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WEAK HARNACK'S INEQUALITY FOR NON-NEGATIVE SOLUTIONS OF ELLIPTIC EQUATIONS WITH POTENTIAL

AHMED MOHAMMED

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Abstract. We present an alternative and shorter proof to a weak Harnack inequality for non-negative solutions of divergence structure elliptic equations with potentials from the Kato class.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, and let

$$L := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right), \quad x \in \Omega,$$

be a uniformly elliptic operator, where $A(x) := (a_{ij}(x))$ is a symmetric matrix with real-valued measurable entries satisfying the uniform ellipticity condition

$$\lambda|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda|\xi|^2, \qquad x \in \Omega, \quad \xi \in \mathbb{R}^n,$$

for some $0 < \lambda \le \Lambda$.

Given $V \in L^1_{loc}(\Omega)$, we say that $u \in H^1(\Omega)$ is a weak solution of $Lu + Vu = 0$ in Ω if and only if $Vu \in L^1_{loc}(\Omega)$ and

$$\int_\Omega \langle A\nabla u, \nabla\psi\rangle + \int_\Omega Vu\psi = 0 \quad \text{for all } \psi \in H^1_0(\Omega).$$

In the important paper [1], F. Chiarenza, E. Fabes and N. Garofalo developed a real variable method to prove Harnack's inequality for non-negative weak solutions of $Lu + Vu = 0$, where the potential V belongs to the Kato class (see definition below). These techniques were subsequently used by several authors to prove Harnack's inequality for other types of uniformly elliptic and degenerate elliptic operators (see, for example, [2, 4] for such results).

Our purpose here is to suggest an alternative and a shorter proof of the Weak Harnack Inequality obtained by F. Chiarenza, E. Fabes, N. Garofalo in [1]. We also avoid the use of a deep result on weights that was used in [1]. We should mention here that in [4] Kurata gives another proof of the Weak Harnack's Inequality without resort to the methods of [1]. His method rests on first showing that $|\nabla u|^2$ belongs to a local Kato class whenever u is a weak solution of $Lu + Vu = 0$.

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Let us now fix some fairly standard notations. The dependence of a constant C on parameters α, β, γ, etc., will be denoted by $C(\alpha, \beta, \gamma, \cdots)$. Sometimes the dependence may be supressed if this is clear from the context. $B_r(x)$ will represent a ball of radius r and centered at x. We will write B_r for such a ball, if the center is understood. Given a ball B we write tB for the ball concentric with B, but of radius t times as big. The average value of f over B will be denoted by

$$\fint_B f(x)\,dx := \frac{1}{|B|}\int_B f(x)\,dx.$$

Finally, for a real-valued function f, we write $f^+ := \max\{f, 0\}$.

As the concept of approximate Green function is used in the proof let us briefly recall the definition and some of the basic properties. For further details and proofs we refer the reader to [3].

Corresponding to $x_0 \in \Omega$, and $\rho > 0$ with $B_\rho := B_\rho(x_0) \subseteq \Omega$, there is $G^\rho := G^\rho(x_0, .) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega \langle A\nabla G^\rho, \nabla\psi\rangle = \fint_{B_\rho} \psi \quad \text{for any } \psi \in H_0^1(\Omega). \tag{1.1}$$

G^ρ will be called an approximate Green function of L on Ω with pole x_0. The following properties of an approximate Green function G^ρ and the Green function G of L on Ω will prove useful in our subsequent discussion:

$$G^\rho(x_0, y) \to G(x_0, y) \qquad \text{for} \quad \rho \to 0, \quad \text{and } y \neq x_0, \tag{1.2}$$

$$0 \le G^\rho(x_0, y) \le C_1(n, \lambda, \Lambda)|x_0 - y|^{2-n} \qquad \text{if } 0 < \rho < \operatorname{dist}(x_0, \partial\Omega), \tag{1.3}$$

$$C_2(n, \lambda, \Lambda)|x - y|^{2-n} \le G(x, y) \qquad \text{for } |x - y| \le \frac{2}{3}\operatorname{dist}(x, \partial\Omega). \tag{1.4}$$

Given $V \in L^1_{loc}(\Omega)$, we let

$$\eta(V)(r) := \sup_{x\in\Omega}\int_{B_r(x)\cap\Omega} |V(y)||x - y|^{2-n}\,dy.$$

V is said to be in the Kato class $K_n(\Omega)$ if $\eta(V)(r) \to 0$ as $r \to 0^+$.

We shall need the following mean-value inequality from [1].

Theorem 1.1. *Let u be a weak solution of $Lu + Vu = 0$ in Ω, and let B be a ball of radius r with $2B \subseteq \Omega$. Given $0 < p < \infty$, there are positive constants $\delta = \delta(n, \lambda, \Lambda)$, $C = C(n, p, \lambda, \Lambda)$ such that*

$$\sup_B |u| \le C\left(\fint_{2B} |u|^p\right)^{\frac{1}{p}},$$

whenever $\eta(V)(r) \le \delta$.

2. Proof of the Weak Harnack's Inequality

Let $J : \mathbb{R} \to \mathbb{R}$ be a smooth function. In our next lemma we consider weak solutions of $Lu + VJ(u) = 0$ in Ω such that $0 \le J(u) \le u$ in Ω. The Lemma is proved in the same way as the corresponding Lemma in [1] (see also [2, 4]).

Lemma 2.1. *Let u be a non-negative weak solution of $Lu + VJ(u) = 0$ for which $0 \le J(u) \le u$. If $\eta_0 = \eta(V)(r_0) < \infty$ for some r_0, then there exists a constant*

$C = C(n, \lambda, \Lambda, \eta_0)$ *such that*

$$\fint_{B_r} \left| \log u - \fint_{B_r} \log u \, dx \right|^2 dx \le C$$

for $B_{2r} \subseteq \Omega$, *with* $0 < r \le r_0$.

The above Lemma, together with John-Nirenberg's lemma, shows that there is $\alpha = \alpha(n, \lambda, \Lambda)$ with $0 < \alpha < 1$ such that

$$\left(\fint_{B_r} u^\alpha \, dx \right) \left(\fint_{B_r} u^{-\alpha} \, dx \right) \le C \quad \text{for } B_{4r} \subseteq \Omega. \tag{2.1}$$

Remark 2.2 (Doubling property). From (2.1), it easily follows that

$$\int_{B_{2r}} u^\alpha \le C \int_{B_r} u^\alpha,$$

whenever $B_{4r} \subseteq \Omega$.

Theorem 2.3 (Weak Harnack's Inequality). *Let u be a non-negative weak solution of $Lu + Vu = 0$ in Ω, and let B be a ball of radius r such that $4B \subseteq \Omega$. Then there are positive constants $\delta_0 = \delta_0(n, \lambda, \Lambda)$ and $C = C(n, \lambda, \Lambda, \eta)$ such that*

$$\left(\fint_B u^\alpha \, dx \right)^{\frac{1}{\alpha}} \le C \inf_B u,$$

whenever $\eta(V)(r) \le \delta_0$. Here α is the constant in (2.1).

Proof. For $t > 0$, put

$$\Omega_t^\rho(x_0) = \{x \in \Omega : G^\rho(x) > t\} \text{ and } \quad \Omega_t(x_0) = \{x \in \Omega : G(x_0, x) > t\},$$

where G^ρ is an approximate Green function of L on Ω with pole x_0 and G is the Green function of L on Ω. We will write Ω_t^ρ and Ω_t for these sets respectively. First we claim that given $0 < \tau < 1$ there is a positive constant $C = C(\tau, \lambda, \Lambda)$ such that for any $t > 0$

$$\int_{\Omega_t^\rho} \left| \nabla \left(u^{\tau/2} \log^+(G^\rho/t) \right) \right|^2 \le \frac{C}{t} \left[\int_{\Omega_t^\rho} |V| G^\rho u^\tau + \fint_{B_\rho(x_0)} u^\tau \right]. \tag{2.2}$$

We first prove the claim for a solution of $Lu + VJ(u) = 0$ such that $0 \le J(u) \le u$ and $\inf_\Omega u > 0$. In the sequel we will use the notation Γ_t^ρ for the function

$$\left(\frac{G^\rho}{t} - 1 \right)^+ - \log^+ \left(\frac{G^\rho}{t} \right).$$

Since $(\log^2 s)/2 \le s - 1 - \log s$ for $s \ge 1$, let us first observe that

$$\frac{1}{2} \left[\log^+(G^\rho/t) \right]^2 \le \Gamma_t^\rho \le G^\rho/t \tag{2.3}$$

and that Γ_t^ρ is supported on Ω_t^ρ for all $t > 0$.

In the definition (1.1) of the approximate Green function, we take

$$\psi := \left(\frac{1}{t} - \frac{1}{G^\rho} \right)^+ u^\tau$$

as a test function (recall that $\inf_\Omega u > 0$). Then we find that

$$\int_{\Omega_t^\rho} \langle A\nabla G^\rho, \nabla G^\rho\rangle \frac{u^\tau}{(G^\rho)^2} + \tau \int_\Omega \langle A\nabla G^\rho, \nabla u\rangle \left(\frac{1}{t} - \frac{1}{G^\rho}\right)^+ u^{\tau-1}$$
$$= \fint_{B_\rho(x_0)} \left(\frac{1}{t} - \frac{1}{G^\rho}\right)^+ u^\tau. \tag{2.4}$$

Using

$$\nabla(\Gamma_t^\rho u^{\tau-1}) + (1-\tau)u^{\tau-2}\Gamma_t^\rho \nabla u = u^{\tau-1}\left(\frac{1}{t} - \frac{1}{G^\rho}\right)^+ \nabla G^\rho$$

in (2.4), followed by an application of (2.3), we find that

$$\int_{\Omega_t^\rho} \langle A\nabla G^\rho, \nabla G^\rho\rangle \frac{u^\tau}{(G^\rho)^2} + \frac{2(1-\tau)}{\tau} \int_{\Omega_t^\rho} \langle A\nabla(u^{\tau/2}), \nabla(u^{\tau/2})\rangle \left[\log^+(G^\rho/t)\right]^2$$
$$\leq \fint_{B_\rho(x_0)} \frac{u^\tau}{t} - \tau \int_\Omega \langle A\nabla u, \nabla(\Gamma_t^\rho u^{\tau-1})\rangle .$$

That is,

$$\int_{\Omega_t^\rho} \left|\nabla\left(u^{\tau/2}\log^+(G^\rho/t)\right)\right|^2 \leq C(\tau,\lambda)\left[\fint_{B_\rho(x_0)} \frac{u^\tau}{t} - \tau\int_\Omega VJ(u)\Gamma_t^\rho u^{\tau-1}\right].$$

Recalling that $0 \leq J(u) \leq u$, and using (2.3), we get (2.2).

Now let u be a non-negative weak solution of $Lu + Vu = 0$. Then for any $\epsilon > 0$, and $J(s) = s - \epsilon$, we see that $w := u + \epsilon$ is a weak solution of $Lw + VJ(w) = 0$ in Ω with $0 \leq J(w) \leq w$ such that $\inf_\Omega w > 0$. Therefore using the above conclusion for w, letting $\epsilon \to 0$, and using the fact that u is locally bounded, we apply Fatou's Lemma and the Lebesgue dominated convergence theorem to get the full statement of the claim.

The following inclusions are direct consequences of the inequalities (1.3) and (1.4):

$$B_{R_2}(x_0) \subseteq \Omega_t(x_0) \quad \text{for } \ t \geq C_2\left(\frac{2}{3}\mathrm{dist}(x_0,\partial\Omega)\right)^{2-n}, \tag{2.5}$$

and

$$\Omega_t^\rho(x_0) \subseteq B_{R_1}(x_0) \quad \text{for sufficiently small } \ \rho. \tag{2.6}$$

In the above inclusions we have used the notations

$$R_j = \left(\frac{C_j}{t}\right)^{\frac{1}{n-2}} \quad \text{for } \ j = 1, 2,$$

and C_1, and C_2 are the constants in (1.3) and (1.4), respectively.

Since $u, G^\rho \in L^\infty_{loc}(\Omega)$, we apply the Sobolev inequality to (2.2) to obtain the following chain of inequalities:

$$\frac{C}{R_1^2}\int_{\Omega^\rho_{2t}} |\log^+(G^\rho/t)|^2 u^\tau \le \frac{C}{R_1^2}\int_{\Omega^\rho_t} |\log^+(G^\rho/t)|^2 u^\tau$$
$$\le \int_{\Omega^\rho_t} |\nabla\left(u^{\tau/2}\log^+(G^\rho/t)\right)|^2 \le \frac{C}{t}\left[\int_{\Omega^\rho_t} |V|G^\rho u^\tau + \frac{1}{|B_\rho|}\int_{B_\rho(x_0)} u^\tau\right].$$

Using (1.3), and (2.6) in the last inequality, we obtain

$$\frac{1}{R_1^2}\int_{\Omega^\rho_{2t}} u^\tau \le \frac{C}{t}\sup_{B_{R_1}}(u^\tau)\int_{B_{R_1}} |V|(y)|x_0-y|^{2-n}\,dy + \frac{C}{|B_\rho|}\int_{B_\rho(x_0)}\frac{u^\tau}{t}.$$

As a result of (1.2), we observe that $\chi_{\Omega_t} \le \liminf_\rho \chi_{\Omega^\rho_t}$. From this and Fatou's lemma we deduce, on letting $\rho \to 0$,

$$\frac{1}{R_1^2}\int_{\Omega_{2t}} u^\tau \le \frac{C}{t}\left[\eta(V)(R_1)\sup_{B_{R_1}}(u^\tau) + u^\tau(x_0)\right],$$

and thus by (2.5) we obtain

$$\frac{1}{R_1^2}\int_{B_{R_2}} u^\tau \le \frac{C}{t}\left[\eta(V)(R_1)\sup_{B_{R_1}}(u^\tau) + u^\tau(x_0)\right].$$

Now, let $r > 0$ such that $B_{4r}(x_0) \subseteq \Omega$. We choose t such that $t = \max\{C_1, C_2\}r^{2-n}$, and we take $\tau := \alpha$, α being the constant in (2.1). Observe that (2.5) holds. If $C_2 \ge C_1$, then $R_2 = r$ and $R_1 \le r$. If $C_2 < C_1$, then $R_2 = (C_2/C_1)^{1/(n-2)}r$ and $R_1 = r$. In either case, we use the doubling property (Remark 2.2) of u^α and Theorem 1.1 to conclude that

$$\fint_{B_r} u^\alpha \le \eta(V)(r)\fint_{B_r} u^\alpha + Cu^\alpha(x_0).$$

By choosing r sufficiently small, we conclude that

$$\fint_{B_r} u^\alpha \le Cu^\alpha(x_0)$$

for some constant $C = C(n, \lambda, \Lambda, \eta)$. Now if $x \in B_r(x_0)$, then $B_r(x_0) \subseteq B_{2r}(x)$ and thus

$$\fint_{B_r(x_0)} u^\alpha \le C\fint_{B_{2r}(x)} u^\alpha \le Cu^\alpha(x),$$

which gives the desired result.

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Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122

E-mail address: ahmed@math.temple.edu