E \cup (z, +\infty)_f$. Thus, G is clopen in X. This however is impossible because $t \notin G$ and $z \in G$, while $t, z \in \mathcal{C}^*[x]$. A contradiction. ☐

5. Proof of Theorem 1.5

Let X be such that the set $D = \{f(X) : f \in \mathcal{S}e\ell(X)\}$ is dense in X. Suppose if possible that X is not totally disconnected. Then, by Theorem 4.1, X must contain an infinite closed connected set A. Let $B = \{g(A) : g \in \mathcal{S}e\ell(A)\}$, and let $f \in \mathcal{S}e\ell(X)$. Note that, by Lemmas 3.2 and 3.3, there exists $a = \min_{\preceq_f} A$. According to the same lemmas, we consider the subset $H = \bigcup\{(a, x)_f : x \in A\}$ of $A \backslash B$. Note that it is non-empty because A is infinite while, by Lemma 3.1, it is open in X. Then, by hypothesis, there exists a selection $h \in \mathcal{S}e\ell(X)$ with $h(X) \in H$. Since h is continuous, there now exists a finite open cover $\mathcal{U}$ of X such that $h(\langle \mathcal{U} \rangle) \subset H$. Take a finite $F \subset X \backslash A$ so that $Z = F \cup A \in \langle \mathcal{U} \rangle$. Then, we have that $h(Z) \in H \subset A$. Set $k = h | \mathcal{F}(Z)$ and $\mathcal{A} = \{S \cup F : S \in \mathcal{F}(A)\}$. Note that $\psi : \mathcal{F}(A) \to \mathcal{A}$, defined by $\psi(S) = S \cup F$, $S \in \mathcal{F}(A)$, becomes a continuous onto map. Since $\mathcal{F}(A)$ is connected (because so is A; see [7, Theorem 4.10]), this implies that $\mathcal{A}$ is also connected. On the other hand, $k^{-1}(A)$ is a τ_V-clopen subset of $\mathcal{F}(Z)$ because A is clopen in Z. Also, $A \cup F = Z \in k^{-1}(A)$ which finally implies that $\mathcal{A} \cap k^{-1}(A) \neq \emptyset$. Thus, $\mathcal{A} \subset k^{-1}(A)$. We may now define a continuous selection $g : \mathcal{F}(A) \to A$ by letting $g = k \circ \psi$. However, this selection has the property that $g(A) = k(\psi(A)) = k(A \cup F) = h(A \cup F) \in H \subset A \backslash B$. A contradiction.

6. Proof of Theorem 1.4

Let X be as in Theorem 1.4, and let $\operatorname{ind}(X) = 0$. Take an $f \in \mathcal{S}e\ell(X)$, a point $x \in X$, and a decreasing clopen base $\mathcal{U} = \{U_n : n < \omega\}$ of x in X such that $U_0 = X$. For reasons of convenience, let $U_\omega = \{x\}$. Now, for every $n \leq \omega$ define a subset $\mathcal{F}_n = \{S \in \mathcal{F}(X) : S \cap U_n \neq \emptyset\}$. Note that $\mathcal{F}_\omega = \bigcap\{\mathcal{F}_n : n < \omega\}$ because $\mathcal{U}$ is a base at x, while $\mathcal{F}_0 = \mathcal{F}(X)$ because $U_0 = X$. Then, define a map $\theta : \mathcal{F}(X) \to \omega + 1$ by setting $\theta(S) = \max\{n \leq \omega : S \in \mathcal{F}_n\}$ for every $S \in \mathcal{F}(X)$. Also, for every $n \leq \omega$, define a map $\varphi_n : \mathcal{F}_n \to \mathcal{F}(X)$ by $\varphi_n(S) = S \cap U_n$, $S \in \mathcal{F}_n$. Finally, define a map $g : \mathcal{F}(X) \to X$ by $g(S) = f(\varphi_{\theta(S)}(S))$ for every $S \in \mathcal{F}(X)$. Thus, we get a selection g for $\mathcal{F}(X)$ such that $g^{-1}(x) = \mathcal{F}_\omega$. It only remains to show that g is continuous. Clearly, g is continuous at the singleton $\{x\}$. So, take an $F \in \mathcal{F}(X)$ with $F \neq \{x\}$. We distinguish the following two cases. If $g(F) = x$, then for every $n < \omega$ let $\mathcal{V}_n = \{U_n, X\backslash U_n\}$. In this case, $g(\langle\mathcal{V}_n\rangle) \subset U_n$. Indeed, let $S \in \langle\mathcal{V}_n\rangle$. Then, $S \cap U_n \neq \emptyset$ implies $\theta(S) \geq n$. Hence, $g(S) = f(\varphi_{\theta(S)}(S)) \in U_{\theta(S)} \subset U_n$ because $\mathcal{U}$ is decreasing. Therefore, g is continuous at F because $\mathcal{U}$ is a local base at $g(F)$ while the set $\{n < \omega : F \notin \langle\mathcal{V}_n\rangle\}$ is finite. Suppose now that $g(F) \neq x$. The definition of g implies that $m = \theta(F) < \omega$. Then, set $\mathcal{W} = \{U_m\backslash U_{m+1}, X\backslash U_{m+1}\}$. We have that $F \in \langle\mathcal{W}\rangle$ because $F \cap U_{m+1} = \emptyset$. On the other hand, $S \in \langle\mathcal{W}\rangle$ implies that $\theta(S) = m$ because $\mathcal{U}$ is decreasing, $S \cap U_m \neq \emptyset$ and $S \cap U_{m+1} = \emptyset$. This finally implies that g is continuous at F because $\langle\mathcal{W}\rangle \subset \mathcal{F}_m$ and $\varphi_m|\langle\mathcal{W}\rangle$ is continuous.

Suppose now that, for every $x \in X$, there exists $f_x \in \mathcal{S}e\ell(X)$ such that $f_x^{-1}(x) = \{S \in \mathcal{F}(X) : x \in S\}$. Next, take a point $x \in X$ and then set $g = f_x$. Also, let V be a neighbourhood of x such that $F = X\backslash V$ is non-empty, i.e. $F \in \mathcal{F}(X)$. Then, by hypothesis, $g(F \cup \{x\}) = x$ and $g(F) \neq x$. Hence, by Theorem 1.5, there exists a clopen set U such that $g(F) \in U$ and $x \notin U$. In this way, we get a τ_V-clopen neighbourhood $g^{-1}(U)$ of F in $\mathcal{F}(X)$. Let $\mathcal{M} \subset g^{-1}(U)$ be a chain which is maximal with respect to the usual set-theoretical inclusion and $F \in \mathcal{M}$. Then, there exists $M = \max \mathcal{M}$ because $g^{-1}(U)$ is τ_V-closed (see [2], [4], [6]). Indeed, it suffices to show that $M = \overline{\bigcup \mathcal{M}} \in g^{-1}(U)$. Take a basic τ_V-neighbourhood $\langle\mathcal{W}\rangle$ of M. Then, for every $W \in \mathcal{W}$ there exists $M_W \in \mathcal{M}$ with $M_W \cap W \neq \emptyset$ because $(\bigcup \mathcal{M}) \cap W \neq \emptyset$. Then, $M_{\mathcal{W}} = \bigcup\{M_W : W \in \mathcal{W}\} \in \langle\mathcal{W}\rangle \cap \mathcal{M}$ because $\mathcal{M}$ is a chain. Hence, in particular, $\langle\mathcal{W}\rangle \cap g^{-1}(U) \neq \emptyset$ which finally implies that $M \in g^{-1}(U)$ because $g^{-1}(U)$ is τ_V-closed. Having established this, let us also observe that M is open because $g^{-1}(U)$ is τ_V-open. Namely, $M \in \langle\mathcal{U}\rangle \subset g^{-1}(U)$ for some finite family $\mathcal{U}$ of open subsets of X. Then, $M = \bigcup \mathcal{U}$ because $\mathcal{M}$ is maximal with respect to the inclusion. Thus, M is a clopen subset of X which contains F because $F \in \mathcal{M}$. Finally note that $g(M) \in U$ which implies $x \notin M$. Hence, $G = X\backslash M$ is a clopen neighbourhood of x with $G \subset V$. This completes the proof of Theorem 1.4.

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