EQUIVALENCE OF POSITIVE HAUSDORFF MEASURE AND THE OPEN SET CONDITION FOR SELF-CONFORMAL SETS

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Abstract. A compact set \( K \) is self-conformal if it is a finite union of its images by conformal contractions. It is well known that if the conformal contractions satisfy the “open set condition” (OSC), then \( K \) has positive \( s \)-dimensional Hausdorff measure, where \( s \) is the solution of Bowen’s pressure equation. We prove that the OSC, the strong OSC, and positivity of the \( s \)-dimensional Hausdorff measure are equivalent for conformal contractions; this answers a question of R. D. Mauldin. In the self-similar case, when the contractions are linear, this equivalence was proved by Schief (1994), who used a result of Bandt and Graf (1992), but the proofs in these papers do not extend to the nonlinear setting.

1. Introduction

Let \( V \subseteq \mathbb{R}^d \). Recall that a map \( S : V \to V \) is contracting if there exists \( 0 < \gamma(S) < 1 \) such that \( |S(x) - S(y)| \leq \gamma(S) \cdot |x - y| \) for all \( x, y \in V \); if equality holds here for all \( x, y \in V \), then \( S \) is a contracting similitude. Let \( \{S_i\}_{i=1}^m \) be a collection of contracting maps on an open set \( V \subseteq \mathbb{R}^d \) and suppose that for some closed set \( X \subseteq V \) we have \( S_i(X) \subseteq X \) for all \( i \leq m \). By [6], there is a unique non-empty compact set \( K \subseteq X \) such that

\[
K = \bigcup_{i=1}^m S_i K.
\]

If all \( S_i \) are similitudes, then \( K \) satisfying (1.1) is called self-similar.

The contracting maps \( \{S_i\}_{i=1}^m \) of \( V \) are said to satisfy the Open Set Condition (OSC) if there is a non-empty open set \( U \subseteq V \) such that \( S_i U \subseteq U \) for all \( i \), and \( S_i U \cap S_j U = \emptyset \) for \( i \neq j \). The strong Open Set Condition holds if the set \( U \) in the definition of the OSC can be chosen with \( U \cap K \neq \emptyset \), where \( K \) is a compact set satisfying (1.1).

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Next, consider a collection of contracting similitudes \( \{S_i\}_{i=1}^m \) and let \( K \) be the corresponding self-similar set. The \textbf{similarity dimension} for this collection is defined as the unique positive solution \( s \) of the equation \( \sum_{i=1}^m \gamma(S_i)^s = 1 \). It is immediate that the Hausdorff measure \( \mathcal{H}^s(K) \) is finite. Hutchinson [11] proved that if the OSC holds, then \( \mathcal{H}^s(K) \) is positive and hence the Hausdorff dimension of \( K \) equals \( s \).

Bandt and Graf [1] gave a very useful characterization of self-similar sets with positive Hausdorff measure in the similarity dimension. Let \( A^* \) be the set of finite “words” in the alphabet \( A = \{1, \ldots, m\} \) and denote \( S_u = S_{u_1} \circ \cdots \circ S_{u_n} \) for \( u = u_1 \ldots u_n \in A^* \). For \( u \in A^* \) let \( K_u = S_u(K) \). We say that two maps \( S_u \) and \( S_v \) are \( \varepsilon \)-relatively close if

\[
|S_u(x) - S_v(x)| \leq \varepsilon \min\{\text{diam}(K_u), \text{diam}(K_v)\} \quad \text{for all} \quad x \in K.
\]

Bandt and Graf [1] proved that \( \mathcal{H}^s(K) > 0 \) if and only if there exists \( \varepsilon > 0 \) such that for distinct \( u, v \) in \( A^* \), the maps \( S_u \) and \( S_v \) are not \( \varepsilon \)-relatively close. Building on [1], Schief [11] proved that \( \mathcal{H}^0(K) > 0 \) is equivalent to the OSC and also to the strong OSC.

Much of the theory has been extended from self-similar to self-conformal sets (see, e.g., [10, 2]). Let \( V \subset \mathbb{R}^d \) be an open set. A \( C^1 \)-map \( S : V \to \mathbb{R}^d \) is \textbf{conformal} if the differential \( S'(x) : \mathbb{R}^d \to \mathbb{R}^d \) satisfies \( |S'(x)y| = |S'(x)| \cdot |y| \neq 0 \) for all \( x \in V \) and \( y \in \mathbb{R}^d \), \( y \neq 0 \). We say that \( \{S_i : X \to X\}_{i=1}^m \) is a \textbf{conformal iterated function system} on a compact set \( X \subset \mathbb{R}^d \) if each \( S_i \) extends to an injective conformal map \( S_i : V \to V \) on an open connected set \( V \supset X \) and \( \sup\{|S_i'(x)| : x \in V\} < 1 \). We assume Hölder continuity of the differentials, that is, there exists \( \alpha > 0 \) such that for all \( i \leq m \),

\[
||S_i'(x)| - |S_i'(y)|| \leq \text{const} \cdot |x - y|\alpha \quad \text{for all} \quad x, y \in V.
\]

We should note that for \( d \geq 2 \) Hölder continuity (and, in fact, real analyticity) of \( |S_i'(\cdot)| \) follows from conformality and injectivity.

Under these assumptions the unique non-empty compact set \( K \subset X \) satisfying (1.1) is called \textbf{self-conformal}. The role of similarity dimension is played by the unique solution \( s \) of the Bowen equation \( P(s) = 0 \), where the pressure \( P(t) \) is defined by

\[
P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in K} \sum_{u \in A^n} |S_u'(x)|^t, \quad \text{for} \quad t > 0.
\]

It is well-known that \( \mathcal{H}^s(K) < \infty \). The definitions of \( \varepsilon \)-relatively close maps (1.2) and of the compositions \( S_u \) extend to this setting.

We say that the \textbf{Bandt-Graf condition} holds if there exists \( \varepsilon > 0 \) such that for distinct \( u, v \) in \( A^* \), the maps \( S_u \) and \( S_v \) are not \( \varepsilon \)-relatively close. Our main result is the complete equivalence theorem for self-conformal sets.

**Theorem 1.1.** For a conformal i.f.s. \( \{S_i\}_{i \leq m} \), satisfying the Hölder condition, and the associated self-conformal set \( K \), the following are equivalent:

(a) the OSC;
(b) \( \mathcal{H}^s(K) > 0 \) where \( s > 0 \) is such that \( P(s) = 0 \);
(c) the Bandt-Graf condition;
(d) the strong OSC.

The implication (a) \( \Rightarrow \) (b) is standard (see, e.g., [21, p. 89]), so we just need to prove that (b) \( \Rightarrow \) (c) \( \Rightarrow \) (d).
Perhaps surprisingly, the existing proofs of these implications in the self-similar case do not extend to the nonlinear setting. The elegant method of Bandt and Graf \cite{BG} for the proof of (b) $\iff$ (c) is very much dependent on the set $K$ being precisely self-similar. In several places of \cite{BG} it was crucial that $\sum_{i} |S'_{g}(x)|^{s} = 1$ for all $x$. We have to use a more "robust" method to allow for distortion.

The implication (a) $\Rightarrow$ (d) answers a question of R. D. Mauldin (see \cite{Mauldin} Question 9.1). This implication was stated by Fan and Lau in \cite{FanLau} Lemma 2.6. Although their approach is very promising, unfortunately, the proof in \cite{FanLau} contains a gap, as was pointed out by N. Patzschke (personal communication). A more detailed comment on this is given at the end of the paper.

We also obtain the following corollary, which extends Schief’s result \cite{Schief} Cor. 2.3:

**Corollary 1.2.** If $K \subset \mathbb{R}^{d}$ is self-conformal and the solution of the pressure equation $s$ equals $d$, then $\mathcal{H}^{d}(K) > 0$ implies that $K$ is the closure of its interior.

### 2. Generalizing the Bandt-Graf Theorem

After some preliminaries, which will be needed in Section 3 as well, we prove the implication (b) $\Rightarrow$ (c) in Theorem \cite{BG} generalizing the result of Bandt and Graf \cite{BG}.

We consider a conformal contracting i.f.s. $\{S_{i}\}_{i=1}^{m}$ satisfying the Hölder condition (1.3) on an open set $V$, such that $S_{i}(X) \subset X$ for a compact set $X \subset V$. Let $A = \{1, \ldots, m\}$ and equip the sequence space $A^{\mathbb{N}}$ with the product topology. We write $A^{*} = \bigcup_{n \geq 1} A^{n}$ for the set of finite "words" in the alphabet $A$. The symbol $\sigma$ denotes the left shift on $A^{\mathbb{N}}$ and $A^{*}$. The map $\Pi : A^{\mathbb{N}} \to \mathbb{R}^{d}$ defined by

$$
\Pi(\omega) = \lim_{n \to \infty} S_{\omega_{1}\ldots\omega_{n}}(x), \ x \in V,
$$

is called the natural projection map (clearly, it does not depend on $x$). The self-conformal set associated with the i.f.s. is $K = \Pi(A^{\mathbb{N}})$. Let

$$
\mathcal{O}(F, r) = \{x \in \mathbb{R}^{d} : \text{dist}(x, F) < r\}
$$

denote the $r$-neighborhood of a compact set $F \subset \mathbb{R}^{d}$. The closed ball of radius $r$ centered at $x \in \mathbb{R}^{d}$ is denoted by $B(x, r)$. We write $[x, y]$ to denote the line segment connecting $x$ and $y$ in $\mathbb{R}^{d}$.

Fix $\delta_{0} > 0$ so that $\mathcal{O}(X, 3\delta_{0}) \subset V$ and let

$$
V' = \mathcal{O}(X, \delta_{0}), \quad V'' = \mathcal{O}(X, 2\delta_{0}).
$$

Since $S_{i}X \subset X$ and $|S'_{i}(x)| < 1$ for all $x \in V$, we also have $S_{i}V' \subset V'$ and $S_{i}V'' \subset V''$ for all $i$.

Next we recall the standard bounded distortion property of conformal i.f.s. satisfying the Hölder condition (see, e.g., \cite{Mauldin} Lemma 2.1): there exists $C_{1} \geq 1$ such that for all $u \in A^{*}$,

$$
|S_{u}'(x)| \leq C_{1}|S_{u}'(y)| \quad \text{for all } x, y \in V''. \tag{2.1}
$$

Denote

$$
\|S_{u}'\| = \sup_{x \in V''} |S_{u}'(x)|.
$$
The property (2.1) yields (see, e.g., \[8\] Lemma 2.2) that there exists \( C_2 \geq 1 \) such that for all \( u \in A^* \),
\[
(2.2) \quad C_2^{-1} ||S'_u|| \cdot |x - y| \leq |S_u(x) - S_u(y)| \leq C_2 ||S'_u|| \cdot |x - y| \quad \text{for all } x, y \in V'.
\]
This implies
\[
(2.3) \quad B(x, r) \subset V' \Rightarrow S_u B(x, r) \supset B(S_u(x), C_2^{-1} ||S'_u|| r) \quad \text{for all } u \in A^*
\]
(see, e.g., \[8\] Cor. 2.3). Denote \( d_u = \text{diam}(K_u) \) for \( u \in A^* \). By (2.2), there exists \( C_3 \geq 1 \) such that
\[
(2.4) \quad C_3^{-1} ||S'_u|| \leq d_u \leq C_3 ||S'_u|| \quad \text{for all } u \in A^*.
\]
By (2.1) and (2.4), there exists \( C_4 \geq 1 \) such that for all \( u, v \in A^* \),
\[
(2.5) \quad C_4^{-1} \max \{ \|S'_u\| d_v, \|S'_v\| d_u \} \leq d_{uv} \leq C_4 \min \{ \|S'_u\| d_v, \|S'_v\| d_u \}.
\]
Let \( \omega \wedge \tau \) denote the common initial block (possibly empty) of two sequences \( \omega, \tau \in A^N \). We equip the space \( A^N \) with a metric
\[
(2.6) \quad \rho(\omega, \tau) = d_{\omega \wedge \tau} \quad \text{for } \omega \neq \tau.
\]
It follows from the bounded distortion properties that the product topology on \( A^N \) coincides with the one defined by \( \rho \). Clearly, the natural projection map \( \Pi : (A^N, \rho) \to \mathbb{R}^d \) is Lipschitz.

The reader is referred to \[3, 4\] for the background on thermodynamic formalism. Define a Hölder continuous function on \( A^N \) by \( \phi(\omega) = \log |S'_j| (\Pi(\omega)) \). The pressure function \( P(t) \) of \( t \phi \) with respect to the shift \( \sigma \) can be expressed by (1.4). There is a unique value \( s \) such that \( P(s) = 0 \). Let \( \mu \) be the Gibbs measure on \( A^N \) for the potential \( s \phi \). Denoting by \([u]\) the cylinder set corresponding to \( u \in A^* \), we have by the definition of the Gibbs measure and the bounded distortion principle (2.1) that there exists \( C_5 \geq 1 \) such that
\[
(2.7) \quad C_5^{-1} ||S'_u||^s \leq \mu[u] \leq C_5 ||S'_u||^s \quad \text{for all } u \in A^*.
\]

**Lemma 2.1.** (i) The measure \( \mu \) is equivalent to the \( s \)-dimensional Hausdorff measure on \( A^N \) with the metric \( \rho \).

(ii) The restriction of the Hausdorff measure \( H^s |_K \) is absolutely continuous with respect to the measure \( \nu = \mu \circ \Pi^{-1} \) on \( K \).

**Proof.** (i) A ball in the metric \( \rho \) is a cylinder \([u]\) for some \( u \in A^* \). Any collection of cylinders in \( A^N \) contains a disjoint subcollection with the same union. Now the claim is immediate by comparing (2.4), (2.6) and (2.7).

(ii) Suppose that \( \nu(B) = 0 \) for some Borel set \( B \subset K \). Then \( \mu(\Pi^{-1} B) = 0 \); hence the \( s \)-dimensional Hausdorff measure of \( \Pi^{-1} B \subset A^N \) is zero by part (i) of this lemma. It follows that \( H^s(B) = 0 \) since \( \Pi \) is Lipschitz.

**Proof of (b) ⇒ (c) in Theorem 1.1.** We are going to prove that if for any \( \varepsilon > 0 \) there exist \( u \neq v \) such that \( S_u \) and \( S_v \) are \( \varepsilon \)-relatively close, then \( H^s(K) = 0 \). First we make a few useful observations concerning \( \varepsilon \)-relatively close maps.

CLAIM 1. If \( S_u, S_v \) are \( \varepsilon \)-relatively close, then \( S_{wu} \) and \( S_{wv} \) are \( C_2 C_4 \varepsilon \)-relatively close for every \( w \in A^* \). Indeed, we have by (2.2), (2.4) and (2.5) for \( x \in K \):
\[
|S_{wu}(x) - S_{wv}(x)| \leq C_2 ||S'_u|| \cdot |S_u(x) - S_v(x)| \leq C_2 ||S'_u|| \cdot \varepsilon \min\{d_u, d_v\} \leq C_2 C_4 \varepsilon \min\{d_{wu}, d_{wv}\}.
\]
CLAIM 2. If $S_{w_1}, S_{w_2}$ are $\varepsilon$-relatively close, then $S_{w_1u}$ and $S_{w_2u}$ are $C_4\|S'_u\|^{-1}\varepsilon$-relatively close for every $u \in A^*$. Indeed, in view of (1.2) and (2.5),

$$|S_{w_1u}(x) - S_{w_2u}(x)| = |S_{w_1}(S_u(x)) - S_{w_2}(S_u(x))| \leq \varepsilon \min\{d_{w_1}, d_{w_2}\} \leq C_4\|S'_u\|^{-1}\varepsilon \cdot \min\{d_{w_1u}, d_{w_2u}\}.$$  

CLAIM 3. If $S_u, S_v$ are $\varepsilon$-relatively close, then

$$d_v \leq (1 + 2\varepsilon) \cdot d_u.$$  

This is immediate from the definition (1.2).

CLAIM 4. If $S_u, S_v$ are $\delta$-relatively close and $S_w, S_v$ are $\delta$-relatively close, then $S_u, S_v$ are $2\delta(1 + 2\delta)$-relatively close. Indeed, by (1.2) and Claim 3, $\min\{d_u, d_v\} \leq (1 + 2\delta) \min\{d_u, d_w\}$ and $\min\{d_v, d_w\} \leq (1 + 2\delta) \min\{d_u, d_w\}$. The rest is immediate.

**Lemma 2.2.** Suppose that for any $\varepsilon > 0$ there exist $u \neq v$ such that $S_u$ and $S_v$ are $\varepsilon$-relatively close. Then for any $N \in \mathbb{N}$ and any $\varepsilon > 0$ there exist distinct $u_1, \ldots, u_N$ such that $S_{u_1}, S_{u_2}$ are $\varepsilon$-relatively close for all $1 \leq i < j \leq N$.

**Proof.** It is enough to show that if the statement holds for $N$, then it holds for $2N$. Assuming it holds for $N$, find distinct $u_1, \ldots, u_N$ such that $S_{u_1}, S_{u_2}, \ldots, S_{u_N}$ are pairwise $\delta_1$-relatively close where $\delta_1 = (1/4)C_4^{-1}\varepsilon$. Next let

$$\delta_2 = (1/4)C_4^{-1}\min_{j \leq N} \|S'_{u_j}\| \cdot \varepsilon$$

and find $w_1 \neq w_2$ such that $S_{w_1}, S_{w_2}$ are $\delta_2$-relatively close. Then the $2N$ words $w_ku_j$, $k = 1, 2$, $1 \leq j \leq N$, are all distinct, and we claim that the maps $\{S_{w_ku_j} : k = 1, 2; 1 \leq j \leq N\}$ are pairwise $\varepsilon$-relatively close. Indeed, $S_{w_1u_j}, S_{w_2u_j}$ are $\frac{\varepsilon}{4}$-close by Claim 1 and $S_{w_1u_j}, S_{w_2u_j}$ are $\frac{\varepsilon}{4}$-close by Claim 2. Now Claim 4 implies that $S_{w_1u_j}, S_{w_2u_j}$ are $\delta_3$-close, with $\delta_3 = \frac{\varepsilon}{4}(1 + \frac{\varepsilon}{4})$. We have $\delta_3 \leq \varepsilon$ for $\varepsilon \leq 2$, which we can certainly assume, and the lemma is proved.

Now we resume the proof of (b) $\Rightarrow$ (c) in Theorem 1.1. Fix $N \in \mathbb{N}$ and find distinct $u_1, \ldots, u_N$ such that $S_{u_1}, \ldots, S_{u_N}$ are pairwise 1-relatively close. Recall that $\nu = \mu \circ \Pi^{-1}$ is the push-down measure on $\mathcal{K}$. We claim that for $\nu$-a.e. $x$,

$$\limsup_{r \to 0} \frac{\nu B(x, r)}{r^s} \geq c \cdot N,$$

with a constant $c > 0$ independent of $N$.

It is well-known (see [3] or [4] Cor. 5.6) that the Gibbs measure $\mu$ is an ergodic invariant measure for the shift $\sigma$ on $A^\mathbb{N}$. Since $\mu[u_1] > 0$, the block $u_1$ occurs infinitely often in $\mu$-a.e. sequence $\omega$ by the Ergodic Theorem. Let $\Omega \subset A^\mathbb{N}$ be the set of all such $\omega$. Fix $\omega \in \Omega$. We know that there exist arbitrarily large $N$ such that $\sigma^nu \in [u_1]$. Fix such $n$, let $w = \omega_1 \ldots \omega_n$, and consider the words $v_j = wu_j$ for $j = 1, \ldots, N$. By Claim 1, the maps $S_{wu_j}$ are pairwise $C_2C_4$-relatively close. By (1.2), this implies that for $x = \Pi(\omega) \in \mathcal{K}_{wu_1}$ we have

$$B(x, r) \supset \bigcup_{j=1}^{N} \mathcal{K}_{wu_j}, \quad \text{where} \quad r = (2 + 2C_4) \max_{j \leq N} d_{wu_j}.$$
Thus, by (2.7) and (2.8),

$$\nu B(x, r) \geq \sum_{j=1}^{N} \mu[w_{uj}] \geq C_3^{-1} C_5^{-1} \min_{j \leq N} d^{n}_{w_{uj}},$$

Combining this with Claim 3, we obtain

$$\frac{\nu B(x, r)}{r^s} \geq \frac{C_3^{-1} C_5^{-1} N}{(2 + C_2 C_4)^s(1 + 2C_2 C_4)^s}. $$

Since $r$ in the last formula can be arbitrarily small, (2.8) follows.

We have verified (2.8) for $x \in \Pi(\Omega)$ which is a set of full $\nu$-measure. Now $\mathcal{H}^s(\Pi(\Omega)) \leq 2^s(eN)^{-1}\nu(\Pi(\Omega))$ by the Rogers-Taylor density theorem (see [9] or [4, Proposition 2.2]). On the other hand, $\nu(K \setminus \Pi(\Omega)) = 0$, so $\mathcal{H}^s(K \setminus \Pi(\Omega)) = 0$ by Lemma (3.1(ii)). Thus, $\mathcal{H}^s(K) \leq 2^s(eN)^{-1}\nu(K)$, and since $N$ was arbitrary we conclude that $\mathcal{H}^s(K) = 0$.

$$\square$$

3. Generalizing Schief’s theorem

In this section we prove the implication (c) $\Rightarrow$ (d) in Theorem 1.1 and Corollary 1.2 generalizing results of Schief [11]. For $T \geq 1, a \geq 0$ and $u \in \mathcal{A}^*$ let

$$(3.1) \quad W_{a, T}(u) = \left\{ v \in \mathcal{A}^* : \frac{1}{T} \leq \frac{d_v}{d_u} \leq T, \text{ dist}(K_v, K_u) \leq ad_u \right\}.$$ 

Lemma 3.1. Suppose that the Bandt-Graf condition holds, that is, there exists $\varepsilon > 0$ such that for any distinct $v, w \in \mathcal{A}^*$,

$$(3.2) \quad \exists x \in \mathcal{K} : |S_v(x) - S_w(x)| \geq \varepsilon \min\{d_v, d_w\}. $$

Then for any $a > 0$ and $T \geq 1$ there exists $C(a, T) < \infty$ such that

$$\# W_{a, T}(u) \leq C(a, T) \quad \text{for all} \quad u \in \mathcal{A}^*.$$ 

Remark. This lemma is the only place in this section where the Bandt-Graf condition is used. It is easy to see that the statement of the lemma holds if the Bandt-Graf condition is replaced by the OSC, thus providing a direct derivation of the implication OSC $\Rightarrow$ SOSC (the strong OSC).

Proof of Lemma 3.1. Let $\delta = \frac{\varepsilon}{4C_2 C_3 T^2}$. It follows from (3.2) that if $\tilde{x} \in K$ and $|x - \tilde{x}| \leq \delta$, then for $v, w \in W_{a, T}(u)$, in view of (2.2) and (2.4),

$$|S_v(x) - S_w(x)| \geq |S_v(x) - S_w(x)| - |S_v(x) - S_v(\tilde{x})| - |S_w(x) - S_w(\tilde{x})|$$

$$\geq \varepsilon \min\{d_v, d_w\} - C_2 \delta (\|S_v\| + \|S_w\|)$$

$$\geq \varepsilon \min\{d_v, d_w\} - C_2 C_3 \delta (d_v + d_w)$$

$$\geq d_u (\varepsilon T^{-1} - C_2 C_3 \cdot 2 \delta T)$$

$$= (1/2) d_u \varepsilon T^{-1}.$$ 

Fix a finite set $\{x_1, \ldots, x_N\} \subset \mathcal{K}$ so that $\bigcup_{i=1}^{N} B(x_i, \delta) \supset \mathcal{K}$. For each $v \in W_{a, T}(u)$ let $\xi_v = \{S_v(x_i)\}_{1 \leq i \leq N} \in \mathbb{R}^{dN}$. By (3.3),

$$|\xi_v - \xi_w| \geq (1/2) d_u \varepsilon T^{-1} \quad \text{for all} \quad v, w \in W_{a, T}(u).$$ 

On the other hand, if $v \in W_{a, T}(u)$, then $\text{dist}(K_v, K_u) \leq ad_u$; hence

$$|S_v(x) - S_w(x)| \leq ad_u + d_u + d_v \leq (a + 1 + T)d_u \quad \text{for all} \quad x \in K.$$
It follows that $|\xi_u - \xi_v| \leq \sqrt{N(a + 1 + T)d_u}$. Thus, open balls in $\mathbb{R}^{dN}$ of radius $\frac{1}{4}d_u \varepsilon T^{-1}$ around $\xi_v$ for $v \in W_{a,T}(u)$ are all disjoint and lie in the ball of radius $(\sqrt{N(a + 1 + T)} + \frac{1}{4}d_u T^{-1})$ around $\xi_u$. It follows that

$$\#W_{a,T}(u) \leq \left(\frac{\sqrt{N(a + 1 + T)} + \frac{1}{4}d_u T^{-1}}{\frac{1}{4}d_u \varepsilon T^{-1}}\right)^{dN},$$

which is a constant independent of $u$. \qed

We need a lemma on “local” bounded distortion. Recall that $V'' = \mathcal{O}(X, 2\delta_0) \subset V$.

**Lemma 3.2.** (i) There exists $L_1 > 0$ such that for all $x, y \in V''$,

$$\frac{|S'_u(x)|}{|S'_u(y)|} \leq \exp[L_1|x - y|^\alpha] \quad \text{for all } u \in \mathcal{A}^*.$$

(ii) There exists $L_2 > 0$ such that for all $u \in \mathcal{A}$ such that $d_u \leq \delta_0$ and all $w \in \mathcal{A}^*$,

$$\text{dist}(z, K_u) \leq d_u \Rightarrow \exp[-L_2d_u^\alpha] \leq \frac{d_{wu}}{d_u |S'_{wu}(z)|} \leq \exp[L_2d_u^\alpha].$$

**Proof.** (i) is folklore; it is obtained in the course of the standard proof of “global” bounded distortion (see, e.g., [2] or [8, Lemma 2.1]).

(ii) Note that $z \in \mathcal{O}(X, \delta_0) \subset V$; hence $|S'_w(z)|$ is well-defined. We can assume that $d_u$ is sufficiently small, since otherwise (3.5) follows from (2.4) and (2.5). Suppose that $C_2C_4d_u \leq \delta_0$. Then for any $x, y \in K$ we have $[S_u(x), S_u(y)] \subset V'$; hence

$$|S_{wu}(x) - S_{wu}(y)| \leq |S'_{wu}(\zeta)| \cdot |S_u(x) - S_u(y)|$$

for some $\zeta$ satisfying $\text{dist}(\zeta, K_u) \leq d_u$. If $\text{dist}(z, K_u) \leq d_u$, then $|\zeta - z| \leq 3d_u$ and $\zeta, z \in V'$. Thus,

$$d_{wu} \leq d_u |S'_{wu}(z)| \exp[L_1(3d_u)^\alpha]$$

by (3.4). To obtain the other inequality, observe that by (2.3) and (2.5),

$$S_w B(S_u(x), C_2C_4d_u) \supset B(S_{wu}(x), C_2\|S'_w\|d_u) \supset B(S_{wu}(x), d_{wu}).$$

Therefore, $[S_{wu}(x), S_{wu}(y)] \subset V'$ and we have

$$|S_u(x) - S_u(y)| \leq |(S_u^{-1})'(x) - |S_{wu}(x) - S_{wu}(y)|,$$

for some $x \in B(S_{wu}(x), d_{wu}) \subset S_w B(S_u(x), C_2C_4d_u)$. We have $|z - S_u^{-1}(\xi)| \leq 2d_u + C_2C_4d_u$; hence by (3.4),

$$d_u \leq d_{wu} |S'_w(S_u^{-1}(\xi))^{-1}| \leq d_{wu} |S'_w(\xi)|^{-1} \exp[L_2d_u^\alpha],$$

with $L_2 = L_1(2 + C_2C_4)^\alpha$, as desired. \qed

**Lemma 3.3.** Let $T_0 \geq 1$ and $\varepsilon > 0$. There exists $\delta = \delta(T_0, \varepsilon) > 0$ such that for all $u \in \mathcal{A}^*$ with $d_u \leq \delta$, for all $a \in [0, 1]$ and all $T \in [T_0, 2T_0]$,

$$v \in W_{a,T}(u) \Rightarrow uv \in W_{a(1+\varepsilon), T(1+\varepsilon)}(wu) \quad \text{for all } w \in \mathcal{A}^*.$$
Proof. Suppose that $d_u \leq \delta < \delta_0/(2T_0)$ and $v \in W_{a,T}(u)$. Fix $w \in A^*$. We need to check that (i) $T^{-1}(1+\varepsilon)^{-1} \leq \frac{d_u}{d_w} \leq T(1+\varepsilon)$ and (ii) $\text{dist}(K_{wu},K_{wu}) \leq a(1+\varepsilon)d_{wu}$.

(i) Let $z \in K_u$ be such that $\text{dist}(z,K_u) \leq ad_u \leq d_u$. Then by (3.5), using that $d_uT^{-1} \leq d_v \leq d_uT \leq 2\delta_0 < \delta_0$, we obtain

$$\frac{d_{wu}}{d_w} \leq \frac{d_u|S'_w(z)| \exp[L_2d_u^n]}{d_v|S'_w(z)| \exp[-L_2d_v^n]} \leq Te^{L_2\delta^n(1+(2T_0)^n)} \leq T(1+\varepsilon),$$

for $\delta > 0$ sufficiently small. The other inequality is obtained similarly.

(ii) Since $v \in W_{a,T}(u)$, there exist $x, y \in K$ such that $|S_u(x) - S_v(y)| \leq ad_u$. Then $[S_u(x), S_v(y)] \subset V'$; hence

$$|S_{wu}(x) - S_{wu}(y)| \leq |S'_w(z)| \cdot |S_u(x) - S_v(y)|$$

for some $z$ with $\text{dist}(z,K_u) \leq ad_u \leq d_u$. Therefore, by (3.5),

$$\text{dist}(K_{wu},K_{wu}) \leq a|S'_w(z)| \cdot d_u \leq ad_{wu} \exp[L_2\delta^n] \leq a(1+\varepsilon)d_{wu},$$

for $\delta > 0$ sufficiently small, and we are done.

Proof of (c) $\Rightarrow$ (d) in Theorem 1.1. The scheme of the proof generally follows that of Schief's [11], but we have to be careful with distortion.

Fix $T_0 \geq 1$ so large that for all $j \in A$ and all $v \in A^*$,

$$d_v \leq T_0^2 d_{v_j} \quad \text{and} \quad T_0 d_j \geq 1$$

(in fact, one can take $T_0 = \max\{d_j^{-1}, C_{A}^{1/2}\|S'_j\|^{-1/2}, \ j \in A\}$ by (2.3)). It follows from (3.6) that for any $r \leq 1$ and any $w = w_1 \ldots w_n \in A^*$, with $d_w \leq r$, there is $1 \leq k \leq n$ such that

$$T_0^{-1} \leq d_w/r \leq T_0 \quad \text{where} \quad w' = w_1 \ldots w_k$$

(just take maximal $1 \leq k \leq n$ such that $d_{w'} \geq rT_0^{-1}$). To simplify notation, let

$$W_a(u) := W_{a,(1+a)T_0}(u) \quad \text{and} \quad M_a(u) = \#W_a(u).$$

By Lemma 3.1 there exists $C = C(1, 2T_0) > 0$ such that

$$M_a(u) \leq C \quad \text{for all} \ u \in A^* \text{ and all } a \in [0, 1].$$

By the definition (3.1), the function $a \mapsto M_a(u)$ is non-decreasing. For $r > 0$ consider

$$\widetilde{M}_a(r) := \sup\{M_a(u) : u \in A^*, d_u \leq r\}.$$

Let $\varepsilon = \frac{1}{2C}$ and fix $r = \min\{1, \delta(T_0, \varepsilon)\}$ where $\delta(T_0, \varepsilon)$ is from Lemma 3.3. The function $a \mapsto \widetilde{M}_a(r)$ on $[0, 1]$ is non-decreasing, integer-valued, and is bounded above by $C$. Thus, we can find an interval $[a_1, a_2] \subset [0, 1]$ with $a_2 - a_1 \geq \frac{1}{C}$ such that $\widetilde{M}_{a_1}(r) = \widetilde{M}_{a_2}(r)$. Clearly, the supremum in (3.8) is attained, so we can find $u \in A^*$ with $d_u \leq r$ such that

$$M_{a_1}(u) = \widetilde{M}_{a_1}(r).$$

Fix this $u$ for the rest of the proof. Since, in addition, $\widetilde{M}_{a_2}(r) = \widetilde{M}_{a_1}(r)$ and $M_{a_2}(u) \geq M_{a_1}(u)$, we deduce that $M_{a_2}(u) = \widetilde{M}_{a_2}(r) = M_{a_1}(u)$. Observe that

$$a_2 \geq (1 + (2C)^{-1})a_1 = a_1(1 + \varepsilon)$$
and
\[ 1 + a_2 \geq (1 + a_1)(1 + (2C)^{-1}) = (1 + a_1)(1 + \varepsilon); \]
hence
\[ v \in W_{a_1}(u) \Rightarrow qv \in W_{a_2}(qu) \quad \text{for all } q \in A^* \]
by Lemma 3.3. It follows that \( M_{a_2}(qu) \geq M_{a_1}(u) \). But
\[ M_{a_2}(qu) \leq M_{a}(r) = M_{a_1}(u); \]
therefore, \( M_{a_2}(u) = M_{a_1}(u) = M_{a_2}(qu) \) for all \( q \in A^* \). Thus,
\[ (3.9) \quad W_{a_1}(qu) = \{qv: \, v \in W_{a_2}(u)\} \quad \text{for all } q \in A^*. \]

Consider
\[ (3.10) \quad U = \bigcup_{v \in A^*} S_v \mathcal{O}(K_{vu}, \varepsilon''), \]
where \( \varepsilon' > 0 \) will be chosen later. This will be our open set in the strong OSC. Clearly, \( U \cap K \neq \emptyset \) and \( S_i U \subset U \) for all \( i \leq m \). It remains to check that \( S_i U \cap S_j U = \emptyset \) for all \( i \neq j \). This will follow if we prove that for all \( v, w \in A^* \) and all \( i \neq j \),
\[ (3.11) \quad S_{iw} \mathcal{O}(K_{wu}, \varepsilon') \cap S_{jw} \mathcal{O}(K_{wu}, \varepsilon') = \emptyset. \]

If \( \varepsilon' \leq \delta_0 \), then
\[ S_{iw} \mathcal{O}(K_{wu}, \varepsilon') \subset \mathcal{O}(K_{ivu}, \|S_{iu}w\| \varepsilon') \subset \mathcal{O}(K_{ivu}, \varepsilon''d_{ivu}), \]
with \( \varepsilon'' = C_d u^{-1}\varepsilon' \),
in view of (2.8). Similarly,
\[ S_{jw} \mathcal{O}(K_{wu}, \varepsilon') \subset \mathcal{O}(K_{jwu}, \varepsilon''d_{jwu}). \]
Assume that \( d_{ivu} \geq d_{jwu} \) without loss of generality. By (3.7), there is a prefix (initial block) \( jw' \) of the word \( jwu \) such that \( T^{-1} \leq d_{ivu} \leq T_0 \) (here \( w' \) may range from empty to \( wu \)). Now \( jw' \) satisfies the diameter condition for membership in \( W_{a_2}(ivu) \) but \( jw' \notin W_{a_2}(ivu) \) by (3.3). Therefore,
\[ \text{dist}(K_{ivu}, K_{jwu}) \geq \text{dist}(K_{ivu}, K_{jwu}) > a_2 d_{ivu}. \]
Thus, if \( \varepsilon'' \leq a_2/2 \), then (3.11) holds. It suffices to take \( \varepsilon' = \min\{\delta_0, a_2 C_d^{-1}d_u\} \), and the proof is complete. \( \square \)

Proof of Corollary 1.2. We want to show that if \( s = d \), the dimension of the space, and \( \mathcal{H}(K) > 0 \), then \( K = \text{clos}(\text{int} \, K) \). The proof is quite similar to the proof of [11, Cor. 2.3]. By Theorem 1.1, the OSC holds, and moreover, the open set \( U \) can be chosen so that \( U \subset V' = \mathcal{O}(X, \delta_0) \) (see (3.11)). The OSC means that \( S_i U \) are pairwise disjoint subsets of \( U \), for \( i \leq m \). Let
\[ W = U \setminus \bigcup_{i=1}^m S_i U. \]
We claim that \( \mathcal{L}_d(W) = \mathcal{H}_d(W) = 0 \) where \( \mathcal{L}_d \) is the Lebesgue measure in \( \mathbb{R}^d \). Indeed, it is easy to see that the sets \( S_i W \) are pairwise disjoint for all \( v \in A^* \), and they all lie in \( U \). Thus,
\[ (3.12) \quad \sum_{n \geq 1} \sum_{|v| = n} \mathcal{L}_d(S_v W) \leq \mathcal{L}_d(U) < \infty. \]
We have
\[ \mathcal{L}_d(S_v W) = \int_W |S'_v(x)|^d \, dx \geq C_1^{-d}\|S'_v\|^d \mathcal{L}_d(W), \]
in view of (2.1). Therefore, by (2.7),
\[ \sum_{|v|=n} \mathcal{L}_d(S_v W) \geq C_1^{-d} C_5^{-1} \mathcal{L}_d(W) \sum_{|v|=n} \mu[v] = C_1^{-d} C_5^{-1} \mathcal{L}_d(W); \]
hence \( \mathcal{L}_d(W) = 0 \) by (3.12). This implies that the open set \( U \setminus \text{clos}(\bigcup_{i=1}^m S_i U) \) is empty. Therefore,
\[ \bigcup_{i=1}^m S_i (\text{clos} U) = \bigcup_{i=1}^m \text{clos}(S_i U) = \text{clos} \left( \bigcup_{i=1}^m S_i U \right) = \text{clos} U, \]
so \( \text{clos} U \) is an invariant compact set for the i.f.s. \( \{S_i\}_{i \leq m} \). By uniqueness, \( \text{clos} U = \mathcal{K} \), and the proof is complete.

In conclusion, we should comment on the paper of Fan and Lau [5] where the implication OSC \( \Rightarrow \) SOSC is stated in Lemma 2.6. However, as pointed out by N. Patzschke (personal communication), the proof of [5, Lemma 2.6] contains a gap. The formula on the second line of [5, p. 335] is unjustified; proving it involves checking two facts, one of which, that \( IJ \in A[U, I_0] \), may fail, due to distortion. Perhaps one could fix the proof, but this would require further arguments.

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REFERENCES


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