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ON AN ASYMPTOTIC BEHAVIOR OF ELEMENTS OF ORDER p IN IRREDUCIBLE REPRESENTATIONS OF THE CLASSICAL ALGEBRAIC GROUPS WITH LARGE ENOUGH HIGHEST WEIGHTS

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Abstract. The behavior of the images of a fixed element of order p in irreducible representations of a classical algebraic group in characteristic p with highest weights large enough with respect to p and this element is investigated. More precisely, let G be a classical algebraic group of rank r over an algebraically closed field K of characteristic $p > 2$. Assume that an element $x \in G$ of order p is conjugate to that of an algebraic group of the same type and rank $m < r$ naturally embedded into G. Next, an integer function σ_x on the set of dominant weights of G and a constant c_x that depend only upon x, and a polynomial d of degree one are defined. It is proved that the image of x in the irreducible representation of G with highest weight ω contains more than $d(r-m)$ Jordan blocks of size p if m and $r-m$ are not too small and $\sigma_x(\omega) \geq p-1+c_x$.

Asymptotic lower estimates for the number of Jordan blocks of size p in the images of a fixed element of order p in irreducible representations of a classical algebraic group in characteristic p with highest weights large enough with respect to p and this element are obtained. More precisely, let G be a classical algebraic group of rank r over an algebraically closed field K of characteristic $p > 2$. Assume that an element $x \in G$ of order p is conjugate to that of an algebraic group G_m of the same type and rank $m < r$ naturally embedded into G. Set

$$d(r-m) = \begin{cases} r-m & \text{for } G = A_r(K), \\ 2r-2m & \text{for other types.} \end{cases}$$

Let Δ_x be the labelled Dynkin diagram of the conjugacy class containing x in the sense of Bala and Carter [1] and let c_x be the sum of the labels at Δ_x for $G \neq A_r(K)$ and the half of this sum for $G = A_r(K)$. For brevity, throughout the article we refer to Δ_x as the labelled Dynkin diagram of x. Next, an integer function σ_x on the set of dominant weights of G that depends only upon Δ_x is defined. For p-restricted weights σ_x coincides with the canonical homomorphism determined by

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Δ_x. It is proved that the image of x in the irreducible representation of G with highest weight ω contains more than $d(r-m)$ Jordan blocks of size p if m and $r-m$ are not too small and $\sigma_x(\omega) \geq p-1+c_x$.

We need some more notation to formulate the main results. Let ω_i and α_i be the fundamental weights and the simple roots of G (with respect to a fixed maximal torus T) labelled as in [2]. Denote by δ_i, $1 \leq i \leq r$, the label on Δ_x corresponding to its ith node. We have $0 \leq \delta_i \leq 2$. In what follows $\mathbb{Z}$ is the set of integers, $\mathbf{X} = \mathbf{X}(G)$ is the set of weights of G, $\mathbf{X}^+ \subset \mathbf{X}$ is the set of dominant weights, $\mathrm{Irr} = \mathrm{Irr}(G)$ is the set of irreducible rational representations of G (considered up to the equivalence) and $\omega(\varphi)$ is the highest weight of a representation φ. There exists a uniquely determined homomorphism $\tau_x : \mathbf{X} \to \mathbb{Z}$ such that $\tau_x(\alpha_i) = \delta_i$. The weight $\mu \in \mathbf{X}^+$ is called p-restricted if $\mu = \sum_{i=1}^r a_i\omega_i$ with all $a_i < p$. Each weight $\omega \in \mathbf{X}^+$ can be represented in the form $\sum_{j=0}^s p^j\omega^j$ where ω^j are p-restricted. Set $\sigma_x(\omega) = \tau_x(\sum_{j=0}^s \omega^j)$. Now we can state our main result.

Theorem 1. *Let $\varphi \in \mathrm{Irr}$ and $\sigma_x(\omega(\varphi)) \geq p-1+c_x$. Assume that $m > 1$ for $G = B_r(K)$ or $D_r(K)$, $r-m > 1$ for $G = A_r(K)$, and > 3 for $G = B_r(K)$ or $D_r(K)$. Then the element $\varphi(x)$ has more than $d(r-m)$ Jordan blocks of size p.*

Proposition 2 below shows that one cannot weaken the inequality for $\sigma_x(\omega(\varphi))$ in Theorem 1 and that the estimates obtained are asymptotically exact.

Proposition 2. *Let $\varphi \in \mathrm{Irr}$ and $\omega = \omega(\varphi) = a\omega_1$ with $a < p$. Assume that m and r are such as in Theorem 1 and x is a regular unipotent element in G_m. Set $c = m$ for $G = A_r(K)$, $2m$ for $G = B_r(K)$, $2m-1$ for $G = C_r(K)$, and $2m-2$ for $G = D_r(K)$. Suppose that $p > c$. Then $|x| = p$, $\sigma_x(\omega) = ac$ and $c_x = c$. There exist constants $N_G(a,m,p)$ and $Q_G(a,m,p)$ that depend upon the type of G, a, m, and p and do not depend upon r such that $\varphi(x)$ contains at most $N_G(a,m,p)$ Jordan blocks of size p if $p < ac < p+c-1$ and at most $d(r-m)+Q_G(a,m,p)$ such blocks if $ac = p+c-1$.*

Put $l = [(m+2)/2]$ for $G = A_r(K)$ and $l = m$ otherwise. By Lemma 3 below, $c_x = \sum_{i=1}^l \delta_i$ and $c_x \leq p-1$. Hence $c_x \leq 2l$.

For $\varphi \in \mathrm{Irr}$ define the weight $\bar{\omega}(\varphi)$ as follows: write down the p-adic expansion for the weight $\omega = \omega(\varphi)$ considered before the statement of Theorem 1 and set $\bar{\omega}(\varphi) = \sum_{j=0}^s \omega^j$. So $\sigma_x(\omega(\varphi)) = \tau_x(\bar{\omega}(\varphi))$.

The study of an asymptotic behavior of elements of order p in representations of the classical groups in characteristic p was begun by the author in [12] where a notion of a p-large representation was introduced. In our present notation a representation $\varphi \in \mathrm{Irr}$ is p-large if and only if $\sigma_x(\omega(\varphi)) \geq p$ for a long root element $x \in G$ (an equivalent definition from [12]: the value of $\bar{\omega}(\varphi)$ on the maximal root is $\geq p$). The common goal of [12] and the present article is to investigate the behavior of elements of order p in irreducible representations in characteristic p for a fixed p and $r \to \infty$ and to discover asymptotic regularities which are specific for prime characteristics but do not (or almost do not) depend upon p. Such properties can find applications in recognizing representations and linear groups. According to [12, Theorem 1.1], the image of any element of order p in a p-large representation has at least $f(r)$ Jordan blocks of size p where

$$f(r) = \begin{cases} 2r-2 & \text{for } G = A_r(K), \\ 6r-7 & \text{for } G = B_r(K),\ p=3, \\ 8r-10 & \text{for } G = B_r(K),\ p>3, \\ 4r-4 & \text{for } G = C_r(K), \\ 4r-8 & \text{for } G = D_r(K),\ p=2, \\ 6r-10 & \text{for } G = D_r(K),\ p=3, \\ 8r-16 & \text{for } G = D_r(K),\ p>3. \end{cases}$$

In [12, Theorem 1.3] for $p > 2$ and all types of the classical groups examples of representations $\varphi \in \mathrm{Irr}$ such that $\varphi(x)$ has only one Jordan block of size p for a long root element x and $\sigma_x(\omega(\varphi)) = p-1$ are given. It is also shown [12, Theorem 1.4] that the estimates in [12, Theorem 1.1] are asymptotically exact for the groups of type A, B, and D provided $p > 2$ in the last two cases.

In what follows $\mathbb{C}$ is the complex field, $G_{\mathbb{C}}$ is the simple algebraic group over $\mathbb{C}$ of the same type and rank as G, and $\mathrm{Irr}_{\mathbb{C}}$ is the set of irreducible rational representations of $G_{\mathbb{C}}$ (considered up to the equivalence). For $\rho \in \mathrm{Irr}$ or $\mathrm{Irr}_{\mathbb{C}}$ and a unipotent element $z \in G$ or $G_{\mathbb{C}}$ denote by $k_\rho(z)$ the degree of the minimal polynomial of $\rho(z)$. It is well known that $k_\rho(z)$ is equal to the maximal size of a Jordan block of $\rho(z)$. If $\varphi \in \mathrm{Irr}$, then $\varphi_{\mathbb{C}}$ is the irreducible representation of $G_{\mathbb{C}}$ with highest weight $\bar{\omega}(\varphi)$. For unipotent $x \in G$ put $k_{\varphi_{\mathbb{C}}}(x) = k_{\varphi_{\mathbb{C}}}(y)$ where $y \in G_{\mathbb{C}}$ is a unipotent element with the labelled Dynkin diagram Δ_x (this is correctly determined). Now let $|x| = p$. By the results [11, Theorem 1.1, Lemma 2.5, and Proposition 2.12], $k_{\varphi_{\mathbb{C}}}(x) = \sigma_x(\omega(\varphi)) + 1$ and $k_\varphi(x) = \min\{p, k_{\varphi_{\mathbb{C}}}(x)\}$. Hence if z is a long root element, then $k_\varphi(z) = k_{\varphi_{\mathbb{C}}}(z)$ if and only if φ is not p-large. The results of [11] imply that for not very small p and r there exists a wide class of representations $\varphi \in \mathrm{Irr}$ such that $k_\varphi(z) = k_{\varphi_{\mathbb{C}}}(z) < p$ for a long root element $z \in G$, but $k_\varphi(x) = p < k_{\varphi_{\mathbb{C}}}(x)$ for many other elements $x \in G$ of order p. In this connection in [13, Section 2] a notion of a p-large representation for a given element x of order p was introduced. A representation $\varphi \in \mathrm{Irr}$ was called p-large for x if $\sigma_x(\omega(\varphi)) \geq p$. It has been conjectured ([13, Conjecture 1] that if $x \in G_m$, r is large enough with respect to m and φ is p-large for x, then $\varphi(x)$ has at least $F(r)$ blocks of size p where F is an increasing function. Our Proposition 2 formally disproves this conjecture, but Theorem 1 actually proves a refined version of it with a stronger assumption on $\sigma_x(\omega(\varphi))$. Thus for arbitrary elements x there is a gap between the class of representations $\varphi \in \mathrm{Irr}$ with $k_\varphi(x) = k_{\varphi_{\mathbb{C}}}(x)$ and that of representations where asymptotic estimates for the number of Jordan blocks of size p in $\varphi(x)$ hold. Perhaps for some classes of elements of order p stronger estimates than those of Theorem 1 are possible, but now it is not clear how to determine such classes.

The case $p = 2$ is not considered here, but in this situation $\varphi \in \mathrm{Irr}$ is 2-large if $\omega(\varphi) \neq 2^j\omega_i$. For 2-large representations the estimates from [12, Theorem 1.1] are available. For remaining representations certain estimates could be obtained as well, but this article does not seem a proper place for this. We plan to handle this question in a subsequent paper which will be devoted to refining some estimates in [12].

The results of this article as well as those of [12] can be easily transferred to irreducible K-representations of finite classical groups in characteristic p.

1. Notation and preliminary comments

Throughout the article for a semisimple algebraic group S the symbols $\mathrm{Irr}(S)$, $\mathbf{X}(S)$, and $\mathbf{X}^+(S)$ mean the same as the similar ones for G introduced earlier; $R(S)$ is the set of roots of S, $\langle S_1, \dots, S_j\rangle$ is the subgroup in S generated by subgroups $S_1, \dots, S_j$; $\mathrm{Irr}_p(S) \subset \mathrm{Irr}(S)$ is the set of p-restricted representations, i.e. irreducible representations with p-restricted highest weights; $\mathbf{X}(\varphi)$ ($\mathbf{X}(M)$) is the set of weights of a representation φ (a module M); $\dim M$ is the dimension of M; $M(\omega)$ is the irreducible S-module with highest weight ω; L is the Lie algebra of G; $R = R(G)$, $R^+ \subset R$ is the set of positive roots; $\mathrm{Irr}_p = \mathrm{Irr}_p(G)$; $\mathcal{X}_\beta \subset G$ and $X_\beta \in L$ are the root subgroup and the root element associated with $\beta \in R$, $\mathcal{X}_{\pm i} = \mathcal{X}_{\pm\alpha_i}$, and $X_{\pm i} = X_{\pm\alpha_i}$. Set $H(\beta_1, \dots, \beta_j) = \langle \mathcal{X}_{\beta_1}, \mathcal{X}_{-\beta_1} \dots, \mathcal{X}_{\beta_j}, \mathcal{X}_{-\beta_j}\rangle$. For $\omega \in \mathbf{X}(S)$ and $\alpha \in R(S)$ denote by $\langle\omega, \alpha\rangle$ the value of the weight ω on the root α. For an S-module M and a unipotent element $x \in S$ define $k_M(x)$ similarly to $k_\varphi(x)$. If $|x| = p$, then $n_\varphi(x)$ is the number of Jordan blocks of size p of the matrix $\varphi(x)$ for a representation φ of S and $n_M(x)$ denotes the same number for a module M affording φ.

An element $x \in G$ of order p can be embedded into a closed connected subgroup Γ of type A_1 whose labelled diagram coincides with Δ_x (see [6, Theorem 4.2]). Set $\mathbf{X}_1 = \mathbf{X}(A_1(K))$ (the simply connected group of this type) and identify $\mathbf{X}_1$ with $\mathbb{Z}$ mapping $a\omega_1 \in \mathbf{X}_1$ into $a \in \mathbb{Z}$. Then $\mathbf{X}(\Gamma)$ can be identified with a subset of $\mathbb{Z}$. The canonical homomorphism τ_x can be obtained as the restriction of weights from a maximal torus $T \subset G$ to a maximal torus $T_1 \subset \Gamma$ such that $T_1 \subset T$. From now on we fix the tori T and T_1, and all weights and roots of G and Γ are considered with respect to T and T_1. Throughout the text ε_i with $1 \le i \le r+1$ for $G = A_r(K)$ and $1 \le i \le r$ otherwise are weights of the standard realization of G labelled as in [3, ch. VIII, §13]. Set $e_i = \tau_x(\varepsilon_i)$. One can choose Γ, T and T_1 such that the restriction to Γ of the natural representation of G is a direct sum of irreducible components with p-restricted highest weights (see comments in [14, Section 3]); $e_i \ge e_j$ for $i < j$; $e_i \ge 0$ if $G = A_r(K)$ and $i \le (r+1)/2$; and $e_i \ge 0$ for all $i \le r$ if $G \ne A_r(K)$. If $H \subset G$ is a semisimple subgroup generated by some root subgroups, then $T_H = T \cap H$ is a maximal torus in H. If $T_1 \subset T_H$, we denote by the same symbol τ_x the homomorphism $\mathbf{X}(H) \to \mathbb{Z}$ determined by restricting weights from T_H to T_1. This causes no confusion. If an element v of some G-module is an eigenvector for T, we denote its weights with respect to T, T_H, and T_1 by $\omega(v)$, $\omega_H(v)$, and $\omega_\Gamma(v)$. In what follows x is conjugate to an element of G_m, $|x| = p$, m and $r - m$ are such as in the assertion of Theorem 1, and $\delta_i = \tau_x(\alpha_i)$, $1 \le i \le r$.

Lemma 3. *Set $l = [(m+2)/2]$ for $G = A_r(K)$ and $l = m$ otherwise. Then $c_x = \sum_{i=1}^{l} \delta_i$ and $c_x \le p - 1$.*

Proof. Put $k = l - 1$ for $G = A_r(K)$, $m = 2t$, and $k = l$ for $G = A_r(K)$, $m = 2t+1$. Our assumptions on e_i, $m - r$, and x imply that $e_i = 0$ for $k < i < r + 2 - k$ if $G = A_r(K)$ and $e_i = 0$ for $i > m$ otherwise; notice that $e_{k+1} = e_{k+2} = 0$ for $G = A_r(K)$. Now it follows from the definition of c_x and the formulae in [3, ch. VIII, §13] that $c_x = \sum_{i=1}^{l} \delta_i = e_1 - e_{l+1} = e_1$. As e_1 is a weight of a p-restricted Γ-module, we have $e_1 < p$. This yields the lemma. □

Proof of Theorem 1. Set $\omega = \omega(\varphi)$ and let $\omega = \sum_{i=1}^{r} a_i\omega_i$. It is clear that $\omega \ne 0$ as $\tau_x(\omega) \ne 0$. Define subgroups H_1 and $H_2 \subset G$ as follows. For $G = A_r(K)$ set

$u = r - t + 2$ if $m = 2t$ and $r - t + 1$ if $m = 2t + 1$, $\beta = \varepsilon_{t+1} - \varepsilon_u$,

$$H_1 = H(\alpha_1, \dots , \alpha_t, \beta, \alpha_u, \dots , \alpha_r), \quad H_2 = H(\alpha_{t+2}, \dots , \alpha_{u-2})$$

(we have $H_1 = H(\alpha_1, \varepsilon_2 - \varepsilon_{r+1})$ for $m = 2$ and $H_1 = H(\varepsilon_1 - \varepsilon_{r+1})$ for $m = 1$). For $G = B_r(K)$, $C_r(K)$, or $D_r(K)$ put $\beta = \varepsilon_m$, $2\varepsilon_m$, or $\varepsilon_{m-1} + \varepsilon_m$, respectively, and

$$H_1 = H(\alpha_1, \dots , \alpha_{m-1}, \beta).$$

Next, set

$$H_2 = H(\alpha_{m+1}, \dots, \alpha_{r-1}, \varepsilon_{r-1} + \varepsilon_r)$$

for $G = B_r(K)$ and

$$H_2 = H(\alpha_{m+1}, \dots, \alpha_r)$$

for $G = C_r(K)$ or $D_r(K)$ (here $H_1 = H(\beta)$ for $G = C_r(K)$ and $m = 1$). One easily observes that the sets of roots in brackets used to define H_1 and H_2 yield bases of the systems $R(H_1)$ and $R(H_2)$, respectively. Denote these bases by $\mathcal{B}_i$. In all cases H_1 is conjugate to G_m in G. We have $H_2 \cong A_{r-m-1}(K)$, $D_{r-m}(K)$, $C_{r-m}(K)$, or $D_{r-m}(K)$ for $G = A_r(K)$, $B_r(K)$, $C_r(K)$, or $D_r(K)$, respectively. It is clear that the subgroups H_1 and H_2 commute. Set $H = H_1H_2$. Let $U_i = \langle \mathcal{X}_\gamma \mid \gamma \in R^+, \quad \mathcal{X}_\gamma \subset H_i \rangle$, $i = 1, 2$, and $U = U_1U_2$. It is not difficult to conclude that U_i is a maximal unipotent subgroup in H_i and U is such a subgroup in H. We can assume that $x \in U_1$, $\Gamma \subset H_1$ and $T_1 \subset T_{H_1}$. We shall write a weight $\mu \in \mathbf{X}(H)$ in the form (μ_1, μ_2) where $\mu_i \in \mathbf{X}(H_i)$ is the restriction of μ to T_{H_i}. Set $M = M(\omega)$.

It is clear that $n_V(x) = \dim(x - 1)^{p-1}V$ for each H-module V. Taking this into account, it is not difficult to conclude the following. If $0 \subset W_1 \subset \dots \subset W_t = V$ is a filtration of V, $F_i = W_i/W_{i-1}$, $1 \le i \le t$, and $n_{F_i}(x) = n_i$, then

$$n_V(x) \ge \sum_{i=1}^{t} n_i. \tag{1}$$

First suppose that $\varphi \in \mathrm{Irr}_p$. Since passing to the dual representation does not influence the Jordan form of $\varphi(x)$, one can assume that $a_i \neq 0$ for some $i \le (r+1)/2$ if $G = A_r(K)$. As for p-large representations the estimates of [12, Theorem 1.1] hold; we also assume that φ is not p-large. Hence $\langle \mu, \alpha \rangle < p$ for all $\mu \in \mathbf{X}(\varphi)$ and long roots α (for all α if $G = A_r(K)$ or $D_r(K)$). By the formulae for the maximal roots of the classical groups in [2, Tables 1-4], this forces that

$$\begin{aligned} a_1 + \dots + a_r < p \quad &\text{for} \quad G = A_r(K) \quad \text{or} \quad C_r(K),\\ a_1 + 2a_2 + \dots + 2a_{r-1} + a_r < p \quad &\text{for} \quad G = B_r(K),\\ a_1 + 2a_2 + \dots + 2a_{r-2} + a_{r-1} + a_r < p \quad &\text{for} \quad G = D_r(K). \end{aligned} \tag{2}$$

Now we proceed to construct two composition factors M_1 and M_2 of the restriction $M|H$ such that $n_{M_1}(x) \ge d(r - m)$ and $n_{M_2}(x) > 0$. This will be done for almost all ω. In exceptional cases we shall find one factor M_1 such that $n_{M_1}(x) > d(r-m)$. By (1), this would yield the assertion of the theorem.

Let $v \in M$ be a nonzero highest weight vector. Put $\mu_i = \omega_{H_i}(v)$. The vector v generates an indecomposable H-module V_1 with highest weight $\mu = (\mu_1, \mu_2)$. Using (2), one can deduce that $\langle \mu_1, \beta \rangle < p$ for all $\beta \in \mathcal{B}_1$. Here for $G = B_r(K)$ we take into account that $m > 1$. Hence μ_1 is p-restricted. Now assume that either $G \neq B_r(K)$, or $a_i \neq 0$ for some $i < r$. For such representations we construct another weight vector $w \in M$ that is fixed by U. Set $l = t + 1$ for $G = A_r(K)$, $m = 2t$;

otherwise take l as in Lemma 3. First suppose that $a_j \neq 0$ for some $j \leq l$ (Case 1). Choose maximal such j and put $w = X_{-l} \dots X_{-(j+1)} X_{-j} v$. Now let $a_j = 0$ for all $j \leq l$ (Case 2). Our assumptions on a_i imply that $a_i \neq 0$ for some $i > l$; furthermore, one can take $i \leq (r+1)/2$ for $G = A_r(K)$ and $i < r$ for $G = B_r(K)$. Choose minimal such i and set $w = X_{-l} \dots X_{-(i-1)} X_{-i} v$ if $G \neq D_r(K)$ or $i < r$ and $w = X_{-l} \dots X_{-(r-3)} X_{-(r-2)} X_{-r} v$ for $G = D_r(K)$ and $i = r$. It follows from [12, Lemma 2.1(iii) and Lemma 2.9] that in all cases $w \neq 0$. Using [10, Lemma 72] and analyzing the roots in $\mathcal{B}_1$ and $\mathcal{B}_2$ and the weight system $\mathbf{X}(\varphi)$, we get that U fixes w in all situations. Here it is essential that the case $G = B_r(K)$ with $\omega = a_r \omega_r$ is excluded. In the latter case we cannot assert that $\mathcal{X}_\beta$ fixes w. Set $\lambda_i = \omega_{H_i}(w)$, $i = 1, 2$. Now it is clear that w generates an indecomposable H-module V_2 with highest weight $\lambda = (\lambda_1, \lambda_2)$. We claim that λ_1 is p-restricted. Write down all the situations where $\langle \lambda_1, \gamma \rangle \neq \langle \mu_1, \gamma \rangle$ for some $\gamma \in \mathcal{B}_1$. We have $\langle \lambda_1, \beta \rangle = \langle \mu_1, \beta \rangle - 1$ in Case 1 if $j = l$ and $G \neq B_r(K)$ or $j = l-1$ and $G = D_r(K)$ and in Case 2 for $G \neq B_r(K)$ and all i; and $\langle \lambda_1, \beta \rangle = \langle \mu_1, \beta \rangle - 2$ for $G = B_r(K)$ both in Case 1 with $j = l$ and in Case 2. In Case 1 we also have $\langle \lambda_1, \alpha_{j-1} \rangle = \langle \mu_1, \alpha_{j-1} \rangle + 1$ if $j > 1$ and $\langle \lambda_1, \alpha_j \rangle = \langle \mu_1, \alpha_j \rangle - 1$ if $j < l$. In Case 2 one gets $\langle \lambda_1, \alpha_{l-1} \rangle = \langle \mu_1, \alpha_{l-1} \rangle + 1$ if $l > 1$. In all other situations we have $\langle \lambda_1, \gamma \rangle = \langle \mu_1, \gamma \rangle$. Now apply (2) to conclude that λ_1 is p-restricted.

Set $M_1 = M(\mu)$, $M_2 = M(\lambda)$, $M_1^j = M(\mu_j)$, and $M_2^j = M(\lambda_j)$, $j = 1, 2$. Obviously, M_i is a composition factor of V_i. It is well known that $M_i = M_i^1 \otimes M_i^2$. It is clear that $\tau_x(\mu_1) = \tau_x(\omega) \geq p$. Since $x \in H_1$, we have $\delta_t = 0$ if $\alpha_t \in \mathcal{B}_2$. So by Lemma 3,

$$\tau_x(\lambda_1) = \tau_x(\omega(w)) \geq \tau_x(\omega) - \sum_{i=1}^{l} \delta_i = \tau_x(\omega) - c_x \geq p - 1.$$

It follows from [11, Theorem 1.1, Lemma 2.5, and Proposition 2.12] that $k_{M_i^1}(x) = p$. Hence $n_{M_i}(x) \geq \dim M_i^2$. One easily observes that M_1^2 and M_2^2 cannot both be trivial H_2-modules. Our assumptions on $r - m$ and [5, Proposition 5.4.13] imply that the dimension of a nontrivial irreducible H_2-module is at least $d(r-m)$. In the exceptional case where $G = B_r(K)$ and $\omega = a_r \omega_r$ we need to evaluate $\dim M_1^2$. First let $a_r \neq 1$. As above, $X_r v \neq 0$. This implies that $\mathbf{X}(M_1^2)$ contains a dominant weight $\mu_2 - \alpha_r$ and $\dim M_1^2$ is greater than the size of the orbit of μ_2 under the action of the Weyl group of H_2. The latter is equal to $2^{r-m-1} \geq d(r-m)$ for our values of $r - m$. By (1), this yields the assertion of the theorem for almost all $\varphi \in \mathrm{Irr}_p$. It remains to consider the case where $G = B_r(K)$ and $\omega = \omega_r$. It is well known that then the restriction $M \downarrow H_1$ is a direct sum of 2^{r-m} H_1-modules $N = M(\omega_m)$. Since $k_m(x) = p$, we get $k_N(x) = p$ and $n_M(x) \geq 2^{r-m} > d(r-m)$.

Now suppose that $\varphi \in \mathrm{Irr} \setminus \mathrm{Irr}_p$. By the Steinberg tensor product theorem [9, Theorem 1.1], φ can be represented in the form $\bigotimes_{j=1}^{s} \varphi_j \operatorname{Fr}^j$ where Fr is the Frobenius morphism of G associated with raising elements of K to the pth power and all $\varphi_j \in \mathrm{Irr}_p$. It is clear that the morphism Fr does not influence the Jordan form of $\varphi(x)$. Hence one can assume that $\varphi = \psi \otimes \theta$ where $\theta = \varphi_j \operatorname{Fr}^j$ for some j and both ψ and θ are nontrivial. Set $a = \sigma_x(\omega(\psi))$, $\nu = \omega(\varphi_j)$, $b = \tau_x(\nu)$ and define by μ the restriction of ν to T_H. Now it follows from the definitions of σ_x and τ_x that $\sigma_x(\omega) = a + b$. By [11, Theorem 1.1, Lemma 2.5 and Proposition 2.12], $k_\psi(x) = \min\{a+1, p\}$ and $k_\theta(x) = \min\{b+1, p\}$. First suppose that a or $b \geq p - 1$. Set $\rho = \psi$ if $a \geq p-1$ and $\rho = \theta$ otherwise and denote by π the remaining representation from the pair (ψ, θ). Then $k_\rho(x) = p$ and [4, ch. VIII, Lemma 2.2] implies that $n_\varphi(x) \geq \dim \pi$. Let $d(r)$ be the value of $d(r-m)$ if one formally sets $m = 0$. Then by [5, Proposition 5.4.13], $\dim \pi \geq d(r) > d(r-m)$ which settles the case under consideration.

Now assume that both a and $b < p-1$. Then $k_\psi(x) = a+1$ and $k_\theta(x) = b+1$. Since $\sigma_x(\omega) \geq p-1+c_x$, we have $b > c_x$. Arguing as for p-restricted φ, we can and shall suppose that $\langle \nu, \alpha_i \rangle \neq 0$ for some $i \leq (r+1)/2$ if $G = A_r(K)$. Put $M' = M(\nu)$ and construct the composition factors M_i, $i = 1, 2$, of the restriction $M'|H$ as for p-restricted M before. Transfer the notation μ_1, λ_1, and M_j^i, $i, j = 1, 2$, to M'. Again we have the exceptional case $G = B_r(K)$ and $\nu = a_r\omega_r$ where we do not construct M_2 and consider M_1 only. Obviously, $\tau_x(\mu_1) = b$. As before, we deduce that $\tau_x(\lambda_1) \geq b - c_x$. By [11, Theorem 1.1, Lemma 2.5, and Proposition 2.12], $k_{M_1}(x) = b+1$ and $k_{M_2}(x) \geq b+1-c_x$. Let n_i be the number of Jordan blocks of the maximal size in the canonical form of x as an element of $\operatorname{End} M_i$, $i = 1, 2$. Looking at the realizations of M_i as tensor products, one easily observes that $n_i \geq \dim M_i^2$. Set $F_1 = M(\omega(\psi)) \otimes M_1$, $F_2 = M(\omega(\psi)) \otimes M_2$ and consider F_i as H-modules in the natural way. In the general case the H-module M has a filtration two of whose quotients are isomorphic to F_1 and F_2, respectively. In the exceptional case F_1 is a quotient of a submodule in M. Observe that $a + k_{M_2}(x) \geq p$. Using [4, ch. VIII, Theorem 2.7] that describes the canonical Jordan form of a tensor product of unipotent blocks, we obtain that $k_{M_i}(x) = p$ and $n_{F_i}(x) \geq \dim M_i^2$. As for p-restricted M, we show that $n_1 \geq 2^{r-m}$ if $G = B_r(K)$ and $\nu = \omega_r$ and conclude that $\dim M_1^2 + \dim M_2^2 > d(r-m)$ in the general case and $\dim M_1^2 > d(r-m)$ in the exceptional cases with $a_r \neq 1$. Now (1) completes the proof. $\square$

Proof of Proposition 2. Let a, x, m, and c be such as in the assertion of the proposition. Assume that $p < ac \leq p + c - 1$. Therefore we have $(a-1)c \leq p-1$. Set $M = M(\omega)$ and denote by M_t the weight subspace of weight $t \in \mathbb{Z}$ in the Γ-module M. It is clear that the Weyl group of Γ interchanges M_t and M_{-t}; hence $\dim M_t = \dim M_{-t}$. Put $e = (a-1)c$, $V_1 = \bigoplus_{t>e} M_t$, $V_2 = \bigoplus_{t>e} M_{-t}$, and $V = M_e$. Set $f = [(m+1)/2]$ for $G = A_r(K)$, $f = m$ for $G = B_r(K)$ or $C_r(K)$, and $f = m-1$ for $G = D_r(K)$. Let $v \in M$ be a nonzero highest weight vector and put $w = X_{-f} \dots X_{-2}X_{-1}v$. By [12, Lemma 2.9], $w \neq 0$. We need a subgroup S which can be defined as follows. Put $I = \{i \mid 1 \leq i \leq r, \quad \delta_i = 0\}$ and $S = \langle \mathcal{X}_i, \mathcal{X}_{-i} \mid i \in I \rangle$. The canonical Jordan forms of x in the standard realizations of G_m and G are well known. We have $|x| = p$ since the dimension of the first realization is at most p due to our assumptions. Taking into account these Jordan forms, one easily obtains the values of δ_i, $1 \leq i \leq r$, and using Lemma 3, deduces the following facts: $I = \{i \mid f+1 \leq i \leq r-f\}$ for $G = A_r(K)$ and $m = 2f$, $I = \{i \mid f+1 \leq i \leq r\}$ for $G = B_r(K)$ and $D_r(K)$, and $S = H_2$ in all other cases where H_2 is the subgroup defined in the proof of Theorem 1; $c_x = \sum_{i=1}^{f} \delta_i = c$, $\tau_x(\omega) = ac$; and $w \in V$. Next, observe that $S \cong A_{r-m}$ for $G = A_r(K)$ and $m = 2f$, $S \cong B_{r-m}(K)$ for $G = B_r(K)$, and $S \cong D_{r-m+1}$ for $G = D_r(K)$. Our construction of the vector w shows that $\mathcal{X}_i$ fixes w if $i \in I$. This forces that w generates an indecomposable S-module M_S with highest weight $\omega_S(w)$. Then one immediately concludes that $M_S \cong M(\omega_1)$. This yields that $\dim M_S = r-m+1 = d(r-m)+1$ for $G = A_r(K)$ and $m = 2f$, $\dim M_S = 2(r-m)+1 = d(r-m)+1$ for $G = B_r(K)$, $\dim M_S = 2(r-m+1) = d(r-m)+2$ for $G = D_r(K)$, and $\dim M_S = d(r-m)$ otherwise. It is clear that $M_S \subset V$. Denote by $\mathbf{X}_f \subset \mathbf{X}(M)$ the subset of weights of the form $\omega - \sum_{i=1}^{f} b_i\alpha_i$ and by M_A the irreducible $A_f(K)$-module with highest weight $a\omega_1$. By Smith's theorem [8], for each $\mu \in \mathbf{X}_f$ the dimension of the weight subspace $M_\mu \subset M$ coincides with that of the weight subspace in M_A whose weight differs from $a\omega_1$ by the same linear combination of the simple roots. Hence $\dim M_\mu$ does not depend upon r. Set $W = \bigoplus_{\mu\in\mathbf{X}_f} M_\mu$. Since M is an irreducible L-module and $p > 2$, observe that M is a linear span of vectors of the form $X_{-i_s}\dots X_{-i_2}X_{-i_1}v$. Now, analyzing the weight structure of M, we conclude that $V_1 \subset W$ and $V = (V \cap W) \oplus M_S$. This implies that $\dim V_1$ $(= \dim V_2)$ and $\dim(V \cap W)$ do not depend upon r.

It follows from [10, Lemma 72] that

$$(x-1)^{p-1}M_t \subset \bigoplus_{i\geq t+2p-2} M_i. \tag{3}$$

Let $M_t \not\subset V_2$. Then $t \geq -e$. Obviously, $e < p-1$ if $ac < p-1+c_x$ and $e = p-1$ for $ac = p-1+c_x$. Thus (3) implies that

$$(x-1)^{p-1}M_t \subset \bigoplus_{t>p-1} M_t \subset V_1$$

in the first case and

$$(x-1)^{p-1}M_t \subset \bigoplus_{t\geq p-1} M_t \subset V_1 \oplus V$$

in the second case. This forces that $n_M(x) \leq \dim V_2 + \dim V_1 = 2\dim V_1$ in the first case and $n_M(x) \leq 2\dim V_1 + \dim(V\cap W) + \dim M_S$ in the second case. We have seen before that $\dim M_S = d(r-m)+u$ with $u = 0$, 1, or 2. Hence one can take $N_G(a,m,p) = 2\dim V_1$ and $Q_G(a,m,p) = 2\dim V_1 + \dim(V\cap W) + u$ to complete the proof. □

Remark 4. For $G = A_r(K)$ or $C_r(K)$ we could give a shorter proof of Proposition 2 using the realization of φ in the ath symmetric power of the standard module (see [7, 1.14 and 8.13]), but we need the proof above for $B_r(K)$ and $D_r(K)$.

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