ROUGH SINGULAR INTEGRALS ASSOCIATED TO SURFACES OF REVOLUTION

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Abstract. Let $1 < p < \infty$ and $n \geq 2$. The authors establish the $L^p(\mathbb{R}^{n+1})$-boundedness for a class of singular integral operators associated to surfaces of revolution, $\{(t, \phi(|t|)) : t \in \mathbb{R}^n\}$, with rough kernels, provided that the corresponding maximal function along the plane curve $\{(t, \phi(|t|)) : t \in \mathbb{R}\}$ is bounded on $L^p(\mathbb{R}^2)$.

1. Introduction

Let $n \geq 2$ and $y \in \mathbb{R}^n$. For the Calderón-Zygmund type kernel

$$K(y) = \frac{\Omega(y)}{|y|^n} b(|y|),$$

and a suitable function $\phi$ on $[0, \infty)$, we define the singular integral operator $T$ along the surface

$$\Gamma = \{(y, \phi(|y|)) : y \in \mathbb{R}^n\}$$

by

$$Tf(x,s) = \text{p.v.} \int_{\mathbb{R}^n} f(x - y, s - \phi(|y|)) K(y) \, dy.$$  

Here and in what follows, we always assume that $b$ is a measurable function on $[0, \infty)$, $\Omega$ is homogeneous of order zero on $\mathbb{R}^n$, integrable on $\mathbb{S}^{n-1}$ and satisfies

$$\int_{\mathbb{S}^{n-1}} \Omega(y) \, d\sigma(y) = 0.$$  

The kernel $K(y)$, which has radial roughness introduced by the factor $b(|y|)$, was first studied by R. Fefferman in the context of singular integrals on $\mathbb{R}^n$ ([9]).

In [10], Kim, Wainger, Wright and Ziesler proved the following theorem.
Theorem A (10). Let \( \phi \in C^2((0, \infty)) \) be convex, increasing and \( \phi(0) = 0 \). Let \( \Omega \in \mathcal{C}^\infty(S^{n-1}) \) satisfy (2) and \( b \equiv 1 \). Then \( T \) in (1) is bounded on \( L^p(\mathbb{R}^{n+1}) \) for \( 1 < p < \infty \).

In (3), Chen and Fan generalized the above result by requiring that \( \Omega \) belongs to a Block space introduced in (11) and \( b \in L^\infty([0, \infty)) \).

Theorem B (3). Suppose \( \Omega \in B^\beta_r(S^{n-1}) \) for some \( \beta > 0 \) and \( r > 1 \). If the maximal operator \( \nu_\phi \) on \( \mathbb{R} \) given by

\[
(\nu_\phi g)(x) = \sup_{k \in \mathbb{Z}} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |g(x - \phi(t))| dt
\]

is a bounded operator on \( L^p(\mathbb{R}) \) for \( 1 < p < \infty \), then \( T \) is bounded on \( L^p(\mathbb{R}^{n+1}) \) for \( 1 < p < \infty \).

The main purpose of this paper is to consider the \( L^p \) boundedness of \( T \) when \( \Omega \in \mathcal{H}^1(S^{n-1}) \), the Hardy space on the sphere; see \([5]\) and \([4]\) for the definition. The method that we use in this paper comes from the work of Duoandikoetxea and Rubio de Francia (6) and its extension obtained in Fan-Pan (8).

To state our main result, we need to introduce the maximal function \( \mathcal{M}_\phi \) associated to the plane curve \( \{(x, \phi(|x|)) : x \in \mathbb{R}\} \). For any measurable function \( f \) on \( \mathbb{R}^2 \), \( \mathcal{M}_\phi f \) is defined by

\[
(\mathcal{M}_\phi f)(x_1, x_2) = \sup_{k \in \mathbb{Z}} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |f(x_1 - t, x_2 - \phi(|t|))| dt.
\]

Here is our main theorem.

Theorem 1. Let \( \phi : [0, \infty) \to \mathbb{R} \) be continuously differentiable on \((0, \infty)\) and satisfy

\[
|\phi(t) - \phi(0)| \leq C t^\alpha
\]

for some \( \alpha > 0 \) and small \( t \), where \( C \) is a constant independent of \( t \). Let \( \Omega \in \mathcal{H}^1(S^{n-1}) \), \( b \in L^\infty([0, \infty)) \) and \( T \) be given by (1). Then \( T \) is bounded on \( L^p(\mathbb{R}^{n+1}) \) for \( 1 < p < \infty \), provided that \( \mathcal{M}_\phi \) in (3) is bounded on \( L^p(\mathbb{R}^2) \).

The condition imposed on \( \phi(t) \) for \( t \sim 0 \) ensures that the integral in (1) exists in principle-value sense when, say, \( f \in S(\mathbb{R}^{n+1}) \).

The \( L^p(\mathbb{R}^2) \) boundedness of \( \mathcal{M}_\phi \) is known for many \( \phi \)'s. Below we shall mention a few prominent cases:

(i) If \( \phi \) is a real-valued polynomial, then \( \mathcal{M}_\phi \) is bounded on \( L^p(\mathbb{R}^2) \) for \( p > 1 \); see \([13]\).

(ii) Let \( h(t) = t \phi'(t) - \phi(t) \) for \( t > 0 \). If \( \phi : \mathbb{R} \to \mathbb{R} \) is of class \( C^2(0, \infty) \), convex on \([0, \infty)\) and \( \phi(0) = \phi'(0) = 0 \) and there exists an \( \varepsilon > 0 \) so that for each \( t > 0 \), \( h''(t) > \varepsilon h(t)/t \), then \( \mathcal{M}_\phi \) in (3) is bounded on \( L^p(\mathbb{R}^2) \) for \( p > 1 \); see Theorem 1.5 in \([2]\). Moreover, if \( \phi \) is either even or odd, convex on \([0, \infty)\), and there exists a \( 0 < C < \infty \) so that for each \( t > 0 \), \( \phi''(Ct) \geq 2\phi'(t) \), then \( \mathcal{M}_\phi \) in (3) is bounded on \( L^p(\mathbb{R}^2) \) for \( p > 1 \). For details, see \([11]\) or \([2]\).

(iii) For \( \phi(t) = t^\alpha \) with \( \alpha \in (0, 1] \), \( \mathcal{M}_\phi \) is bounded on \( L^p(\mathbb{R}^2) \) for \( p > 1 \); see \([12]\).
2. Proof of Theorem 1

We begin with the definition of the space $H^1(S^{n-1})$. For $f \in L^1(S^{n-1})$ and $x \in S^{n-1}$, we define

$$P^+ f(x) = \sup_{0 < t < 1} \left| \int_{S^{n-1}} P_{tx}(y) f(y) \, d\sigma(y) \right|,$$

where

$$P_{tx}(y) = \frac{1 - t^2}{|y - tx|^n}$$

for $y \in S^{n-1}$.

**Definition 1.** An integrable function $f$ on $S^{n-1}$ is in the space $H^1(S^{n-1})$ if and only if

$$\|P^+ f\|_{L^1(S^{n-1})} = \int_{S^{n-1}} |P^+ f(x)| \, d\sigma(x) < \infty$$

and we define

$$\|f\|_{H^1(S^{n-1})} = \|P^+ f\|_{L^1(S^{n-1})}.$$

A very useful characterization of the space $H^1(S^{n-1})$ is its atomic decomposition. Let us first recall the definition of atoms.

**Definition 2.** A function $a(\cdot)$ on $S^{n-1}$ is a regular atom if there exist $\xi \in S^{n-1}$ and $\rho \in (0, 2]$ such that

(i) $\text{supp} \, a \subset S^{n-1} \cap B(\xi, \rho)$, where $B(\xi, \rho) = \{ y \in \mathbb{R}^n : |y - \xi| < \rho \}$;

(ii) $\|a\|_{L^\infty(S^{n-1})} \leq \rho^{-n+1}$;

(iii) $\int_{S^{n-1}} a(y) \, d\sigma(y) = 0$.

A function $a(\cdot)$ on $S^{n-1}$ is an exceptional atom if $a(\cdot) \in L^\infty(S^{n-1})$ and

$$\|a\|_{L^\infty(S^{n-1})} \leq 1.$$

The following can be found in [5] and [4].

**Lemma 1.** For any $f \in H^1(S^{n-1})$ there are complex numbers $\lambda_j$ and atoms (regular or exceptional) $a_j$ such that

$$f = \sum_j \lambda_j a_j$$

and

$$\|f\|_{H^1(S^{n-1})} \sim \sum_j |\lambda_j|.$$

The following lemma is a simple corollary of Theorem B.

**Lemma 2.** Let $\phi$ be the same as in Theorem 1. Let $\Omega \in L^r(S^{n-1})$ for some $1 < r \leq \infty$, $n \geq 2$ and $T$ be given by (1). Then $T$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, provided that $M_\phi$ in (3) is bounded on $L^p(\mathbb{R}^2)$.
boundedness of the maximal operator \( \mathcal{M}_\phi \) implies the \( L^p(\mathbb{R}) \) boundedness of the maximal operator \( \nu_\phi \).

For \( N \in \mathbb{N} \), let

\[
(\nu_\phi^N g)(x) = \sup_{-\infty < k \leq N} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |g(x - \phi(t))| \, dt.
\]

Then for \( f(x, y) = \chi_{[0,2^{N+2}]}(x)g(y) \),

\[
\chi_{[0,2^{N+1}]}(x)(\nu_\phi^N g)(y) \leq (\mathcal{M}_\phi f)(x, y).
\]

Thus

\[
2^{(N+1)/p}\|\nu_\phi^N g\|_{L^p(\mathbb{R})} \leq \|\mathcal{M}_\phi f\|_{L^p(\mathbb{R}^2)} \leq C_p\|f\|_{L^p(\mathbb{R}^2)} = C_p2^{(N+2)/p}\|g\|_{L^p(\mathbb{R})}.
\]

By letting \( N \to \infty \) (after dividing both sides by \( 2^{(N+1)/p} \)), we obtain

\[
\|\nu_\phi g\|_{L^p(\mathbb{R})} \leq C_p2^{1/p}\|g\|_{L^p(\mathbb{R})}.
\]

This finishes the proof of Lemma 2. \( \square \)

The following lemma in [7] is one of our main tools.

**Lemma 3.** Let \( l, m \in \mathbb{N} \) and \( \{\sigma_{s,k} : 0 \leq s \leq l \text{ and } k \in \mathbb{Z}\} \) be a family of measures on \( \mathbb{R}^m \) with \( \sigma_{0,0} = 0 \) for every \( k \in \mathbb{Z} \). Let \( \{\alpha_j : 1 \leq s \leq l \text{ and } 1 \leq j \leq 2\} \subset (0, \infty) \), \( \{\eta_k : 1 \leq s \leq l \} \subset (0, \infty) \setminus \{1\} \), \( \{M_s : 1 \leq s \leq l \} \subset \mathbb{N} \), and \( L_s : \mathbb{R}^m \to \mathbb{R}^M \) be linear transformations for \( 1 \leq s \leq l \). Suppose

(i) \( \|\sigma_{s,k}\| \leq 1 \text{ for } k \in \mathbb{Z} \text{ and } 1 \leq s \leq l; \)

(ii) \( |\sigma_{s,k}(\xi)| \leq C(\eta_k^s L_s \xi)^{-\alpha_{s,j}} \text{ for } \xi \in \mathbb{R}^m, k \in \mathbb{Z} \text{ and } 1 \leq s \leq l; \)

(iii) \( |\sigma_{s,k} - \sigma_{s-1,k}(\xi)| \leq C(\eta_k^s L_s \xi)^{\alpha_{s,j}} \text{ for } \xi \in \mathbb{R}^m, k \in \mathbb{Z} \text{ and } 1 \leq s \leq l; \)

(iv) For some \( q > 1 \), there exists \( A_q > 0 \) such that

\[
\left\| \sup_{k \in \mathbb{Z}} |\sigma_{s,k} * f| \right\|_{L^q(\mathbb{R}^m)} \leq A_q\|f\|_{L^q(\mathbb{R}^m)}
\]

for all \( f \in L^q(\mathbb{R}^m) \) and \( 1 \leq s \leq l \).

Then for every \( p \in \left( \frac{2q}{q+1}, \frac{2q}{q-1} \right) \), there exists a positive constant \( C_p \) such that

(a) \( \left\| \sum_{k \in \mathbb{Z}} \sigma_{l,k} * f \right\|_{L^p(\mathbb{R}^m)} \leq C_p\|f\|_{L^p(\mathbb{R}^m)} \)

and

(b) \( \left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{l,k} * f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)} \leq C_p\|f\|_{L^p(\mathbb{R}^m)} \)

hold for all \( f \in L^p(\mathbb{R}^m) \). The constant \( C_p \) is independent of the linear transformations \( \{L_s\}_{s=1}^l \).

The following result is just Lemma 5.1 in [7], which follows immediately from Lemma 6.2 in [8] and is an extension of an earlier theorem due to Duoandikoetxea and Rubio de Francia in [7].

**Lemma 4.** Let \( s, m \in \mathbb{N}, \eta \in (0, \infty) \setminus \{1\}, \delta_1, \delta_2 > 0 \), and \( L : \mathbb{R}^s \to \mathbb{R}^m \) be a linear transformation. Suppose that \( \{\sigma_k\}_{k \in \mathbb{Z}} \) is a sequence of measures on \( \mathbb{R}^m \)
satisfying:

(i) \( \|\sigma_k\| \leq 1 \) for \( k \in \mathbb{Z} \);
(ii) \( \overline{\sigma_k}(\xi) \leq C[\min\{(\eta^k|L\xi|)^{\delta_1}, (\eta^k|L\xi|)^{-\delta_2}\}] \) for \( \xi \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \);
(iii) For some \( q > 1 \), there exists \( A_q > 0 \) such that

\[
\|\sigma^s(f)\|_{L^q(\mathbb{R}^m)} = \left\| \sup_{k \in \mathbb{Z}} |\sigma_k| \ast f \right\|_{L^q(\mathbb{R}^m)} \leq A_q \|f\|_{L^q(\mathbb{R}^m)}
\]

for all \( f \in L^q(\mathbb{R}^m) \).

Then for \( p \in \left( \frac{2q}{q+1}, \frac{2q}{q-1} \right) \), there exists a positive constant \( C_p = C(p, s, m, \eta, \delta_1, \delta_2) \) such that

(a) \[
\left\| \sum_{k \in \mathbb{Z}} \sigma_k \ast f \right\|_{L^p(\mathbb{R}^m)} \leq C_p \|f\|_{L^p(\mathbb{R}^m)}
\]

and

(b) \[
\left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_k \ast f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)} \leq C_p \|f\|_{L^p(\mathbb{R}^m)}
\]

hold for all \( f \in L^p(\mathbb{R}^m) \). The constant \( C_p \) is independent of the linear transformation \( L \).

In order to handle truncation in the phase space, we need the following useful lemma, which is Lemma 6.4 in [8].

Lemma 5. For \( s \leq d \), let \( H : \mathbb{R}^s \to \mathbb{R}^s \) and \( G : \mathbb{R}^d \to \mathbb{R}^d \) be two nonsingular linear transformations and \( \varphi \in \mathcal{S}(\mathbb{R}^s) \). Define \( J \) and \( X_r = X_r(\varphi, G, H) \) by

\[
(Jf)(x) = f(G^t(H^t \otimes \text{id}_{\mathbb{R}^{d-s}}))(x)
\]

and

\[
X_r f(x) = J^{-1}((|\Phi_r| \otimes \delta_{\mathbb{R}^{d-s}}) \ast f)(x),
\]

where \( x \in \mathbb{R}^d \), \( r > 0 \), \( G^t \) and \( H^t \) are respectively the transposes of \( G \) and \( H \), \( \text{id}_{\mathbb{R}^{d-s}} \) is the identity operator on \( \mathbb{R}^{d-s} \), \( \delta_{\mathbb{R}^{d-s}} \) is the Dirac delta operator on \( \mathbb{R}^{d-s} \), and \( \Phi \in \mathcal{S}(\mathbb{R}^s) \) satisfies \( \Phi = \varphi \). Let \( X = X(\varphi, G, H) \) be given by

\[
X f(x) = \sup_{r > 0} |X_r f(x)|.
\]

Then for \( 1 < p \leq \infty \), there exists a positive constant \( C_p = C(p, \varphi, s, d) \) such that

\[
\|X f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}
\]

for all \( f \in L^p(\mathbb{R}^d) \). The constant \( C_p \) is independent of the linear transformations \( G \) and \( H \).

Now let \( \Delta_1(0, \infty) \) denote the set of functions \( b \) on \( (0, \infty) \) satisfying

\[
\sup_{R > 0} \frac{1}{R} \int_0^R |b(t)|^\gamma dt < \infty.
\]

For \( y = (y_1, \cdots, y_n) \in \mathbb{R}^n \), let \( \tilde{y} = (y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1} \). We denote the north pole \((0, \cdots, 0, 1)\) on \( S^{n-1} \) by \( \rho_1 \). Let \( F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) be of the form

\[
F(t, y) = t^l q(\tilde{y}) + W_1(t, y) + W_2(t),
\]
where \( q : \mathbb{R}^{n-1} \to \mathbb{R} \) is a polynomial, \( W_1 \) satisfies
\[
(5) \quad \frac{\partial^j W_1}{\partial t^j}(t, y) \equiv 0,
\]
and \( W_2(\cdot) \) is an arbitrary function.

The following estimate on the oscillatory integrals is Proposition 5.3 in [8] and is of considerable importance to us.

**Lemma 6.** Let \( \rho \in (0, 1/4), \ l \in \mathbb{N}, \ m \geq 0, \ q(y) = \sum_{j=0}^{m} q_j(y) \), where \( q_j(\cdot) \) is a homogeneous polynomial of degree \( j \) on \( \mathbb{R}^{n-1} \) for \( 0 \leq j \leq m \). Let \( F(t, y) \) be given by (4) and (5). Suppose that \( b \in \Delta_{\gamma} \) for some \( \gamma > 1 \) and \( \Omega(\cdot) \) is a function satisfying

(a) \( \text{supp}(\Omega) \subset B(\rho_1, \rho) \); 
(b) \( \|\Omega\|_{L^\infty(S^{n-1})} \leq \rho^{-n+1} \).

If we assume \( q_m(y) = \sum_{|\beta|=m} a_\beta y^\beta \) and \( \|q_m\| = \sum_{|\beta|=m} |a_\beta| \), then there exists a positive constant \( C \) such that
\[
\int_{\mathbb{R}_2^k} \left| \int_{S^{n-1}} e^{i F(t, y)} \Omega(y) \, d\sigma(y) \right| \frac{\|b(t)\|}{t} \, dt \leq C \left( 2^{\kappa m} \rho^m \|q_m\| \right)^{\frac{1-k}{m+1}}.
\]
The constant \( C \) may depend on \( \kappa, \ m, \ n, \) and \( b(\cdot) \), but it is independent of \( k, \rho, W_1(\cdot, \cdot), W_2(\cdot), \) and the coefficients of \( q(\cdot) \).

**Proof of Theorem 1.** Since \( \Omega \in H^1(S^{n-1}) \) and \( \int_{S^{n-1}} \Omega(y) \, d\sigma(y) = 0 \), there are regular atoms \( a_j(\cdot) \) and \( \{C_j\} \subset \mathbb{C} \) such that
\[
\Omega(y) = \sum_j C_j a_j(y)
\]
by Lemma 1.

Therefore, we only need to be concerned with the case where \( \Omega(y) \) is a regular atom on \( S^{n-1} \). By Lemma 2 and using a rotation if necessary, we may assume that there is a \( \rho \in (0, 1/4) \) such that
\[
\text{supp}(\Omega) \subset B(\rho_1, \rho), \ \text{where} \ \rho_1 = (0, \cdots, 0, 1);
\]
and
\[
\|\Omega\|_{L^\infty(S^{n-1})} \leq \rho^{-(n-1)} \int_{S^{n-1}} \Omega(y) \, d\sigma(y) = 0.
\]

For any integrable function \( a(\cdot) \) on \( S^{n-1} \) and a suitable mapping \( \Gamma : \mathbb{R}^n \to \mathbb{R}^{n+1} \), we define the sequence of measures \( \{\sigma_{a, \Gamma, k}\}_{k \in \mathbb{Z}} \) by
\[
\int_{\mathbb{R}^{n-1}} F \, d\sigma_{a, \Gamma, k} = \int_{\{y \in \mathbb{R}^{n-2} : |y| < 2^{k+1}\}} F(\Gamma(y)) \frac{a(y)}{|y|^n} \, b(|y|) \, dy.
\]

For \( y \in \mathbb{R}^n \setminus \{0\} \), let \( \bar{y} = (y_1/|y|, \cdots, y_{n-1}/|y|) \). Let \( N = \lfloor \frac{3(n-1)}{2} \rfloor + 2 \) (this \( N \) is chosen so that we can have both (9) and (10) for \( j = N \)). For \( j = 1, \cdots, N - 2 \), let \( b_j = (-1)^j \frac{j}{2} (\frac{1}{2} - 1) \cdots (\frac{1}{2} - j + 1)/j! \). Thus
\[
|(1 - t)^{1/2} - 1 - \sum_{l=1}^{j-1} bt^l| \leq C_j t^j
\]
for \( t \in [0, 1/4] \).
We now define the mappings $\Gamma_0$, $\Gamma_1$, $\cdots$, $\Gamma_N$ by

$$
\Gamma_N(y) = (y, \phi(|y|)),
$$

$$
\Gamma_j(y) = (|y|^2, |y|(1 + b_1|y|^2 + \cdots + b_{j-1}|y|^{2(j-1)}), \phi(|y|)), \quad j = 2, \cdots, N - 1,
$$

$$
\Gamma_1(y) = (|y|^2, |y|, \phi(|y|)),
$$

and

$$
\Gamma_0(y) = (0, |y|, \phi(|y|)).
$$

For $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$, we shall establish estimates (ii) and (iii) in Lemma 3 for $\{\tilde{\sigma}_{\Omega, \Gamma_j}(\xi, \eta) : 1 \leq j \leq N$ and $k \in \mathbb{Z}\}$. By an inequality on page 551 of [6], we have

$$
|\tilde{\sigma}_{\Omega, \Gamma_N,k}(\xi, \eta)| \leq \int_{2k}^{2k+1} \left| \int_{S^{n-1}} e^{-i[(\xi \cdot \eta + \eta \phi(t))]\Omega(y)\,d\sigma(y)} \right| |b(t)| \frac{dt}{t} \leq C_j |\xi|^{-1/6} ||\Omega||_{L^2(S^{n-1})}
$$

$$
\leq C_2 |\xi|^{-1/6}.
$$

One observes that the variable $\eta$ does not appear in the previous inequality. The same is true for the Fourier estimates obtained from here on.

Now, for $2 \leq j \leq N - 1$, we have

$$
|\tilde{\sigma}_{\Omega, \Gamma_j,k}(\xi, \eta)| \leq \int_{2k}^{2k+1} \left| \int_{S^{n-1}} e^{-i[(\xi \cdot \eta + \eta \phi(t))]\Omega(y)\,d\sigma(y)} \right| |b(t)| \frac{dt}{t}.
$$

By applying Lemma 6 with $q(\eta) = -[(\xi_1, \cdots, \xi_{n-1}) \cdot \eta + \xi_n \sum_{s=1}^{j-1} b_s|\eta|^{2s}], m = 2(j-1), \gamma = 2$ and $l = 1$, we obtain

$$
|\tilde{\sigma}_{\Omega, \Gamma_j,k}(\xi, \eta)| \leq C \left[ 2^k |\xi|^{-\frac{1}{6}} \right].
$$

Finally, by Lemma 6 with $m = 1, \gamma = 2$ and $l = 1$, we have

$$
|\tilde{\sigma}_{\Omega, \Gamma_j,k}(\xi, \eta)| \leq \int_{2k}^{2k+1} \left| \int_{S^{n-1}} e^{-i[(\xi \cdot \eta + \eta \phi(t))]\Omega(y)\,d\sigma(y)} \right| |b(t)| \frac{dt}{t} \leq C \left[ 2^k |\xi|^{-\frac{1}{6}} \right].
$$

Let

$$
L_1(\xi, \eta) = \rho(\xi_1, \cdots, \xi_{n-1}), \quad \theta_1 = \frac{1}{8};
$$

$$
L_j(\xi, \eta) = \rho^{2(j-1)}(\xi_n), \quad \theta_j = \frac{1}{16(j-1)}, \quad 2 \leq j \leq N - 1;
$$

$$
L_N(\xi, \eta) = \rho^{3(n-1)}(\xi_n), \quad \theta_N = \frac{1}{6}.
$$

Then by (6)–(8), we have

$$
|\tilde{\sigma}_{\Omega, \Gamma_j,k}(\xi, \eta)| \leq C |L_j(\xi, \eta)|^{-\theta_j}
$$

for $1 \leq j \leq N, k \in \mathbb{Z}, \xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$. Next we shall verify that for $(\xi, \eta) \in \mathbb{R}^{n+1}, k \in \mathbb{Z}$ and $1 \leq j \leq N$,

$$
|\tilde{\sigma}_{\Omega, \Gamma_j,k}(\xi, \eta) - \tilde{\sigma}_{\Omega, \Gamma_{j-1},k}(\xi, \eta)| \leq C 2^k |L_j(\xi, \eta)|.
$$
Let us begin with $j = N$. In this case, we have
\[
|\tilde{\sigma}_{\Omega, N, k}(\xi, \eta) - \tilde{\sigma}_{\Omega, N-1, k}(\xi, \eta)| \\
\leq \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \left| e^{-it\xi_n b_1} \frac{1}{\rho_2(N-1)} - 1 \right| \left| \sigma(y) \right| \left| b(t) \right| \frac{dt}{t} \\
\leq C \int_{2^k}^{2^{k+1}} \rho_2(N-1) \left| \sigma(y) \right| \left| b(t) \right| \frac{dt}{t} \\
\leq C 2^k \rho_2(N-1) \xi_n = C 2^k N(X, \xi, \eta).
\]
For $2 \leq j \leq N - 1$, we have
\[
|\tilde{\sigma}_{\Omega, j, k}(\xi, \eta) - \tilde{\sigma}_{\Omega, j-1, k}(\xi, \eta)| \\
\leq \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \left| e^{-it\xi_n b_1} \frac{1}{\rho_2(j-1)} - 1 \right| \left| \sigma(y) \right| \left| b(t) \right| \frac{dt}{t} \\
\leq C 2^k \rho_2(j-1) \xi_n = C 2^k L_j(X, \xi, \eta).
\]
Finally, for $j = 1$, we have
\[
|\tilde{\sigma}_{\Omega, 1, k}(\xi, \eta) - \tilde{\sigma}_{\Omega, 0, k}(\xi, \eta)| \\
\leq \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \left| e^{-it\xi_n b_1} \xi_n - 1 \right| \left| \sigma(y) \right| \left| b(t) \right| \frac{dt}{t} \\
\leq C 2^k \rho_1(\xi_1, \cdots, \xi_{n-1}) = C 2^k L_1(X, \xi, \eta).
\]
This completes the proof of (10). \hfill \square

We still need to verify condition (iv) in Lemma 3. It suffices to establish the $L^p(\mathbb{R}^n)$ boundedness of the operators $\sigma_{\Omega, j}^*$ defined by
\[
\sigma_{\Omega, j}^*(f)(x, s) = \sup_{k \in \mathbb{Z}} |(\sigma_{\Omega, \gamma, j, k} \ast f)(x, s)|,
\]
where $j = 1, \cdots, N$, $x \in \mathbb{R}^n$, $s \in \mathbb{R}$ and $1 < p < \infty$.

Let us begin with $\sigma_{\Omega, 1}^*$ which is given by
\[
\sigma_{\Omega, 1}^*(f)(x, s) = \sup_{k \in \mathbb{Z}} |(\sigma_{\Omega, \gamma, 1, k} \ast f)(x, s)|.
\]
Choose $\theta \in C_0^\infty(\mathbb{R}^{n-1})$ such that $\theta(t) \equiv 1$ for $|t| \leq 1/2$ and $\theta(t) \equiv 0$ for $|t| \geq 1$. For $k \in \mathbb{Z}$, we define $\nu_k$ by
\[
\nu_k(\xi, \eta) = \theta(2^k \rho(\xi_1, \cdots, \xi_{n-1})) \sigma_{\Omega, \gamma, 0, k}(\xi, \eta)
\]
for $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$. Let $\tau_k = \sigma_{\Omega, \gamma, 1, k} - \nu_k$. Then by (10) and $|\sigma_{\Omega, \gamma, 0, k}(\xi, \eta)| \leq C$, we have
\[
|\tau_k(\xi, \eta)| \leq |\sigma_{\Omega, \gamma, 1, k}(\xi, \eta) - \sigma_{\Omega, \gamma, 0, k}(\xi, \eta)| \\
+ |1 - \theta(2^k \rho(\xi_1, \cdots, \xi_{n-1}))| |\sigma_{\Omega, \gamma, 0, k}(\xi, \eta)| \\
\leq C \left[ |2^k L_1(\xi, \eta)| + |2^k \rho(\xi_1, \cdots, \xi_{n-1})| \right] \\
= C 2^k \left| L_1(\xi, \eta) \right|.
\]
If $2^k |L_1(\xi, \eta)| > 1$, by (9), we have
\[
|\tau_k(\xi, \eta)| \leq C (2^k |L_1(\xi, \eta)|)^{-1/8}.
\]
let

\[ |\tau_k(\xi, \eta)| \leq C \left[ \min\{2^k|L(\xi, \eta)|, (2^k|L(\xi, \eta)|)^{-1}\} \right]^{1/8}. \]

Let

\[ \tau^*(f)(x, s) = \sup_{k \in \mathbb{Z}} |(\tau_k * f)(x, s)|, \quad \nu^*(f)(x, s) = \sup_{k \in \mathbb{Z}} |(\nu_k * f)(x, s)| \]

and

\[ g_{r}(f)(x, s) = \left\{ \sum_{k \in \mathbb{Z}} |(\tau_k * f)(x, s)|^2 \right\}^{1/2}. \]

Then

\[ \sigma^*_{r, 1}(f)(x, s) \leq g_{r}(f)(x, s) + \nu^*(f)(x, s) \]

and

\[ \tau^*(f)(x, s) \leq \sigma^*_{r, 1}|f|(x, s) + \nu^*(|f|)(x, s) \leq g_{r}(|f|)(x, s) + 2\nu^*(|f|)(x, s). \]

By the $L^p(\mathbb{R}^2)$ boundedness of $M_\phi$ and Lemma 5, for $1 < p < \infty$, we have

\[ \|\nu^*|f|\|_{L^p(\mathbb{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbb{R}^{n+1})}. \]

Also, from (11), it is easy to deduce that

\[ \|g_{r}(f)\|_{L^2(\mathbb{R}^{n+1})} \leq C\|f\|_{L^2(\mathbb{R}^{n+1})}. \]

Thus, (13) implies that

\[ \|\tau^*(f)\|_{L^2(\mathbb{R}^{n+1})} \leq C\|f\|_{L^2(\mathbb{R}^{n+1})}. \]

By invoking Lemma 4, we obtain

\[ \|g_{r}(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbb{R}^{n+1})} \]

for $4/3 < p < 4$. Thus, by (13) again, we obtain

\[ \|\tau^*(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbb{R}^{n+1})} \]

for $4/3 < p < 4$. By using (14), (13) and repeating the preceding argument, we obtain

\[ \|g_{r}(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbb{R}^{n+1})} \]

for $1 < p < \infty$. Now, from (12), it follows that

\[ \|\sigma^*_{r, 1}|f|\|_{L^p(\mathbb{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbb{R}^{n+1})} \]

for $1 < p < \infty$.

Similarly, we can show that

\[ \|\sigma^*_{r, j}|f|\|_{L^p(\mathbb{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbb{R}^{n+1})} \]

for $1 \leq j \leq N$. Now, by (9), (10), (15) and Lemma 3, we have

\[ \left\| \sum_{k \in \mathbb{Z}} \sigma_{r, \Gamma_N, k} * f \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbb{R}^{n+1})}. \]
for $1 < p < \infty$. Noting that
\[
\sum_{k \in \mathbb{Z}} (\sigma_{\Omega, r_N, k} * f)(x, s) = \int_{\mathbb{R}^n} f(x - y, s - \phi(|y|)) \frac{\Omega(y)}{|y|^n} b(|y|) \, dy,
\]
we thus obtain a proof of our theorem.

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