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SOME DESCRIPTIVE SET-THEORETIC PROPERTIES OF THE ISOMORPHISM RELATION BETWEEN BANACH SPACES

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ABSTRACT. Consider the space $\mathcal{V}(E)$ of closed linear subspaces of a separable Banach space E equipped with the standard Effros Borel structure. The isomorphism relation between Banach spaces being elements of $\mathcal{V}(E)$ determines a partition of $\mathcal{V}(E)$. In this note we prove a result describing the complexity of analytic subsets of $\mathcal{V}(E)$ intersecting a large enough number of the abovementioned parts of $\mathcal{V}(E)$.

1. INTRODUCTION

Given a separable Banach space E, we shall denote by $\mathcal{V}(E)$ the collection of closed linear subspaces of E equipped with the Effros Borel structure. The Effros structure is the σ-algebra generated by the sets $\{F \in \mathcal{V}(E) : F \cap U \neq \emptyset\}$, U being open in E. The Effros Borel space $\mathcal{V}(E)$ is standard, i.e., it is Borel isomorphic to the unit interval I; cf. [Chr], [Ke].

Recall that a set in a finite product $\mathcal{V}(E)^m$ is analytic if it is the projection of a Borel set in $\mathcal{V}(E)^m \times I$.

Bossard [Bo] investigated descriptive set-theoretic properties of the isomorphism relation on $\mathcal{V}(E)$ (we call Banach spaces X and Y isomorphic if X is linearly homeomorphic to Y; cf. [L-T]). Bossard proved that the isomorphism relation is analytic in $\mathcal{V}(E) \times \mathcal{V}(E)$ and, if E is universal, the relation is not Borel and admits no analytic selectors ([Bo], Theorem 2).

The aim of this note is to prove the following.

1.1. Theorem. *Let $\mathcal{A}$ be an analytic set in the Effros Borel space $\mathcal{V}(E)$ such that each Banach space of continuous functions on a countable compact topological space has an isomorphic copy in $\mathcal{A}$. Then $\mathcal{A}$ contains $2^{\aleph_0}$ pairwise isomorphic elements which are isomorphic to a Banach space of continuous functions on some countable compact space.*

The result, strengthening Bossard's theorem on selectors, is related to some topics concerning Lusin's constituents. Let us comment on this aspect of the subject. Let $2^{\mathbb{Q}}$ be the Cantor set of all subsets of the rationals $\mathbb{Q}$ (identified with the characteristic functions), and let us consider in $2^{\mathbb{Q}} \times 2^{\mathbb{Q}}$ the relation of having the same order type. The equivalence class of a well-ordered set of type α is the αth Lusin

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constituent; cf. [Ku], [Ke]. V. G. Kanovei [Ka], answering an old question of Lusin, demonstrated that no analytic set in $2^{\mathbb{Q}}$ can hit every Lusin constituent in exactly one point (cf. [CGP] for a simpler proof). Analogous facts concerning the Cantor-Bendixson derivative were discussed in [Ch-P]. Theorem 1.1 discloses a similar phenomenon for the isomorphism relation between Banach spaces with separable dual (note that this collection forms a coanalytic but not Borel set in $\mathcal{V}(E)$, provided E is universal; see [Bo], Corollaire 7, and [Ke], 33.24).

In fact, our proof of Theorem 1.1 given in section 3, will be based on the results about the Cantor-Bendixson derivative. In section 2 we shall clarify some notions and establish some background needed in this proof.

2. Terminology and some background

Our terminology concerning descriptive set theory follows Kechris [Ke] and Kuratowski [Ku]. For the material related to Banach spaces and ordinals we refer to Semadeni [Se]. Some additional facts about ordinals we shall use can be found in [K-M].

2.1. **Borel spaces and analytic sets.** A Borel (or measurable) space is a pair $(A, \mathcal{S})$, where A is an arbitrary set and $\mathcal{S} \subset 2^A$ is a σ-algebra of subsets of A. The elements of $\mathcal{S}$ are called Borel sets. We usually denote a Borel space $(A, \mathcal{S})$ just as A. In this case a σ-algebra $\mathcal{S}$ is assumed to be already defined or is not important. A Borel space $(A, \mathcal{S})$ is standard if there exists a separable completely metrizable topology on A, such that the Borel sets generated by this topology are exactly the elements of $\mathcal{S}$. According to this definition every topological space can be regarded as a Borel space and every separable complete metric space can be regarded as a standard Borel space. Every Borel subset of a standard Borel space equipped with a relative Borel structure is a standard Borel space.

A map $f : A \to B$ between Borel spaces $(A, \mathcal{S})$ and $(B, \mathcal{T})$ is Borel if $f^{-1}(T) \in \mathcal{S}$ for $T \in \mathcal{T}$.

A set $K \subset B$ is analytic in a standard Borel space $(B, \mathcal{T})$ if it is the projection of a Borel set in $(B \times I, \mathcal{T} \otimes \mathcal{B})$, where I is a unit interval and $\mathcal{T} \otimes \mathcal{B}$ is the σ-algebra generated by the rectangles $T \times B$ with $T \in \mathcal{T}$ and B open in I.

2.2. **The Effros Borel space $\mathcal{V}(E)$.** We shall fix some notation and mention a few basic facts concerning the space of closed linear subspaces of a Banach space and its Borel structure.

Let E be a separable Banach space. We shall denote by $\mathcal{F}(E)$ the Borel space of nonempty closed subsets of E. Borel structure of $\mathcal{F}(E)$ is an Effros structure and is defined as a σ-algebra generated by sets of the form

$$B_U = \{F \in \mathcal{F}(E) : F \cap U \neq \emptyset\},$$

where U is an open subset of E. It is well known that $\mathcal{F}(E)$ is a standard Borel space. In $\mathcal{F}(E)$ we can consider a subset $\mathcal{V}(E)$ consisting of closed Banach subspaces of E. $\mathcal{V}(E)$ is Borel in $\mathcal{F}(E)$; hence $\mathcal{V}(E)$ equipped with the relative Borel structure is a standard Borel space. If E is a Banach algebra with a unit we shall also consider the subset $\mathcal{H}(E) \subset \mathcal{V}(E)$ consisting of subalgebras with a unit. The set $\mathcal{H}(E)$ is Borel in $\mathcal{F}(E)$ and is a standard Borel space (after equipping with the relative Borel structure). Our main area of interest is the Banach algebra $C(2^{\mathbb{N}})$ (i.e., the space of continuous functions on the Cantor set), so denote $\mathcal{F} = \mathcal{F}(C(2^{\mathbb{N}}))$, $\mathcal{V} = \mathcal{V}(C(2^{\mathbb{N}}))$ and $\mathcal{H} = \mathcal{H}(C(2^{\mathbb{N}}))$.

The Banach space $C(2^{\mathbb{N}})$ is universal in the class of separable Banach spaces, i.e., $C(2^{\mathbb{N}})$ is separable and for every separable Banach space E there exists a closed subspace in $C(2^{\mathbb{N}})$ isomorphic to E. Therefore, the Borel space $\mathcal{V}(E)$ can be considered as a Borel subspace of $\mathcal{V}$.

2.3. **The Cantor-Bendixson derivative and the Cantor-Bendixson rank.** For any separable metric space X the hyperspace $\mathcal{K}(X)$ is the space of compact subsets of X with the Hausdorff metric; cf. [Ke]. If X is a separable complete metric space, so is $\mathcal{K}(X)$. We shall denote by $\mathcal{K}$ the hyperspace $\mathcal{K}([-1,1]^{\mathbb{N}})$ of the Hilbert cube.

The Cantor-Bendixson derivative of $K \in \mathcal{K}(X)$ is the set K' of all accumulation points of K. If ξ is an ordinal, we define the ξth iterated Cantor-Bendixson derivative by the transfinite induction: $D^0(K) = K$, $D^\xi(K) = (D^\zeta(K))'$ if $\xi = \zeta + 1$ and $D^\xi(K) = \bigcap_{\zeta<\xi} D^\zeta(K)$ if ξ is a limit ordinal.

The Cantor-Bendixson rank on a hyperspace $\mathcal{K}(X)$ is a map $\delta_{CB} : \mathcal{K}(X) \to \omega_1 \cup \{\infty\}$, such that $\delta_{CB}(K)$ is the minimal ordinal ξ with $D^{\xi+1}(K) = \emptyset$, or $\delta_{CB}(K) = \infty$ if $D^\xi(K) \neq \emptyset$ for every ordinal ξ.

We shall denote by $\mathcal{K}_{<\infty}$ the collection of countable subsets of the Hilbert cube, i.e., $\mathcal{K}_{<\infty} = \{K \in \mathcal{K} : \delta_{CB}(K) < \infty\}$.

2.4. **The ordinals.** We shall denote the minimal infinite ordinal by ω, and the minimal uncountable ordinal by ω_1. As usual, an ordinal ξ will be identified with the segment $\{\alpha : \alpha < \xi\}$, and the segment will be considered with the order topology. By the Mazurkiewicz theorem, each countable compact space K is homeomorphic to an ordinal $\omega^\xi n + 1$, with ξ countable and n natural, and the Cantor-Bendixson rank of K is then equal to ξ, while $D^\xi(K) = n$; cf. [Se].

We shall need a theorem of Bassaga and Pełczyński [B-Pe] providing an isomorphical classification of the Banach spaces $C(\xi+1)$ of continuous functions on countable ordinals.

2.4.1. Theorem. *For any two countable infinite ordinals ξ and ζ Banach spaces $C(\xi+1)$ and $C(\zeta+1)$ are isomorphic if and only if $\xi < \zeta^\omega$ and $\zeta < \xi^\omega$.*

Moreover, for any infinite $\xi < \omega_1$ the space $C(\xi+1)$ has dual l_1.

A set of ordinals $A \subset \omega_1$ is called a club if it is closed and unbounded in ω_1. An intersection of countably many clubs is a club.

Let $\Lambda \subset \omega_1$ be the club of fixed points of the function $\xi \mapsto \omega^\xi$, i.e., $\Lambda = \{\xi < \omega_1 : \xi = \omega^\xi\}$; cf. [K-M]. If $\lambda \in \Lambda$, $\xi < \lambda$ and $n \in \mathbb{N}$, then $(\omega^\xi n)^\omega \leq (\omega^{\xi+1})^\omega = \omega^{(\xi+1)\omega} \leq \omega^{\omega^{\xi+1}\omega} = \omega^{\omega^{\xi+2}} \leq \omega^{\omega^\lambda} = \omega^\lambda$. Therefore, by Theorem 2.4.1 we get

2.4.2. Corollary. *For any $\lambda \in \Lambda$, $\xi < \lambda$ and $n \in \mathbb{N}$ the Banach space $C(\omega^\lambda + 1)$ is not isomorphic to $C(\omega^\xi n + 1)$.*

2.5. **A result concerning the Cantor-Bendixson rank (cf.** [Ch-P]**).** We shall use the following fact concerning the Cantor-Bendixson rank $\delta_{CB} : \mathcal{K}(X) \to \omega_1 \cup \{\infty\}$; cf. section 2.3. The fact is a simple corollary of the main result in [Ch-P].

2.5.1. Theorem. *For any metrizable compact space X and any separable completely metrizable space T let $\delta : T \times \mathcal{K}(X) \to \omega_1 \cup \{\infty\}$ be a function defined by*

$$\delta(t, K) = \delta_{CB}(K) \text{ for } t \in T \text{ and } K \in \mathcal{K}(X), \tag{1}$$

where δ_{CB} is the Cantor-Bendixson rank on $\mathcal{K}(X)$. Let $A \subset T \times \mathcal{K}(X)$ be an analytic set satisfying $A \cap \delta^{-1}(\{\xi\}) \neq \emptyset$ for ξ in some club in ω_1. Then, for all ξ in some club, any F_σ-set containing $A \cap \delta^{-1}(\{\xi\})$ intersects $A \cap \delta^{-1}(\{\alpha\})$ for some $\alpha < \xi$.

3. Proof of Theorem 1.1

Before proving the main theorem let us make a few observations.

First observe that it is enough to prove Theorem 1.1 with $E = C(2^{\mathbb{N}})$. Indeed, let E be arbitrary and let $\mathcal{A} \subset E$ satisfy the assumption of Theorem 1.1. We can embed E into $C(2^{\mathbb{N}})$ as a closed subspace. Then $\mathcal{V}(E)$ is Borel in $\mathcal{V} = \mathcal{V}(C(2^{\mathbb{N}}))$ and $\mathcal{A}$ is analytic in $\mathcal{V}$. This reduces our problem to the case $E = C(2^{\mathbb{N}})$.

We shall define a Borel map $\Phi : \mathcal{H} \to \mathcal{K}$, such that for any $H \in \mathcal{H}$ the algebra $C(\Phi(H))$ is isometric to H. To this end we shall apply the Kuratowski–Ryll-Nardzewski theorem (cf. [Ke], Theorem 12.13) which allows one to select for each $H \in \mathcal{H}$ a collection of elements $d_n^H \in H$, $n \in \mathbb{N}$, dense in the unit ball of H such that each map $d_n^{\cdot} : \mathcal{H} \to C(2^{\mathbb{N}})$ is Borel. (Note that the map $\mathcal{H} \to \mathcal{F}$ assigning to a closed subalgebra of $C(2^{\mathbb{N}})$ its intersection with the closed unit ball in $C(2^{\mathbb{N}})$ is Borel.)

The map Φ is defined as follows:

$$\Phi(H) = \{(d_1^H(x), d_2^H(x), \dots) : x \in 2^{\mathbb{N}}\}. \tag{2}$$

Obviously $\Phi(H)$ is a non-empty closed subset of the Hilbert cube $[-1,1]^{\mathbb{N}}$, for $H \in \mathcal{H}$.

3.1. Lemma. *For any $H \in \mathcal{H}$, the algebra $C(\Phi(H))$ is isometrically isomorphic to the algebra H.*

Proof. For any $h \in H$ and $y \in \Phi(H)$ let $\psi(h)(y) = h((d_1^H, d_2^H, \dots)^{-1}(y))$. It is easy to see that $\psi : H \to C(\Phi(H))$ is well defined and it is a norm preserving homomorphism of the algebras. To check that it is surjective observe that $\psi(H)$ contains constant functions and that $\psi(H)$ separates the elements of $\Phi(H)$, so $\psi(H) = C(\Phi(H))$, by the Stone-Weierstrass theorem. $\square$

3.2. *Remark.* As a matter of fact the space $\Phi(H)$ is homeomorphic to a quotient space $2^{\mathbb{N}}/\simeq$, where $x_1 \simeq x_2$ if and only if $h(x_1) = h(x_2)$ for all $h \in H$. Indeed, the map Φ is identifying exactly these points of Cantor set $2^{\mathbb{N}}$ which are in relation $\simeq$ and it is easy to verify that a topology of $\Phi(H)$ is a topology of quotient space.

3.3. Lemma. *The map $\Phi : \mathcal{H} \to \mathcal{K}$ is Borel.*

Proof. The σ-algebra of Borel sets in $\mathcal{K}$ is generated by the sets $B_{U_1,\dots,U_k} = \{K \in \mathcal{K} : K \cap (U_1 \times \ldots \times U_k \times [-1,1]^{\mathbb{N}}) \neq \emptyset\}$, where $k \in \mathbb{N}$ and $U_1, \dots, U_k$ are open in $[-1,1]$. Let us fix a countable dense subset M of $2^{\mathbb{N}}$. Then

$$\begin{aligned}
\Phi^{-1}(B_{U_1,\dots,U_k}) &= \{H \in \mathcal{H} : \exists_{x \in 2^{\mathbb{N}}}\, d_1^H(x) \in U_1, \dots, d_k^H(x) \in U_k\} \\
&= \{H \in \mathcal{H} : \exists_{x \in M}\, d_1^H(x) \in U_1, \dots, d_k^H(x) \in U_k\} \\
&= \bigcup_{x \in M} \{H \in \mathcal{H} : d_1^H(x) \in U_1, \dots, d_k^H(x) \in U_k\} \\
&= \bigcup_{x \in M} \bigcap_{i=1}^{k} \{H \in \mathcal{H} : d_i^H(x) \in U_i\}.
\end{aligned}$$

The sets $\mathcal{U}_{i,x} = \{H \in \mathcal{H} : d_i^H(x) \in U_i\}$ are Borel because, for any fixed $x \in M$, $\mathcal{U}_{i,x}$ is the preimage of U_i under the composition of the Borel map $H \mapsto d_i^H$ and the evaluation map $f \mapsto f(x)$. Therefore, $\Phi^{-1}(B_{U_1,\dots,U_k})$ is Borel as a countable sum of finite intersections of Borel sets $\mathcal{U}_{i,x}$. □

The Borel space $\mathcal{V}$ is standard and we can fix a completely metrizable separable topology on $\mathcal{V}$ such that the Borel structure of $\mathcal{V}$ is the σ-algebra generated by the open sets. In the sequel, we shall refer to this topology. Let $\mathcal{R} \subset \mathcal{V} \times \mathcal{V}$ be the collection of pairs of Banach subspaces of $C(2^{\mathbb{N}})$ that are isomorphic. By the result of Bossard [Bo], $\mathcal{R}$ is an analytic set.

We shall define a rank $\delta : \mathcal{V} \times \mathcal{K} \to \omega_1 \cup \{\infty\}$ by the formula

$$\delta(L, K) = \delta_{CB}(K) \text{ for } L \in \mathcal{V} \text{ and } K \in \mathcal{K},$$

where $\delta_{CB} : \mathcal{K} \to \omega_1 \cup \{\infty\}$ is the Cantor-Bendixson rank on the hyperspace $\mathcal{K}$ of the Hilbert cube.

We can now pass to the proof of Theorem 1.1.

Proof of Theorem 1.1 (the case $E = C(2^{\mathbb{N}})$). Let $\mathcal{A}$ be an analytic set in $\mathcal{V}$ containing isomorphic copies of every separable Banach space of continuous functions on a countable compact topological space and let

$$\widehat{\mathcal{A}} = (\mathcal{A} \times \mathcal{K}) \cap (\mathrm{Id}_{\mathcal{V}} \times \Phi)(\mathcal{R} \cap (\mathcal{V} \times \mathcal{H})),$$

where $\mathrm{Id}_{\mathcal{V}}$ is the identity on $\mathcal{V}$ and Φ is defined by (2). According to this definition and Lemmas 3.1 and 3.3 set $\widehat{\mathcal{A}}$ is analytic in $\mathcal{V} \times \mathcal{K}$, and it consists of pairs $(L, K) \in \mathcal{V} \times \mathcal{K}$ such that $L \in \mathcal{A}$, L is isomorphic to $C(K)$ and there exists $H \in \mathcal{H}$ satisfying $\Phi(H) = K$.

For any infinite $\xi \in \omega_1$ set $\mathcal{A}$ contains an element L_ξ isomorphic to $C(\omega^\xi + 1)$ and there exists an algebra $H_\xi \in \mathcal{H}$, such that $\Phi(H_\xi)$ is homeomorphic to $\omega^\xi + 1$. To see that define H_ξ as $\{f \circ h : f \in C(\omega^\xi + 1)\}$, where $h : 2^{\mathbb{N}} \to \omega^\xi + 1$ is a continuous surjection and apply Remark 3.2. We have $(L_\xi, \Phi(H_\xi)) \in \widehat{\mathcal{A}}$ and $\delta(L_\xi, \Phi(H_\xi)) = \xi$. This implies that for any infinite $\xi \in \omega_1$ the intersection $\widehat{\mathcal{A}} \cap \delta^{-1}(\{\xi\})$ is nonempty and we can apply Theorem 2.5.1 with $\widehat{\mathcal{A}}$ analytic in $\mathcal{V} \times \mathcal{K}$. The conclusion of Theorem 2.5.1 states that there exists a club $\Theta \subset \omega_1$ such that for any $\xi \in \Theta$ every F_σ-set containing $\widehat{\mathcal{A}} \cap \delta^{-1}(\{\xi\})$ intersects $\widehat{\mathcal{A}} \cap \delta^{-1}(\{\zeta\})$ for some $\zeta < \xi$.

We shall show that if $0 < \xi \in \Lambda \cap \Theta$ (note that $\Lambda \cap \Theta$ is a club in ω_1 and hence is uncountable), then $\mathcal{A}$ contains uncountably many isomorphic copies of $C(\omega^\xi + 1)$. Aiming at a contradiction, assume that there exists $0 < \xi \in \Lambda \cap \Theta$, such that $\mathcal{I}_\xi = \{L \in \mathcal{A} : L \text{ is isomorphic to } C(\omega^\xi + 1)\}$ is countable. For every $K \in \mathcal{K}$ satisfying $\delta(K) = \xi$ the set K is homeomorphic to $\omega^\xi n + 1$ with $n < \omega$ and due to Theorem 2.4.1 $C(K)$ is isomorphic to $C(\omega^\xi + 1)$. Hence the F_σ-set $\mathcal{I}_\xi \times \mathcal{K} \subset \mathcal{V} \times \mathcal{K}$ contains $\widehat{\mathcal{A}} \cap \delta^{-1}(\{\xi\})$ and it must intersect $\widehat{\mathcal{A}} \cap \delta^{-1}(\{\zeta\})$ for some $\zeta < \xi$. The last statement means that there exists $L \in \mathcal{I}_\xi$ isomorphic to $C(\omega^\zeta n + 1)$ with $n \in \mathbb{N}$ and consequently $C(\omega^\xi + 1)$ is isomorphic to $C(\omega^\zeta n + 1)$. This contradicts Theorem 2.4.2. The above contradiction proves that $\mathcal{I}_\xi$ is uncountable and since it is an analytic set in the standard Borel space $\mathcal{V}$ it contains $2^{\aleph_0}$ elements. □

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