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A UNIQUENESS THEOREM WITH MOVING TARGETS WITHOUT COUNTING MULTIPLICITY

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Abstract. In this paper, we prove a uniqueness theorem for holomorphic curves with moving targets without counting multiplicity.

1. Introduction

It is well known that two non-constant polynomials f, g over an algebraic closed field of characteristic zero are identical if there exist two distinct values a, b such that $f(x) = a \Leftrightarrow g(x) = a$ and $f(x) = b \Leftrightarrow g(x) = b$. In 1926, R. Nevanlinna (cf. [Ne]) extended the above result to meromorphic functions. He proved that two non-constant meromorphic functions of a complex variable which attain five distinct values at the same points must be identical.

Wilhelm Stoll (cf. [St]) later extended Nevanlinna's result to holomorphic curves and obtained the following theorem.

Theorem A1. *Let $f_1, f_2, \ldots, f_\lambda : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be linearly non-degenerated holomorphic curves. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Assume that $f_1^{-1}(H_j) = \cdots = f_\lambda^{-1}(H_j)$ and denote by $A_j = f_1^{-1}(H_j), 1 \le j \le q$. Assume further that for each $i \ne j, A_i \cap A_j = \emptyset$. Let $A = \bigcup_{j=1}^q A_j$. Let l, $2 \le l \le \lambda$, be an integer such that for any increasing sequence $1 \le j_1 < j_2 < \cdots < j_l \le \lambda$, $f_{j_1}(z) \wedge \cdots \wedge f_{j_l}(z) = 0$ for every point $z \in A$. If $q \ge \frac{\lambda n}{\lambda - l + 1} + n + 2$, then $f_1, \ldots, f_\lambda$ are algebraically dependent over $\mathbb{C}$, i.e., $f_1 \wedge \cdots \wedge f_\lambda \equiv 0$ on $\mathbb{C}$.*

In the case $\lambda = 2$, Theorem A1 gives the following statement.

Theorem A2. *Let $f,\ g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerated holomorphic curves. Let $H_1, \ldots, H_{3n+2}$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Assume that $f^{-1}(H_j) = g^{-1}(H_j), 1 \le j \le 3n+2$, and for each $i \ne j$, $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$. Let $A = \bigcup_{j=1}^q f^{-1}(H_j)$. If for every point $z \in A$, $f(z) = g(z)$, then $f \equiv g$.*

This paper generalizes W. Stoll's result to moving targets. First, we introduce some notation. For a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, we use $\mathbf{f} : \mathbb{C} \to \mathbb{C}^{n+1} - \{0\}$ to denote a reduced representation of f, that is $\mathbb{P}(\mathbf{f}) = f$. We note that a hyperplane H in $\mathbb{P}^n(\mathbb{C})$ can be regarded as a point in $\mathbb{P}^n(\mathbb{C}^*)$, where $\mathbb{C}^*$ is the dual space

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of $\mathbb{C}$. By a moving target we mean a holomorphic map $g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}^*)$. Let $\mathbf{g} : \mathbb{C} \to {\mathbb{C}^*}^{n+1} - \{0\}$ be a reduced representation of g. Then $\mathbf{g}(\mathbf{f})$ is an entire function on $\mathbb{C}$. Note that although the function $\mathbf{g}(\mathbf{f})$ depends on the choice of representations, the zeros of $\mathbf{g}(\mathbf{f})$ however are independent of the the choice of representations. Moving targets $g_j : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}^*), 1 \le j \le q$, are said to be **in general position** if there is $z_0 \in \mathbb{C}$ such that the hyperplanes $g_j(z_0), 1 \le j \le q$, are located in general position. Let $f_1, f_2, \ldots, f_\lambda : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be non-constant holomorphic curves. Let $g_j : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}^*)$, $1 \le j \le q$, be moving targets located in general position. Assume that $\mathbf{g}_j(\mathbf{f}_t) \not\equiv 0$ for $1 \le j \le q, 1 \le t \le \lambda$, and assume that $(\mathbf{g}_j(\mathbf{f}_1))^{-1}\{0\} = \cdots = (\mathbf{g}_j(\mathbf{f}_\lambda))^{-1}\{0\}$. Let $A_j = (\mathbf{g}_j(\mathbf{f}_1))^{-1}\{0\}$. Denote by $T[n+1, q]$ the set of all injective maps from $\{1, \ldots, n+1\}$ to $\{1, \ldots, q\}$. For every $z \in \mathbb{C} - \bigcup_{\beta \in T[n+1,q]} \{z \mid \mathbf{g}_{\beta(1)}(z) \wedge \cdots \wedge \mathbf{g}_{\beta(n+1)}(z) = 0\}$, we define $\rho(z) = \#\{j \mid z \in A_j\}$. Then $\rho(z) \le n$. For any positive number $r > 0$, define $\rho(r) = \sup\{\rho(z) \mid |z| \le r\}$ where the sup is taken over all $z \in \mathbb{C} - \bigcup_{\beta \in T[n+1,q]} \{z \mid \mathbf{g}_{\beta(1)}(z) \wedge \cdots \wedge \mathbf{g}_{\beta(n+1)}(z) = 0\}$ with $|z| \le r$. Then $\rho(r)$ is a decreasing function. Let

$$d = \lim_{r \to \infty} \rho(r). \tag{1.1}$$

Then $d \le n$. If for each $i \ne j, A_i \cap A_j = \emptyset$, then $d = 1$. A number $r_0 > 1$ exists such that

$$\rho(z) \le d \le n, \text{ if } |z| \ge r_0 \text{ and } z \notin \bigcup_{\beta \in T[n+1,q]} \{z \mid \mathbf{g}_{\beta(1)}(z) \wedge \cdots \wedge \mathbf{g}_{\beta(n+1)}(z) = 0\}. \tag{1.2}$$

The following results are obtained in this paper.

Theorem 1. *Let $f_1, f_2, \ldots, f_\lambda : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be non-constant holomorphic curves. Let $g_j : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}^*)$ be moving targets located in general position and $T_{g_j}(r) = o(\max_{1 \le t \le \lambda}\{T_{f_t}(r)\})$, $1 \le j \le q$. Assume that $\mathbf{g}_j(\mathbf{f}_t) \not\equiv 0$ for $1 \le j \le q, 1 \le t \le \lambda$, and $(\mathbf{g}_j(\mathbf{f}_1))^{-1}\{0\} = \cdots = (\mathbf{g}_j(\mathbf{f}_\lambda))^{-1}\{0\}$. Denote by $A_j = (\mathbf{g}_j(\mathbf{f}_1))^{-1}\{0\}$ and let $A = \bigcup_{j=1}^q A_j$. Let l, $2 \le l \le \lambda$, be an integer such that for any increasing sequence $1 \le j_1 < j_2 < \cdots < j_l \le \lambda$, $\mathbf{f}_{j_1}(z) \wedge \cdots \wedge \mathbf{f}_{j_l}(z) = 0$ for every point $z \in A$. If $q > \frac{dn^2(2n+1)\lambda}{\lambda - l + 1}$, then $f_1, \ldots, f_\lambda$ are algebraically dependent over $\mathbb{C}$, i.e, $\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_\lambda \equiv 0$ on $\mathbb{C}$.*

When $n = 1$ and $\lambda = 2$, the above theorem yields the following result.

Corollary. *Given two non-constant meromorphic functions f, g, assume that there exist seven meromorphic functions $a_1, \ldots, a_7$, with $a_i \not\equiv a_j$ for $i \ne j$, such that $T_{a_j}(r) = o(\max\{T_f(r), T_g(r)\})$, $1 \le j \le 7$, and such that $f(z) = a_j(z) \Leftrightarrow g(z) = a_j(z)$ for $1 \le j \le 7$. Then $f \equiv g$.*

We note that Qinde Zhang [Z] had a better result of six small functions, instead of seven.

If we further assume that $f_t, 1 \le t \le \lambda$, are linearly non-degenerated, then we have a better result as follows.

Theorem 2. *In addition to the assumptions in Theorem 1 we assume further that $f_t, 1 \le t \le \lambda$, are linearly non-degenerated. Then $f_1, \ldots, f_\lambda$ are algebraically dependent over $\mathbb{C}$ (in the sense that $\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_\lambda \equiv 0$ on $\mathbb{C}$) if $q > \frac{dn(n+2)\lambda}{\lambda - l + 1}$.*

2. Proof of Theorem 1 and Theorem 2

We first recall some standard notation and definitions in Nevanlinna theory (cf. [St]).

Let $f = [f_0 : ... : f_n] : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, where $f_0, ..., f_n$ are entire functions without common zeros. The Cartan characteristic function of f is defined by

$$T_f(r) = \int_0^{2\pi} \log \max_i |f_i(re^{i\theta})| \frac{d\theta}{2\pi} - \log \max_i |f_i(0)|.$$

For holomorphic maps $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, $g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}^*)$, let $n_{f,g}(r)$ be the number of zeros of $\mathbf{g}(\mathbf{f})$ in $|z| < r$, counting multiplicity, and let $n^{(n)}_{f,g}(r)$ be the number of zeros of $\mathbf{g}(\mathbf{f})$ in $|z| < r$, where the zero multiplicity k is counted as usual if $k \le n$ and the zero multiplicity is counted only as n if $k > n$. The counting function is defined by

$$N_{f,g}(r) = \int_0^r \frac{n_{f,g}(t)}{t} dt - n_{f,g}(0) \log r,$$

and the truncated counting function is

$$N^{(n)}_{f,g}(r) = \int_0^r \frac{n^{(n)}_{f,g}(t)}{t} dt - n_{f,g}(0) \log r.$$

If f is a meromorphic function, we use $n^{(n)}_f(r, 0)$ to denote the number of zeros of f in $|z| < r$, where the zero multiplicity k is counted as usual if $k \le n$ and the zero multiplicity is counted only as n if $k > n$. Let $N^{(n)}_f(r, 0)$ be the corresponding truncated counting function.

We recall the following Borel's lemma.

Theorem 2.1. *Let $f = [f_0 : ... : f_n] : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, with $f_0, ..., f_n$ entire and no common zeros. Assume that f_{n+1} is a holomorphic function and $f_0 + \cdots + f_n + f_{n+1} = 0$. If $\sum_{i \in I} f_i \not\equiv 0$ for any proper subset I of $\{0, ..., n+1\}$, then*

$$T_f(r) \le \sum_{j=0}^{n+1} N^{(n)}_{f_j}(r, 0) + O_{exc}(\log r + \log^+ T_f(r))$$

for $r \to \infty$, where O_{exc} means the estimate holds except for r in a set of finite Lebesgue measure.

The proof of Theorem 2.1 is standard and can be found in [RW].

We then need the following "second main theorem type" inequality.

Theorem 2.2. *Let $f : \mathbb{C} \to \mathbb{P}^n$ be a non-constant holomorphic map. Let $g_j : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}^*)$ be moving targets in general position. Assume that $\mathbf{g}_j(\mathbf{f}) \not\equiv 0$ for $j = 1, \ldots, q$. If $q \ge 2n + 1$, then*

$$\frac{q}{n(2n+1)} T_f(r) \le \sum_{j=1}^{q} N^{(n)}_{f,g_j}(r) + O\left(\max_{1 \le j \le q} T_{g_j}(r)\right) + O_{exc}(\log r + \log^+ T_f(r)).$$

The proof of Theorem 2.2 also appeared in [RW]. We include the proof here for the completeness.

Proof of Theorem 2.2. Let $I \subset \{2, \ldots, q\}$ be the index set with the property that $i \in I$ if and only if

$$T_{\frac{\mathbf{g}_i(\mathbf{f})}{\mathbf{g}_1(\mathbf{f})}}(r) \le \sum_{j=1}^{q} N_{f,g_j}^{(n)}(r) + O\left(\max_{1\le j\le q} T_{g_j}(r)\right) + O_{exc}(\log r + \log^+ T_f(r)). \tag{2.1}$$

We first show that $\#I \ge n$. Assume that $I = \{2, \ldots, u\}$, and $u \le n$. For dimension reason, $\{\mathbf{g}_1, \mathbf{g}_{n+1}, \ldots, \mathbf{g}_{2n+1}\}$ is always linearly dependent over Hol($\mathbb{C}$), i.e.

$$a_1\mathbf{g}_1 + a_{n+1}\mathbf{g}_{n+1} + \cdots + a_{2n+1}\mathbf{g}_{2n+1} = 0,$$

where $a_1, a_{n+1}, \ldots, a_{2n+1}$ are holomorphic functions on $\mathbb{C}$. By solving the above linear equations for $a_1, a_{n+1}, \ldots, a_{n+2}$,

$$T_{[a_1:a_{n+1}:\cdots:a_{2n+1}]}(r) \le 2(n+1)\left(\max_{1\le j\le q} T_{g_j}(r)\right).$$

After rearranging the index we will have an equation

$$a_1\mathbf{g}_1(\mathbf{f}) + a_{n+1}\mathbf{g}_{n+1}(\mathbf{f}) + \cdots + a_m\mathbf{g}_m(\mathbf{f}) \equiv 0,$$

and no proper subsum is identically zero as a function of z where $m \ge n+1$; moreover we may assume without loss of generality that functions

$$a_1\mathbf{g}_1(\mathbf{f}), \ldots, a_m\mathbf{g}_m(\mathbf{f})$$

have no common zeros. Therefore, by Theorem 2.1, we conclude that

$$T_{\frac{\mathbf{g}_{n+1}(\mathbf{f})}{\mathbf{g}_1(\mathbf{f})}}(r) \le \sum_{j=1}^{q} N_{f,g_j}^{(n)}(r) + O\left(\max_{1\le j\le q} T_{g_j}(r)\right) + O_{exc}(\log r + \log^+ T_f(r)).$$

This contradicts the fact that $n+1$ is not in I. Thus $\#I \ge n$. So (2.1) implies

$$\begin{aligned} T_f(r) &\le \sum_{j=2}^{n+1} T_{\frac{\mathbf{g}_j(\mathbf{f})}{\mathbf{g}_1(\mathbf{f})}}(r) + O\left(\max_{1\le j\le q} T_{g_j}(r)\right) \\ &\le \sum_{j=1}^{q} nN_{f,g_j}^{(n)}(r) + O\left(\max_{1\le j\le q} T_{g_j}(r)\right) + O_{exc}(\log r + \log^+ T_f(r)). \end{aligned} \tag{2.2}$$

Denote by $\mathcal{L}_q$ the collection of moving hyperplanes $g_1, \ldots, g_q$ located in general position. We claim

$$\frac{q}{2n+1}T_f(r) \le \sum_{g\in\mathcal{L}_q} nN_{f,g}^{(n)}(r) + O\left(\max_{g\in\mathcal{L}_q} T_g(r)\right) + O_{exc}(\log r + \log^+ T_f(r)). \tag{2.3}$$

We will prove (2.3) by induction on q. When $q = 2n+1$, (2.3) is just (2.2), so (2.3) holds. We assume (2.3) holds for q and verify (2.3) for $q+1$. In fact, for $\mathcal{L}_{q+1}$, we can choose q moving hyperplanes at a time and apply (2.3). This gives $q+1$ inequalities as (2.3). Summing up these $q+1$ inequalities, we have

$$\frac{q+1}{2n+1}T_f(r) \le \sum_{g\in\mathcal{L}_{q+1}} nN_{f,g}^{(n)}(r) + O\left(\max_{g\in\mathcal{L}_{q+1}} T_g(r)\right) + O_{exc}(\log r + \log^+ T_f(r)).$$

This completes the proof of Theorem 2.2. $\square$

Proof of Theorem 1. We first apply Theorem 2.2 to $f_t, 1 \leq t \leq \lambda$, to get

(2.4)

$$\frac{q}{n(2n+1)} T_{f_t}(r) \leq \sum_{j=1}^{q} N^{(n)}_{f_t,g_j}(r) + O\left(\max_{1\leq j\leq q} T_{g_j}(r)\right) + O_{exc}(\log r + \log^+ T_{f_t}(r)).$$

Assume that $\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_\lambda \not\equiv 0$. We denote by $\mu_{\mathbf{f}_1\wedge\cdots\wedge\mathbf{f}_\lambda}$ the divisor associated with $\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_\lambda$. Denote by $N_{\mu_{\mathbf{f}_1\wedge\cdots\wedge\mathbf{f}_\lambda}}(r)$ the counting function associated with the divisor $\mu_{\mathbf{f}_1\wedge\cdots\wedge\mathbf{f}_\lambda}$. We make the following claim.

Claim. For every $1 \leq t \leq \lambda$ and $r \geq r_0$,

$$\sum_{j=1}^{q} N^{(n)}_{f_t,g_j}(r) \leq \frac{dn}{\lambda - l + 1} N_{\mu_{\mathbf{f}_1\wedge\cdots\wedge\mathbf{f}_\lambda}}(r) + \sum_{\beta} N_{\mu_{\mathbf{g}_{\beta(1)}\wedge\cdots\wedge\mathbf{g}_{\beta(n+1)}}}(r) \tag{2.5}$$

where the sum is over all injective maps $\beta : \{1, \ldots, n+1\} \to \{1, \ldots, q\}$.

We now prove the claim. Let $z \in A$. Then for any increasing sequence $1 \leq j_1 < j_2 < \cdots < j_l \leq \lambda$,

$$\mathbf{f}_{j_1}(z) \wedge \cdots \wedge \mathbf{f}_{j_l}(z) = 0. \tag{2.6}$$

We verify that $\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_\lambda$ vanishes at z with the vanishing order at least $\lambda - l + 1$. In fact, by the power series expansion for each component of $\mathbf{f}_i$, we can write, for $1 \leq i \leq \lambda$,

$$\mathbf{f}_i(\zeta) = \mathbf{a}_i + (\zeta - z)\mathbf{h}_i(\zeta),$$

where $\mathbf{a}_i$ is a constant vector and $\mathbf{h}_i(\zeta)$ is a holomorphic vector-valued function defined around z. Denote by $T[\alpha, \lambda]$ the set of all increasing injective maps from $\{1, 2, \ldots, \alpha\}$ to $\{1, 2, \ldots, \lambda\}$. For each $\eta \in T[\alpha, \lambda]$, there exists a unique $\hat{\eta} \in T[\lambda - \alpha, \lambda]$ such that $(Im\,\eta) \cap (Im\,\hat{\eta}) = \emptyset$. Abbreviate $\epsilon_\eta = \operatorname{sing}\eta$. (2.6) then implies that, for any $\eta \in T[l, \lambda]$,

$$\mathbf{a}_{\eta(1)} \wedge \cdots \wedge \mathbf{a}_{\eta(l)} = 0.$$

Thus,

$$\begin{aligned}\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_\lambda = &\sum_{\alpha=1}^{l-1} (\zeta - z)^{\lambda-\alpha} \sum_{\eta\in T[\alpha,\lambda]} \epsilon_\eta \left(\bigwedge_{j=1}^{\alpha} \mathbf{a}_{\eta(j)}\right) \wedge \left(\bigwedge_{k=1}^{\lambda-\alpha} \mathbf{h}_{\hat{\eta}(k)}\right)\\ &+ (\zeta - z)^\lambda \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_\lambda.\end{aligned}$$

The lowest exponent of $(\zeta - z)$ is $\lambda - l + 1$. This verifies that $\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_\lambda$ vanishes at z with the vanishing order at least $\lambda - l + 1$. This, together with the property of $\min\{m, n\} \leq n = \frac{n}{\lambda-l+1}(\lambda - l + 1)$ and the definition of d, implies that

$$\sum_{j=1}^{q} N^{(n)}_{f_t,g_j}(r) \leq \frac{dn}{\lambda - l + 1} N_{\mu_{\mathbf{f}_1\wedge\cdots\wedge\mathbf{f}_\lambda}}(r) + \sum_{\beta} N_{\mu_{\mathbf{g}_{\beta(1)}\wedge\cdots\wedge\mathbf{g}_{\beta(n+1)}}}(r).$$

So the Claim is proved.

We now proceed with our proof. By the First Main Theorem of the exterior product (cf. [St], P.327),

$$N_{\mu_{\mathbf{f}_1\wedge\cdots\wedge\mathbf{f}_\lambda}}(r) \leq \sum_{i=1}^{\lambda} T_{f_i}(r) + O(1) \text{ and } N_{\mu_{\mathbf{g}_{\beta(0)}\wedge\cdots\wedge\mathbf{g}_{\beta(n)}}}(r) \leq \sum_{j=1}^{q} T_{g_j}(r). \tag{2.7}$$

Combining (2.5) and (2.7) yields

$$\sum_{j=1}^{q} N^{(n)}_{f_t,g_j}(r) \le \frac{dn}{\lambda - l + 1}\sum_{i=1}^{\lambda} T_{f_i}(r) + O\left(\max_{1\le j\le q} T_{g_j}(r)\right) + O(1).$$

This, together with (2.4), gives, for $1 \le t \le \lambda$,

$$\frac{q}{n(2n+1)}T_{f_t}(r) \le \frac{dn}{\lambda - l + 1}\sum_{i=1}^{\lambda} T_{f_i}(r) + O\left(\max_{1\le j\le q} T_{g_j}(r)\right) + O_{exc}(\log r + \log^+ T_{f_t}(r)).$$

Thus, by summing them up,

$$\frac{q}{n(2n+1)}\sum_{t=1}^{\lambda} T_{f_t}(r) \le \frac{dn\lambda}{(\lambda - l + 1)}\sum_{t=1}^{\lambda} T_{f_t}(r) + O\left(\max_{1\le j\le q} T_{g_j}(r)\right) + O_{exc}(\log r + \sum_{t=1}^{\lambda}\log^+ T_{f_t}(r)),$$

which gives a contradiction under the assumption that

$$q > \frac{dn^2(2n+1)\lambda}{\lambda - l + 1}.$$

This completes the proof of Theorem 1. □

To prove Theorem 2, we need replace Theorem 2.2 with Theorem 2.3.

Theorem 2.3. *Let $f : \mathbb{C} \to \mathbb{P}^n$ be a linearly non-degenerated holomorphic map. Let $g_j : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}^*)$ be moving targets located in general position. If $q \ge n+2$, then*

$$\frac{q}{n+2}T_f(r) \le \sum_{j=1}^{q} N^{(n)}_{f,g_j}(r) + O\left(\max_{1\le j\le q} T_{g_j}(r)\right) + O_{exc}(\log r + \log^+ T_f(r)). \tag{2.8}$$

Proof of Theorem 2.3. Denote by $\mathcal{L}_q$ the collection of moving hyperplanes $g_1, \ldots, g_q$ located in general position. We will prove (2.8) by induction on q. For any $n+2$ moving hyperplanes $g_1, \ldots, g_{n+2}$ located in general position, we have the following result due to S. Mori [M]:

$$T_f(r) \le \sum_{j=1}^{n+2} N^{(n)}_{f,g_j}(r) + O\left(\max_{1\le j\le q} T_{g_j}(r)\right) + O_{exc}(\log r + \log^+ T_f(r)). \tag{2.9}$$

We note that the method of proving (2.9) is to transform the moving hyperplanes to coordinate hyperplanes. For example, when n=1, the transformation is just like

$$\frac{f - g_1}{f - g_2}\cdot\frac{g_3 - g_2}{g_3 - g_1}.$$

So (2.8) holds for $q = n+2$. We now assume (2.8) holds for q, and verify (2.8) for $q+1$. In fact, for $\mathcal{L}_{q+1}$, we can choose q moving targets at a time and apply (2.8). This gives $q+1$ inequalities as (2.8). Summing up these $q+1$ inequalities, we have

$$\frac{q+1}{n+2}T_f(r) \le \sum_{g\in\mathcal{L}_{q+1}} N^{(n)}_{f,g}(r) + O\left(\max_{g\in\mathcal{L}_{q+1}} T_g(r)\right) + O_{exc}(\log r + \log^+ T_f(r)).$$

This completes the proof of Theorem 2.3. □

Proof of Theorem 2. Assume that $\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_\lambda \not\equiv 0$. By the same method in the proof of Theorem 1, replacing Theorem 2.2 by Theorem 2.3, we can derive

$$\sum_{t=1}^{\lambda} T_{f_t}(r) \leq \frac{dn(n+2)\lambda}{q(\lambda - l + 1)} \sum_{t=1}^{\lambda} T_{f_t}(r) + O\left(\max_{1\leq j\leq q} T_{g_j}(r)\right) + O_{exc}(\log r + \sum_{t=1}^{\lambda} \log^+ T_{f_t}(r)),$$

which gives a contradiction under the assumption that

$$q > \frac{dn(n+2)\lambda}{\lambda - l + 1}.$$

This completes the proof of Theorem 2. □

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