THE NATURAL MAXIMAL OPERATOR ON BMO

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Abstract. We introduce a generalization of the Hardy-Littlewood maximal operator, the \(M^3\), in some sense the maximal operator which most naturally commutes pointwise with the logarithm on \(A^\infty\). This commutation reveals the behavior of \(M^3\) to directly correspond to that of \(M^2 : BMO \to BLO\); the boundedness of \(M^2 : BMO \to BLO\) is an immediate consequence.

In 1981 Bennett, DeVore, and Sharpley \cite{1} proved the Hardy-Littlewood maximal operator \(M\) to be bounded on \(BMO\). Later \cite{4}, this was improved to yield \(M^2 : BMO \to BLO\) and furthermore was shown to be intimately related to the observation that \(M^2\) maps \(A^\infty\) into \(A^1\). In this paper we shall clarify this relationship and in fact demonstrate a precise equivalence.

Let us begin by defining the natural maximal operator \(M^3\) by
\[
M^3 f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f.
\]
Of course, \(M(f)(x) = M^2(|f|)(x)\); so to prove the boundedness of \(M\) it suffices to show that of \(M^3\). This is easily accomplished with the following commutation lemma, which allows us to pass between the language of \(A^\infty\) and that of \(BMO\).

Lemma. For \(w \in A^\infty\), \(0 \leq \log M^3 - M^3 \log |w(x)| \leq \log A^\infty(w)\).

Proof. By Jensen’s inequality and the reverse inequality for \(A^\infty\),
\[
\frac{1}{|Q|} \int_Q w \leq A^\infty(w) e^{\frac{1}{|Q|} \int_Q \log w} \leq A^\infty(w) \frac{1}{|Q|} \int_Q w.
\]
Take the supremum over all \(Q \ni x\) and then take log.

Theorem. \(M^3\) maps \(BMO\) boundedly into \(BLO\).

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\[\text{1Recall that } BLO \text{ denotes the functions of bounded lower oscillation, i.e. } f \text{ such that over all cubes } Q, \frac{1}{|Q|} \int_Q f - \inf_Q f \leq C; \text{ we denote by } ||f||_{BLO} \text{ the smallest such } C.\]
\[\text{2} A^\infty(w) \text{ denotes the smallest } C \text{ such that for all cubes } Q, \frac{1}{|Q|} \int_Q w \leq C e^{\frac{1}{|Q|} \int_Q \log w}.\]
Proof. Let $\phi \in \text{BMO}$. As a consequence of the John-Nirenberg Inequality, $\phi(x) = \frac{\|\phi\|}{c_n} \log w(x)$ for some $w \in A^2$, with $A^2(w) \leq \sqrt{c}$ and $c_n = \frac{1}{2^{n+1}c}$, where $n$ denotes the dimension. Thus, by the lemma,

$$M^3\phi(x) = \frac{\|\phi\|}{c_n} [\log Mw(x) + b(x)]$$

for some $||b||_\infty \leq 1$. Let’s consider the log $Mw$ term: $M$ maps $A^\infty$ into $A^1$, and log maps $A^1$ into $\text{BLO}$, so it is in $\text{BLO}$. Further, its norm depends on $A^1(Mw)$, which depends on the reverse Hölder class and norm of $w$, which in turn depend on the $A^p$ class and $A^p$ norm of $w$. As noted above, these depend on nothing; thus $||M^3\phi||_{\text{BLO}} \leq C_n||\phi||_*$. \hfill $\square$

Corollary. $M$ maps $\text{BMO}$ boundedly into $\text{BLO}$

Proof. $Mf = M^2(|f|) \in \text{BLO}$, and $||Mf||_{\text{BLO}} = ||M^2(|f|)||_{\text{BLO}} \leq 2C_n||f||_*$. \hfill $\square$

Notice that above, the set inclusion $M(A^\infty) \subset A^1$ implies $M^3(BMO) \subset BLO$, and the dependence of $A^1(Mw)$ on $A^p(w)$ and $p$ implies the boundedness of $M^3$. Repeated application of the commutation lemma, combined with the observation that $\phi \in \text{BLO}$ if and only if $M^3\phi(x) \leq \phi(x) + ||\phi||_{\text{BLO}}$, yields the following converse (assuming the weak result that $M(A^\infty) \subset A^\infty$ with $A^\infty(Mw)$ dependent on $A^p(w)$ and $p$):

Theorem. $M^3 : \text{BMO} \to \text{BLO}$ bounded implies $M(A^\infty) \subset A^1$ with $A^1(Mw)$ dependent on $A^p(w)$ and $p$.

Proof. Let $w \in A^\infty$. The commutation lemma applied to $w$ implies

$$e^{M^3\log w(x)} \leq Mw(x) \leq A^\infty(w)e^{M^3\log w(x)};$$

applied twice to $Mw$ it yields

$$e^{M^3M^3\log w(x)} \leq MMw(x) \leq A^\infty(w)A^\infty(Mw)e^{M^3M^3\log w(x)}.$$ 

By hypothesis, $M^3\log w \in \text{BLO}$; thus

$$MMw(x) \approx e^{M^3M^3\log w(x)} \leq e^{M^3\log w(x) + ||M^3\log w||_{\text{BLO}}} \approx e^{||M^3\log w||_{\text{BLO}}Mw(x)},$$

i.e., $Mw \in A^1$, with $A^1(Mw) \leq A^\infty(w)A^\infty(Mw)e^{C_n||\log w||_*}$.

In other words, the behavior of $M$ on $A^\infty$ corresponds exactly to that of $M^3$ on $\text{BMO}$. For amusement, one can adapt the original proof of Bennett, DeVore, and Sharpley to prove directly the bound for $M^3$ and thus obtain a new, if unwieldy, proof of $M : A^\infty \to A^1$.

References


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4 Added in proof: Chiarenza and Frasca [2] have also proved this bound directly using the John-Nirenberg inequality and the fact that $(Mf)^{1/2} \in A^1$ for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ [2], though without the commutation or noticing the connection with the behavior on $A^\infty$. 

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