A REMARK ON A PAPER OF E. B. DAVIES

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Abstract. We explain the existence of open sets of complex quasi-modes in terms of Hörmander’s commutator condition.

In a recent paper [1], Davies proves the following interesting result: if $P(h) = (hD_x)^2 + V(x)$, $D_x = (1/i)d/dx$, $V \in C^\infty(\mathbb{R})$, then

$$\text{Im} V'(x_0) \neq 0, \quad E = \xi_0^2 + V(x_0), \quad \xi_0 \in \mathbb{R} \setminus \{0\} \implies$$

$$\exists u(h) \in L^2(\mathbb{R}), \quad ||u(h)||_{L^2(\mathbb{R})} = 1, \quad (P(h) - E)u(h) = \mathcal{O}(h^\infty),$$

where $\mathcal{O}(h^\infty)$ is a bound in any norm, for instance $C^k$ for any $k$.

The purpose of this note is to point out that a more general statement than (1) follows immediately from the now standard results in microlocal analysis described in Chapter 26 of [5].

Let $X$ be a manifold and $S^{m,k}(T^*X)$ be the space of semi-classical symbols on $T^*X$: $a \in C^\infty(T^*X \times (0, 1])$,

$$\partial_\xi^\alpha \partial_\eta^\beta a(x, \xi, \eta; h) \leq C_{\alpha, \beta} h^{-m}(1 + ||\xi||^{k-|\beta|}), \quad (x, \xi) \in T^*X, \quad h \in (0, 1].$$

The local quantization formula

$$a(x, hD_x; h)u = \frac{1}{(2\pi h)^n} \int \int a(x, \xi; h) e^{i(x - \xi)u} dy d\xi$$

defines a class of semi-classical pseudo-differential operators on $X$, $\Psi^{m,k}_h(X)$, with a symbol map, $\sigma$, and a short exact sequence,

$$0 \longrightarrow \Psi^{-1,k-1}_h(X) \longrightarrow \Psi^{m,k}_h(X) \xrightarrow{\sigma} S^{m,k}(T^*X) / S^{-1,k-1}(T^*X) \longrightarrow 0.$$

If we strengthen the condition (2) to

$$(hD_h)^\alpha \partial_\xi^\beta a(x, \xi; h) \leq C_{\alpha, \beta} h^{-m}(1 + ||\xi||^{k-|\beta|}), \quad (x, \xi) \in T^*X, \quad h \in (0, 1],$$

we can then use, without any modifications, the $C^\infty$ theory of pseudo-differential operators when working in compact subsets of $T^*X$. We can simply introduce a new variable $t \in \mathbb{R}$ and consider $h = 1/\tau$, where $\tau$ is the dual variable to $t$. Working microlocally in the region $|\xi/\tau| \leq C$ corresponds to working semi-classically in compact subsets of $T^*X$. In particular we have a correspondence between the
frequency set and the wave front set:

\[ (x, \xi) \notin WF_h(u(h)) \iff \exists Q \in \Psi^0(\mathbb{R} \times X) \text{ elliptic at } (t, x, \tau \xi) \text{ for all } (t, \tau) \in T^* \mathbb{R} \setminus 0 \]

\[ \text{such that } QF_{\tau_0}^{-1}(u(1/\bullet)) \in C^\infty(\mathbb{R} \times X), \]

where \( F_{\tau_0}^{-1} \) is the inverse Fourier transform (see [3] and [6] Proposition 9] for the definition and the relation to other wave front sets). The characterization given here follows from the one in [6].

This allows us to adapt directly the results of Hörmander and of Duistermaat and Sjöstrand (see [2] and [5] Sect.26.3]) to obtain the following statement: if \( P(x, hD_x; h) \) is a semi-classical pseudo-differential operator, in the class satisfying [3], and with the principal symbol \( p \), then

\[ p(m) = 0, \quad \{\text{Re } p, \text{Im } p\}(m) < 0, \quad m \in T^* X \implies \]

\[ \exists u(h) \in C^\infty(X) \text{ such that } WF_h(u(h)) = \{m\} \text{ and } WF_h(Pu) = \emptyset. \]

If we take \( P(x, hD_x; h) = (hD_x)^2 + V(x) - E \) we immediately recover Davies’s result: if \( \chi \in C^\infty_c(X) \) is equal to 1 near \( x_0 \), \( m = (x_0, \xi_0) \), then, because of the \( WF_h \) statements, \( P(\chi u) = \mathcal{O}(h^\infty) \) and \( \chi u \in C^\infty_c(X) \). The condition on the Poisson bracket generalizes the one dimensional condition \( \text{Im } V'(x_0) \neq 0 \). For Schrödinger operators in higher dimensions, we need \( E = \xi_0^2 + V(x_0), \quad \text{Im } \xi_0 \cdot \nabla V(x_0) \neq 0. \)

We conclude with a few remarks. The microlocal statement (4) is a generalization of the celebrated commutator condition of Hörmander [4]. The crucial part of his argument was in fact a geometric optics “quasi-mode” construction which, in a very special case, is repeated in [1]. As was pointed out to the author by Victor Ivrii, the microlocal commutator condition [4] has the following well known global analogue: let \( Q \) be an unbounded operator on a Hilbert space \( \mathcal{H} \), with the domain \( \mathcal{D} \). Let us also assume that \( Q^* \) has the same domain as \( Q \) and that \( Q - z, Q^* - \bar{z} : \mathcal{D} \to \mathcal{H}, \) are Fredholm operators for all \( z \). Then

\[ \pm [Q, Q^*] \geq C > 0 \implies \text{spec } Q = \mathbb{C}. \]

In the case of the + sign, \( Q - z \) fails to be injective for all \( z \), and in the case of the − sign, surjective. In fact, we conclude that for all \( z \in \mathbb{C} \) and \( u \in \mathcal{D}, \)

\[ \pm \| (Q - z)u \|^2 \pm \| (Q^* - \bar{z})u \|^2 \geq C\|u\|^2. \]

If \( Q - z_0 \) were invertible at some \( z_0 \), then index \( (Q - z) = \text{index } (Q - z_0) = 0. \) In the − case \( Q - z \) would then be invertible everywhere, and hence the spectrum is either empty or equal to \( \mathbb{C} \). In the + case we draw the same conclusion for the adjoint.

We finally remark that, as pointed out by Davies, the existence of complex quasi-modes implies lower bounds on the resolvent, and that has computation consequences.

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REFERENCES


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