PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 10, Pages 3037–3047
S 0002-9939(01)05926-3
Article electronically published on April 2, 2001

SOME RESULTS RELATED TO THE LOGVINENKO-SEREDA THEOREM

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(Communicated by Albert Baernstein II)

Abstract. We prove several results related to the theorem of Logvinenko and Sereda on determining sets for functions with Fourier transforms supported in an interval. We obtain a polynomial instead of exponential bound in this theorem, and we extend it to the case of functions with Fourier transforms supported in the union of a bounded number of intervals.

The purpose of this work is to study the behavior of functions whose Fourier transforms are supported in an interval or in a union of finitely many intervals on "thick" subsets of the real line. A main result of this type was proven by Logvinenko and Sereda.

By a "thick" subset of the real line we mean a measurable set E for which there exist $a > 0$ and $\gamma > 0$ such that

$$|E \cap I| \geq \gamma \cdot a \tag{1}$$

for every interval I of length a.

The Logvinenko-Sereda Theorem. *Let J be an interval with $|J| = b$. If $f \in L^p$, $p \in [1, +\infty]$, and supp $\hat{f} \subset J$, and if a measurable set E satisfies* (1)*, then*

$$\|f\|_{L^p(E)} \geq \exp(-C \cdot \frac{(ab+1)}{\gamma}) \cdot \|f\|_p. \tag{2}$$

It is a well-known fact that condition (1) is also necessary for an inequality of the form

$$\|f\|_{L^p(E)} \geq C \cdot \|f\|_p$$

to hold. See for example [3], p. 113.

We will improve the estimate (2) by getting a polynomial dependence on γ and show that our estimate is optimal except for the constant C:

Theorem 1.

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{C}\right)^{C\cdot(ab+1)} \cdot \|f\|_p.$$

Received by the editors September 24, 1999 and, in revised form, March 3, 2000.
2000 *Mathematics Subject Classification.* Primary 42A99, 42B99.

We will also generalize the Logvinenko-Sereda theorem to functions whose Fourier transforms are supported on a union of finitely many intervals:

Theorem 2. *Let J_k be intervals with $|J_k| = b$. If $f \in L^p$, $p \in [1, +\infty]$, and supp $\hat{f} \subset \bigcup_1^n J_k$, and if a measurable set E satisfies* (1)*, then*

$$\|f\|_{L^p(E)} \geq c(\gamma, n, ab, p) \cdot \|f\|_p$$

where $c(\gamma, n, ab, p) = \left(\frac{C}{\gamma}\right)^{-ab\left(\frac{C}{\gamma}\right)^n - n + \frac{p-1}{p}}$ *depends only on the number of intervals but not how they are placed.*

Note that the constant C below is not fixed and varies appropriately from one equality or inequality to another even without mentioning it.

Proof of Theorem 1. First we treat the case when $p \in [1, +\infty)$. Without loss of generality we can always assume that $J = [-\frac{b}{2}, \frac{b}{2}]$. By considering $f(\frac{x}{a})$ instead of f we can also assume that $|E \cap [x, x+1]| \geq \gamma \ \forall x$ and supp $\hat{f} \subset [-\frac{ab}{2}, \frac{ab}{2}]$, just say supp $\hat{f} \subset [-\frac{b}{2}, \frac{b}{2}]$. Bernstein's inequality ([1], Theorem 11.3.3) gives that

$$\int |f^{(\alpha)}|^p \leq (C \cdot b)^{\alpha p} \cdot \int |f|^p$$

with $C = \frac{1}{2}$.

Divide the whole $\mathbb{R}$ into intervals of length 1. Choose $A > 1$. Call an interval I bad if $\exists \alpha \geq 1$ such that

$$\int_I |f^{(\alpha)}|^p \geq A^{\alpha p} (C \cdot b)^{\alpha p} \cdot \int_I |f|^p.$$

Then

$$\begin{aligned}
\int_{\bigcup_{I \text{ is bad}} I} |f|^p &\leq \int_{\bigcup_{I \text{ is bad}} I} \sum_{\alpha=1}^{\infty} \frac{1}{A^{\alpha p}(C \cdot b)^{\alpha p}} |f^{(\alpha)}|^p \\
&= \sum_{\alpha=1}^{\infty} \frac{1}{A^{\alpha p}(C \cdot b)^{\alpha p}} \int_{\bigcup_{I \text{ is bad}} I} |f^{(\alpha)}|^p \\
&\leq \sum_{\alpha=1}^{\infty} \frac{1}{A^{\alpha p}(C \cdot b)^{\alpha p}} \int |f^{(\alpha)}|^p \\
&\leq \sum_{\alpha=1}^{\infty} \frac{1}{A^{\alpha p}} \int |f|^p \\
&= \frac{1}{A^p - 1} \int |f|^p. \qquad (3)
\end{aligned}$$

Choose $A = 3$ and apply (3). So

$$\int_{\bigcup_{I \text{ is bad}} I} |f|^p \leq \frac{1}{2} \int |f|^p.$$

Therefore,

$$\int\limits_{\bigcup\limits_{I \text{ is good}} I} |f|^p \geq \frac{1}{2}\int |f|^p. \tag{4}$$

We claim that $\exists B > 1$ such that if I is a good interval, then $\exists x \in I$ with the property that

$$|f^{(\alpha)}(x)|^p \leq 2 \cdot B^{\alpha p}(C \cdot b)^{\alpha p} \cdot \int\limits_I |f|^p \quad \forall \alpha \geq 0.$$

Suppose towards a contradiction that this is not true. Then

$$2 \cdot \int\limits_I |f|^p \leq \sum_{\alpha=0}^{\infty} \frac{1}{B^{\alpha p}(C \cdot b)^{\alpha p}} |f^{(\alpha)}(x)|^p \quad \forall x \in I. \tag{5}$$

Integrate both sides of (5) over I:

$$\begin{aligned} 2 \cdot \int\limits_I |f|^p &\leq \sum_{\alpha=0}^{\infty} \frac{1}{B^{\alpha p}(C \cdot b)^{\alpha p}} \int\limits_I |f^{(\alpha)}(x)|^p \\ &\leq \sum_{\alpha=0}^{\infty} \frac{1}{B^{\alpha p}} \int\limits_I |f|^p \\ &= \frac{1}{1 - \left(\frac{1}{B}\right)^p} \int\limits_I |f|^p. \end{aligned} \tag{6}$$

Choose $B = 3$ and apply (6). So

$$2 \cdot \int\limits_I |f|^p \leq \frac{3}{2} \int\limits_I |f|^p.$$

This contradiction proves our claim.

We will need to prove the following local estimate:

$$\int\limits_{E \cap I} |f|^p \geq \left(\frac{\gamma}{C}\right)^{Cbp+2} \int\limits_I |f|^p$$

for every good interval I. Without loss of generality we can assume that $I = [-\frac{1}{2}, \frac{1}{2}]$ by considering a shift $f(x-n)$ which has $\operatorname{supp} \widehat{f(x-n)} \subset [-\frac{b}{2}, \frac{b}{2}]$. Therefore if $y \in D(0, R) \subset D(x, R + \frac{1}{2})$, then

$$\begin{aligned} |f(y)| &\leq \sum_{\alpha=0}^{\infty} \frac{|f^{(\alpha)}(x)|}{\alpha!} \cdot |y - x|^\alpha \\ &\leq \sum_{\alpha=0}^{\infty} 2^{\frac{1}{p}} \frac{(R + \frac{1}{2})^\alpha \cdot (Cb)^\alpha}{\alpha!} \|f\|_{L^p(I)} \\ &= 2^{\frac{1}{p}} \exp(Cb(R + \frac{1}{2})) \cdot \|f\|_{L^p(I)}. \end{aligned} \tag{7}$$

Now we will give a local estimate for analytic functions.

Lemma 1. *Let $\phi(z)$ be analytic in $D(0,5)$ and let I be an interval of length 1 such that $0 \in I$ and let $E \subset I$ be a measurable set of positive measure. If $|\phi(0)| \geq 1$ and $M = \max_{|z|\leq 4} |\phi(z)|$, then*

$$\sup_{x\in I} |\phi(x)| \leq \left(\frac{C}{|E|}\right)^{\frac{\ln M}{\ln 2}} \sup_{x\in E} |\phi(x)|. \tag{8}$$

Proof of Lemma 1. Let $a_1, a_2, \ldots, a_n$ be the zeros of ϕ in $D(0,2)$. If

$$g(z) = \phi(z) \cdot \prod_{k=1}^{n} \frac{4 - \bar{a}_k z}{2(a_k - z)} = \phi(z) \cdot \frac{Q(z)}{P(z)},$$

then $|g(0)| \geq 1$ and $\max_{|z|\leq 2} |g(z)| \leq M$ by the property of Blaschke products. Applying Harnack's inequality to the positive harmonic function $\ln M - \ln |g(z)|$ in $D(0,2)$ we have

$$\max_{|z|\leq 1} (\ln M - \ln |g(z)|) \leq 3 \ln M.$$

Therefore,

$$\min_{|z|\leq 1} |g(z)| \geq M^{-2}.$$

This gives us

$$\frac{\max_{x\in I} |g(x)|}{\min_{x\in I} |g(x)|} \leq M^3.$$

We can give a similar estimate for Q:

$$\frac{\max_{x\in I} |Q(x)|}{\min_{x\in I} |Q(x)|} \leq \frac{\max_{|z|\leq 1} \prod_{k=1}^{n} |4 - \bar{a}_k z|}{\min_{|z|\leq 1} \prod_{k=1}^{n} |4 - \bar{a}_k z|} \leq 3^n.$$

From the Remez inequality for polynomials ([2], Theorem 5.1.1) it follows that

$$\sup_{x\in I} |P(x)| \leq \left(\frac{4}{|E|}\right)^n \cdot \sup_{x\in E} |P(x)|.$$

Therefore,

$$\begin{aligned}
\sup_{x\in I} |\phi(x)| &\leq \max_{x\in I} |g(x)| \cdot \frac{\max_{x\in I} |P(x)|}{\min_{x\in I} |Q(x)|} \\
&\leq M^3 \cdot 3^n \cdot \left(\frac{C}{|E|}\right)^n \cdot \min_{x\in I} |g(x)| \cdot \frac{\sup_{x\in E} |P(x)|}{\max_{x\in I} |Q(x)|} \\
&\leq M^3 \cdot \left(\frac{C}{|E|}\right)^n \cdot \sup_{x\in E} |\phi(x)|.
\end{aligned}$$

From Jensen's formula it follows that $n \leq \frac{\ln M}{\ln 2}$. Therefore,

$$\sup_{x\in I} |\phi(x)| \leq \left(\frac{C}{|E|}\right)^{\frac{\ln M}{\ln 2}} \sup_{x\in E} |\phi(x)|.$$

□

Corollary. *If $p \in [1, \infty)$, then*

$$\|\phi\|_{L^p(I)} \leq \left(\frac{C}{|E|}\right)^{\frac{\ln M}{\ln 2}+\frac{1}{p}} \|\phi\|_{L^p(E)}. \tag{9}$$

It follows from (8) *that*

$$|\{x \in I : |\phi(x)| < \left(\frac{\epsilon}{C}\right)^{\frac{\ln M}{\ln 2}} \|\phi\|_{L^\infty(I)}\}| \leq \epsilon, \quad \epsilon > 0.$$

If we put $\epsilon = \frac{|E|}{2}$, then

$$|\{x \in I : |\phi(x)| < \left(\frac{|E|}{2C}\right)^{\frac{\ln M}{\ln 2}} \|\phi\|_{L^\infty(I)}\}| \leq \frac{|E|}{2}.$$

Therefore,

$$\begin{aligned}
\int_E |\phi|^p &\geq \int_E \chi_{|\phi| \geq \left(\frac{|E|}{2C}\right)^{\frac{\ln M}{\ln 2}} \cdot \|\phi\|_{L^\infty(I)}} \cdot |\phi|^p \\
&\geq \frac{|E|}{2} \cdot \left(\frac{|E|}{2C}\right)^{p\frac{\ln M}{\ln 2}} \cdot \|\phi\|^p_{L^\infty(I)} \\
&\geq \left(\frac{|E|}{2C}\right)^{p\frac{\ln M}{\ln 2}+1} \cdot \int_I |\phi|^p.
\end{aligned}$$

□

Now we are in a position to proceed with the proof of our theorem. We can assume that $\int_I |f|^p = 1$. Then $\exists x_0 \in I$ such that $|f(x_0)| \geq 1$. Applying (9) to $\phi(z) = f(z + x_0)$, $I - x_0$ and $(E \cap I) - x_0$ we have

$$\int_{E \cap I} |f|^p \geq \left(\frac{|E \cap I|}{C}\right)^{p\frac{\ln M}{\ln 2}+1} \int_I |f|^p.$$

Apply (7) to get

$$\begin{aligned}
M &\leq \max_{|z| \leq 4+\frac{1}{2}} |f(z)| \\
&\leq 2^{\frac{1}{p}} \exp(5Cb).
\end{aligned}$$

Therefore, we have

$$\int_{E \cap I} |f|^p \geq \left(\frac{\gamma}{C}\right)^{Cbp+2} \int_I |f|^p. \tag{10}$$

Summing (10) over all good intervals and applying (4) we have

$$\begin{aligned}
\int_E |f|^p &\geq \int_{E \cap \bigcup_{I \text{ is good}} I} |f|^p \\
&\geq \left(\frac{\gamma}{C}\right)^{Cbp+2} \cdot \int_{\bigcup_{I \text{ is good}} I} |f|^p \\
&\geq \frac{1}{2}\left(\frac{\gamma}{C}\right)^{Cbp+2} \cdot \int |f|^p.
\end{aligned}$$

Replacing b with ab and choosing a new C we have

$$\int\limits_E |f|^p \geq \left(\frac{\gamma}{C}\right)^{Cabp+2} \cdot \int |f|^p.$$

If $p = \infty$, then the proof is almost the same: $\|f\|_{L^\infty(\bigcup\limits_{I \text{ is good}} I)} = \|f\|_\infty$. If I is good, then $\|f\|_{L^\infty(E\cap I)} \geq \left(\frac{\gamma}{C}\right)^{Cb+1} \cdot \|f\|_{L^\infty(I)}$. Hence,

$$\|f\|_{L^\infty(E)} \geq \left(\frac{\gamma}{C}\right)^{Cb+1} \cdot \|f\|_\infty.$$

The proof of Theorem 1 is complete. □

If we keep track of all the constants and do the calculations more accurately, then we can get that if $p \in [1, \infty)$,

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{300}\right)^{33ab+\frac{2}{p}} \cdot \|f\|_p;$$

if $p = \infty$,

$$\|f\|_{L^\infty(E)} \geq \left(\frac{\gamma}{100}\right)^{33ab+1} \cdot \|f\|_\infty.$$

However, if we try to minimize the factor in front of ab, then we can get the following estimate:

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{C}\right)^{(\frac{(1+\epsilon)}{2}+\epsilon)\cdot ab+A(\epsilon)} \cdot \|f\|_p \quad \forall \epsilon > 0.$$

The following example suggests that the right behavior of the estimate in the Logvinenko-Sereda theorem is γ to the power of a linear function of ab and how far we are from the exact factor in front of ab.

Let E be a 1-periodic set such that

$$E \cap [-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{2}, -\frac{1}{2} + \frac{\gamma}{2}] \cap [\frac{1}{2} - \frac{\gamma}{2}, \frac{1}{2}]$$

and let

$$f(x) = \left(\frac{\sin(2\pi x)}{x}\right)^{[\frac{b}{4\pi}]}.$$

If b is large enough, we have

$$\|f\|_{L^p(E)} \leq \left(\frac{\gamma}{C}\right)^{\frac{b}{4\pi}-1} \|f\|_p$$

and $\text{supp}\hat{f} \subset [-\frac{b}{2}, \frac{b}{2}]$.

Remark 1. When ab is sufficiently small, the proof of the theorem is much simpler: if $ab \leq 1$, then $\|f\|_{L^p(E)} \geq \frac{\gamma^{\frac{1}{p}}}{2} \|f\|_p$. This can be proven very easily. If $p \in [1, +\infty)$, we have

$$\begin{aligned} |f(x)|^p = |f(y) - \int\limits_x^y f'(t)dt|^p &\geq \frac{|f(y)|^p}{2^{p-1}} - |\int\limits_x^y f'(t)dt|^p \\ &\geq \frac{|f(y)|^p}{2^{p-1}} - \int\limits_I |f'|^p \cdot a^{p-1} \end{aligned}$$

where $x, y \in I$, $|I| = a$. Hence,

$$\begin{aligned} a \cdot \int\limits_{E\cap I} |f(x)|^p dx &= \int\limits_I \left(\int\limits_{E\cap I} |f(x)|^p dx \right) dy \\ &\geq |E \cap I| \cdot \left(\frac{1}{2^{p-1}} \int\limits_I |f|^p - a^p \int\limits_I |f'|^p \right). \end{aligned}$$

Therefore, $\frac{1}{\gamma} \cdot \int\limits_{E\cap I} |f|^p \geq \frac{1}{2^{p-1}} \int\limits_I |f|^p - a^p \int\limits_I |f'|^p$. Summing over all intervals I we have

$$\begin{aligned} \frac{1}{\gamma} \cdot \int\limits_E |f|^p &\geq \frac{1}{2^{p-1}} \int |f|^p - a^p \int |f'|^p \\ &\geq \frac{1}{2^{p-1}} \int |f|^p - \left(\frac{b}{2}\right)^p a^p \int |f|^p \\ &\geq \frac{1}{2^p} \cdot \int |f|^p. \end{aligned}$$

Using $\|(f^p)'\|_1 \leq \frac{pb}{2} \|f^p\|_1$ we can get a similar result. The proof for $p = \infty$ is even easier.

In a similar way we can treat the case when $1 - \gamma$ is sufficiently small depending on ab: if $p \in [1, \infty)$ and $1 - \gamma \leq \frac{1}{2+pab}$, then $\|f\|^p_{L^p(E)} \geq \frac{1}{2} \|f\|^p_p$.

Proof of Theorem 2. Let $J_k = [\lambda_k - \frac{b}{2}, \lambda_k + \frac{b}{2}]$, $k = 1, 2, ..., n$. First we will prove a special case of Theorem 2:

Theorem 2′. *If $\lambda_{k+1} - \lambda_k \geq 2b > 0$ $(k = 1, 2, ..., n-1)$, then*

$$\|f\|_{L^p(E)} \geq c(\gamma, n, ab, p) \cdot \|f\|_{L^p}$$

where $c(\gamma, n, ab, p) = \left(\frac{\gamma}{C}\right)^{ab\left(\frac{C}{\gamma}\right)^n + n - \frac{p-1}{p}}$.

Proof of Theorem 2′. Let $\hat{f}(x) = \sum\limits_{k=1}^n \hat{f}_k(x - \lambda_k)$ where $\operatorname{supp} \hat{f}_k \subset [-\frac{b}{2}, \frac{b}{2}]$ and $f(x) = \sum\limits_{k=1}^n f_k(x) e^{i\lambda_k x}$. The following lemma gives an estimate of $\|f_k\|_p$ from above.

Lemma 2.

$$\|f_k\|_p \leq C \|f\|_p \quad (k = 1, 2, ..., n). \tag{11}$$

Proof of Lemma 2. Let ϕ be a Schwartz function such that supp $\hat{\phi} \subset [-1, 1]$ and $\hat{\phi}(x) = 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$. Then $\hat{f}_k(x) = \hat{f} \cdot \hat{\phi}(\frac{x - \lambda_k}{b})$. Therefore, $f_k = f * (b\phi(bx) e^{i\lambda_k x})$. Applying Young's inequality we have $\|f_k\|_p \leq \|f\|_p \cdot \|\phi\|_1$. □

We will also need the following auxiliary lemma:

Lemma 3. *If $r(x) = \sum\limits_{k=1}^n p_k(x) e^{i\lambda_k x}$, where $p_k(x)$ is a polynomial of degree $\leq m-1$ and $E \subset I$ is measurable with $|E| > 0$, then*

$$\|r\|_{L^p(I)} \leq \left(\frac{C|I|}{|E|}\right)^{nm - \frac{(p-1)}{p}} \cdot \|r\|_{L^p(E)}. \tag{12}$$

Proof of Lemma 3. First we prove the statement for pure trigonometric polynomials, i.e., if $g(x) = \sum_{k=1}^{n} c_k e^{i\lambda_k x}$, then

$$\|g\|_{L^p(I)} \leq \left(\frac{C|I|}{|E|}\right)^{n-\frac{(p-1)}{p}} \cdot \|g\|_{L^p(E)}. \tag{13}$$

This follows from a theorem on trigonometric polynomials by F. Nazarov ([4], Theorem 1.5) saying that

$$\|g\|_{L^\infty(I)} \leq \left(\frac{C|I|}{|E|}\right)^{n-1} \cdot \|g\|_{L^\infty(E)}. \tag{14}$$

An argument similar to the proof of the Corollary to Lemma 1 shows that (13) follows from (14).

If $p(x) = \sum_{l=0}^{m-1} a_l x^l$ is a polynomial of degree $m-1$, then it can be approximated uniformly on an interval with a trigonometric polynomial of order $\leq m$

$$\tilde{p}(x) = \sum_{l=0}^{m-1} a_l \left(\frac{e^{i\lambda x}-1}{i\lambda}\right)^l = \sum_{l=0}^{m-1} \tilde{a}_l e^{il\lambda x}$$

because $x = \lim_{\lambda\to 0} \frac{e^{i\lambda x}-1}{i\lambda}$ uniformly on an interval. Applying (13) to the trigonometric polynomial of order mn,

$$\tilde{r}(x) = \sum_{k=1}^{n} \tilde{p}_k(x) e^{i\lambda_k x}$$

and taking the limit we have the desired result

$$\|r\|_{L^p(I)} \leq \left(\frac{C|I|}{|E|}\right)^{nm-\frac{(p-1)}{p}} \cdot \|r\|_{L^p(E)}.$$

□

We will need the Taylor formula

$$\begin{aligned} g(x) &= \sum_{l=0}^{m-1} \frac{g^l(s)}{l!}(x-s)^l + \frac{1}{(m-1)!}\int_s^x g^{(m)}(t)(x-t)^{m-1}dt \\ &= p(x) + \frac{1}{(m-1)!}\int_s^x g^{(m)}(t)(x-t)^{m-1}dt \end{aligned}$$

where $p(x)$ is a polynomial of degree $m-1$.

Now we are in a position to proceed with the proof of Theorem 2′.

First we assume that $p \in [1, \infty)$. Divide the whole $\mathbb{R}$ into intervals of length a each. Consider one of them: $I = [s, s+a]$. Then

$$\begin{aligned} f(x) &= \sum_{k=1}^{n} f_k(x) e^{i\lambda_k x} \\ &= \sum_{k=1}^{n} p_k(x) e^{i\lambda_k x} + \frac{1}{(m-1)!} \sum_{k=1}^{n} e^{i\lambda_k x} \int_s^x f_k^{(m)}(t)(x-t)^{m-1} dt \\ &= r(x) + T(x) \end{aligned}$$

Applying Holder's inequality we have

$$\begin{aligned} \int_I |T(x)|^p dx &\leq \frac{n^{p-1}}{[(m-1)!]^p} \sum_{k=1}^{n} \int_I \left| \int_s^x f_k^{(m)}(t)(x-t)^{m-1} dt \right|^p dx \\ &\leq \frac{n^{p-1} a^{pm}}{[m!]^p} \sum_{k=1}^{n} \int_I |f_k^{(m)}|^p. \end{aligned} \tag{15}$$

$$\begin{aligned} \int_I |f|^p &\leq 2^{p-1} \int_I |r|^p + 2^{p-1} \int_I |T|^p \\ &\leq \left(\frac{C|I|}{|E \cap I|} \right)^{pnm-(p-1)} \cdot \int_{E \cap I} |r|^p + 2^{p-1} \int_I |T|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \left(2^{p-1} \int_{E \cap I} |f|^p + 2^{p-1} \int_{E \cap I} |T|^p \right) + 2^{p-1} \int_I |T|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int_{E \cap I} |f|^p + \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int_I |T|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int_{E \cap I} |f|^p + \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \frac{n^{p-1} a^{pm}}{[m!]^p} \sum_{k=1}^{n} \int_I |f_k^{(m)}|^p. \end{aligned}$$

The second inequality is based on Lemma 3. The last follows from (15).

Summing over all intervals I we have

$$\begin{aligned} \int |f|^p &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int_E |f|^p + \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \frac{n^{p-1} a^{pm}}{[m!]^p} \sum_{k=1}^{n} \int |f_k^{(m)}|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int_E |f|^p + \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \frac{n^{p-1} a^{pm} (Cb)^{pm}}{[m!]^p} \sum_{k=1}^{n} \int |f_k|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int_E |f|^p + \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \frac{n^p (ab)^{pm}}{[m!]^p} \int |f|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int_E |f|^p + \left(\frac{C}{\gamma} \right)^{pnm} \frac{(ab)^{pm}}{m^{pm}} \int |f|^p. \end{aligned}$$

The second inequality follows from Bernstein's Theorem. The third is an application of (11). The last inequality is due to Stirling's formula for $m!$ and the fact that $n \leq 2^n$.

Choose m such that it is a positive integer and $\left(\frac{C}{\gamma}\right)^n \frac{ab}{m} \leq \frac{1}{2}$, e.g., $m = 1 + [\left(\frac{C}{\gamma}\right)^n \cdot ab]$ for some $C > 0$. Therefore,

$$\begin{aligned} \int |f|^p &\leq \left(\frac{C}{\gamma}\right)^{pn(1+\left(\frac{C}{\gamma}\right)^n \cdot ab)-(p-1)} \cdot \int\limits_E |f|^p \\ &\leq \left(\frac{C}{\gamma}\right)^{p\left(\frac{C}{\gamma}\right)^n \cdot ab+pn-(p-1)} \cdot \int\limits_E |f|^p. \end{aligned}$$

The proof for $p = \infty$ is similar and even simpler. The proof of Theorem $2'$ is complete. □

Now we can proceed with the proof of Theorem 2. We will apply induction on n. For $n = 1$ the theorem follows from Theorem $2'$ or the usual Logvinenko-Sereda Theorem. Suppose the statement is true for $n \leq m$. Let $n = m + 1$.

If $\lambda_{k+1} - \lambda_k \geq 2b > 0$ $(k = 1, 2...)$, then the result follows from Theorem $2'$.

If $0 < \lambda_{k+1} - \lambda_k < 2b$ for some k, then we can replace b with $3b$ reducing the number of frequencies λ_k. Therefore, by induction

$$\begin{aligned} \|f\|_{L^p(E)} &\geq \left(\frac{C}{\gamma}\right)^{-3ba\left(\frac{C}{\gamma}\right)^m - m + \frac{p-1}{p}} \cdot \|f\|_p \\ &\geq \left(\frac{C}{\gamma}\right)^{-ab\left(\frac{C}{\gamma}\right)^{(m+1)} - (m+1) + \frac{p-1}{p}} \cdot \|f\|_p. \end{aligned}$$

The proof of Theorem 2 is complete. □

The purpose of this theorem is to prove the existence of a constant $c(\gamma, n, ab, p) > 0$ depending only on the number of intervals and not how they are placed rather than to get the best possible estimate.

Final remark. By a "thick" subset of $\mathbb{R}^d$ we mean a measurable set E for which there exist a parallelepiped Π with sides of length $a_1, a_2, ..., a_d$ parallel to coordinate axes and $\gamma > 0$ such that

$$|E \cap (\Pi + x)| \geq \gamma |\Pi| \tag{16}$$

for every $x \in \mathbb{R}^d$. Theorems 1 and 2 can be easily extended to higher dimensions with polynomial dependence on γ for the former. The proofs are analogous to the previous proofs. We can assume that Π is a unit cube. Define good cubes in a similar way. The main issue is how to obtain a local estimate for good cubes. If $|f|$ attains its maximum in a cube Π at $y \in \Pi$, then following an idea of F. Nazarov we can use spherical coordinates centered at y to find a segment I in Π such that $y \in I$ and $\frac{|E \cap I|}{|I|} \geq C(d)\gamma$, and reduce our problem to a 1-dimensional one. In case of Theorem 1 we can define an analytic function of one complex variable which coincides with f on I. In case of Theorem 2 we will approximate f on I with a polynomial defined on I.

Theorem 3. *Let J be a parallelepiped with sides of length $b_1, b_2, ..., b_d$ parallel to coordinate axes. If $f \in L^p(\mathbb{R}^d)$, $p \in [1, +\infty]$, and supp $\hat{f} \subset J$, and if a measurable, set E satisfies* (16)*, then*

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{C^d}\right)^{C(d+\sum\limits_{k=1}^{d} a_k b_k)} \|f\|_p.$$

By an example similar to the one after Theorem 1 (with supp$\hat{f}$ in a neighborhood of a main diagonal of J with the direction of $\mathbf{b} = (b_1, ..., b_d)$ and E periodic along the same direction with period $\sim \mathbf{a} \cdot \mathbf{b}/|\mathbf{b}|$) we can show that this estimate is optimal except for the constant C.

Theorem 4. *Let J_l be parallelepipeds with sides of length $b_1, b_2, ..., b_d$ parallel to coordinate axes. If $f \in L^p(\mathbb{R}^d)$, $p \in [1, +\infty]$, and supp $\hat{f} \subset \bigcup\limits_{1}^{n} J_l$, and if a measurable set E satisfies* (16)*, then*

$$\|f\|_{L^p(E)} \geq c(\gamma, n, \mathbf{a} \cdot \mathbf{b}, p, d)\|f\|_p$$

where $c(\gamma, n, \mathbf{a} \cdot \mathbf{b}, p, d) = \left(\frac{C^d}{\gamma}\right)^{-\left(\frac{C^d}{\gamma}\right)^n \cdot \sum\limits_{k=1}^{d} a_k b_k - n + \frac{p-1}{p}}$ depends only on the number of parallelepipeds but not how they are placed.

Acknowledgments

The author is grateful to Professor Thomas Wolff for his interest in this work.

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