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ACYCLICITY CRITERIA FOR COMPLEXES ASSOCIATED WITH AN ALTERNATING MAP

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Abstract. When I is a Gorenstein ideal of grade 3 in a local ring R, results of Boffi and Sánchez, and of Kustin and Ulrich show that for each $t \geq 1$ one can construct in a canonical way a finite free complex $\mathcal{D}^t$ that is "approximately" a resolution for the ideal I^t. Kustin and Ulrich also provide a sufficient condition that $\mathcal{D}^t$ is acyclic, and a sufficient condition that $\mathcal{D}^t$ is a resolution of I^t. We complete these two acyclicity criteria by showing that the corresponding sufficient conditions are also necessary.

Introduction

Let ξ be a $g \times g$ alternating matrix over a commutative Noetherian ring R, and let I be the ideal of Pfaffians of ξ of maximal even size. In their influential work [5], Buchsbaum and Eisenbud show that when R is local, grade 3 ideals such as I classify the grade 3 Gorenstein ideals of R. It is therefore of considerable interest to obtain information on the homological properties of the ideal I and its powers I^q, $q \geq 1$.

One approach to this problem was pursued by Boffi and Sánchez [2] and Kustin and Ulrich [6]. They construct a family of free complexes $\{\mathcal{D}^q(\xi)\}_{q\geq 0}$ associated with the alternating matrix ξ such that each complex $\mathcal{D}^q(\xi)$ approximates a resolution of the qth symmetric power $\mathrm{S}^q M$ of the cokernel M of ξ; and they show that if ξ is a generic alternating matrix, then each complex $\mathcal{D}^q(\xi)$ is in fact a resolution of the ideal I^q.

In the non-generic case Kustin and Ulrich give a sufficient condition (the *Weak Pfaffian Condition*) for the acyclicity of $\mathcal{D}^q(\xi)$ in terms the grades of certain Pfaffian ideals of ξ. We establish that the Weak Pfaffian Condition is also necessary for the acyclicity of the complex $\mathcal{D}^q(\xi)$:

Theorem A. *Let $q \geq 1$ be an integer, let $r = 2\max\{0, \lfloor (g-q-1)/2 \rfloor\} + 1$, and write $\mathrm{Pf}_{2i}(\xi)$ for the ideal of $2i \times 2i$ Pfaffians of ξ. The following conditions are equivalent:*

(i) *The complex $\mathcal{D}^q(\xi)$ is a free resolution of $\mathrm{S}^q M$.*
(ii) *The complex $\mathcal{D}^t(\xi)$ is a free resolution of $\mathrm{S}^t M$ for $t = 1, \ldots, q$.*
(iii) (*Weak Pfaffian Condition*) $\operatorname{grade} \mathrm{Pf}_{2i}(\xi) \geq g - 2i + 1$ *for* $r + 1 \leq 2i \leq g$.

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Kustin and Ulrich also give a sufficient condition (the *Strong Pfaffian Condition*) for $\mathcal{D}^q(\xi)$ to be a resolution of I^q. As a consequence of Theorem A we obtain that $\mathcal{D}^q(\xi)$ is a resolution of I^q if and only if the Strong Pfaffian Condition holds:

Theorem B. *With assumptions as in Theorem* A, *if $M \neq 0$, then the following conditions are equivalent:*

(i) *The complex $\mathcal{D}^q(\xi)$ is acyclic and $\mathrm{S}^q M$ is torsion free.*
(ii) *The complex $\mathcal{D}^t(\xi)$ is a free resolution of I^t for $t = 1, \dots, q$.*
(iii) (*Strong Pfaffian Condition*) $\operatorname{grade} \mathrm{Pf}_{2i}(\xi) \geq g - 2i + 2$ *for* $r + 1 \leq 2i \leq g$.

1. The complexes $\mathcal{D}^q(\xi)$

Throughout this paper rings are commutative Noetherian with unity, and modules are unitary.

For the convenience of the reader, and in order to establish notation, we recall in this section, after [6], the definition and some properties of the complexes $\mathcal{D}^q(\xi)$.

Let R be a ring, let G be a free R-module of rank g, and let $G^* = \operatorname{Hom}_R(G, R)$ be the dual of G. Throughout this section $\Gamma = \{\gamma_1, \dots, \gamma_g\}$ denotes a basis for G and we write $\Gamma^* = \{\gamma_1^*, \dots, \gamma_g^*\}$ for the dual basis of G^*.

A map $\xi : G^* \to G$ is *alternating* if it is given by an alternating matrix (ξ_{ij}) in the bases Γ and Γ^* (i.e. $\xi_{11} = \cdots = \xi_{gg} = 0$ and $\xi_{ij} = -\xi_{ji}$ for $i \neq j$). It is easy to check that ξ is alternating if and only if $[\gamma^* \circ \xi](\gamma^*) = 0$ for each $\gamma^* \in G^*$, therefore the definition of alternating map is independent of the choice of the basis Γ.

For the rest of this paper $\xi : G^* \to G$ is an alternating map.

Pfaffians. Let $t \geq 1$ be an integer, and let T_t be the polynomial ring over $\mathbb{Z}$ on the set of variables $\{x_{ij} \mid 1 \leq i < j \leq t\}$. It is well known (cf. e.g. [1, Theorem 3.27]) that there exists a unique polynomial $\mathrm{pf}_t \in T_t$ called the $t \times t$ *Pfaffian*, such that $\mathrm{pf}_t(\xi_{ij})^2 = \det(\xi_{ij})$ for any $t \times t$ alternating matrix (ξ_{ij}), and such that $\mathrm{pf}_t = x_{12}x_{34}\dots x_{t-1,t}$ when t is even and the rest of the variables x_{ij} are specialized to 0. When t is odd, $\mathrm{pf}_t \equiv 0$ since the determinant of an alternating matrix of odd size is 0. By convention $\mathrm{pf}_0 \equiv 1 \in \mathbb{Z}$.

We refer the reader to [1, Ch. III, §5] and [6, §1] for the properties of the Pfaffian. In particular, it is shown in [6, (1.19)] that for each $k \geq 1$ the alternating map ξ induces a map of R-modules $\xi^{(k)} : \bigwedge^{2k} G^* \to R$ given for any $\alpha_1^*, \dots, \alpha_{2k}^* \in G^*$ by

$$\xi^{(k)}(\alpha_1^* \wedge \cdots \wedge \alpha_{2k}^*) = \mathrm{pf}_{2k}(M),$$

where $M = (m_{ij})$ is the $2k \times 2k$ alternating matrix with (i,j)th entry $m_{ij} = [\alpha_i^* \circ \xi](\alpha_j^*)$.

We write $\mathrm{Pf}_{2k}(\xi)$ for the image of $\xi^{(k)}$. Clearly, $\mathrm{Pf}_{2k}(\xi) = 0$ when $2k > g$, and we have $\mathrm{Pf}_0(\xi) = R$ by convention.

Let (ξ_{ij}) be an alternating matrix for ξ. A $t \times t$ *principal* submatrix is a submatrix of (ξ_{ij}) on rows and columns indexed by the same set of t integers. Thus a principal submatrix is alternating as well, and it is easy to check that the ideal $\mathrm{Pf}_{2k}(\xi)$ is generated by the Pfaffians of the $2k \times 2k$ principal submatrices of (ξ_{ij}).

The complexes $\mathbb{K}^q(\xi)$. Let $S = \mathrm{S}(G) = \bigoplus_{t \geq 0} \mathrm{S}^t G$ be the symmetric algebra of the R-module G, and let $\mathbb{K}(\xi) = (\bigwedge_S (G^* \otimes_R S), \partial)$ be the Koszul complex over $\mathrm{S}(G)$ on the elements $\xi(\gamma_1^*), \dots, \xi(\gamma_g^*)$. It is easy to check that assigning degree 0 to the elements of G and degree 1 to the elements of G^* gives $\mathbb{K}(\xi) \cong \bigwedge(G^*) \otimes_R \mathrm{S}(G)$ the structure of a DG-*algebra* over R (i.e. for any homogeneous elements $a, b, c \in \mathbb{K}(\xi)$ with $|c|$ odd the differential ∂ of $\mathbb{K}(\xi)$ satisfies the Leibnitz rule $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$, one has $c^2 = 0$, and $ab = (-1)^{|a||b|}ba$).

Note that as a complex $\mathbb{K}(\xi)$ decomposes into a direct sum of subcomplexes

$$\mathbb{K}(\xi) = \bigoplus_{q \geq 0} \mathbb{K}^q(\xi)$$

where the qth *graded strand* $\mathbb{K}^q(\xi)$ has the form

$$\mathbb{K}^q(\xi) = 0 \to \bigwedge^q G^* \to G \otimes \bigwedge^{q-1} G^* \to \cdots \to \mathrm{S}^{q-1}G \otimes G^* \to \mathrm{S}^q G \to 0,$$

and the differential ∂ acts by

$$\partial(f \otimes \gamma^*_{a_1} \wedge \cdots \wedge \gamma^*_{a_k}) = \sum_{i=1}^{k} (-1)^{i-1} \xi(\gamma^*_{a_i}) f \otimes (\gamma^*_{a_1} \wedge \cdots \wedge \gamma^*_{a_{i-1}} \wedge \gamma^*_{a_{i+1}} \wedge \cdots \wedge \gamma^*_{a_k}).$$

The trace element. Let $\eta_\xi \in G \otimes G^*$ be the element which corresponds to $\mathrm{id}_{G^*} \in \mathrm{Hom}_R(G^*, G^*)$ under the canonical isomorphism $G \otimes G^* \cong \mathrm{Hom}_R(G^*, G^*)$. Thus if $\gamma_1, \ldots, \gamma_g$ is a basis of G, and $\gamma_1^*, \ldots, \gamma_g^*$ is the corresponding dual basis of G^*, then

$$\eta_\xi = \gamma_1 \otimes \gamma_1^* + \cdots + \gamma_g \otimes \gamma_g^*.$$

Note that ξ is alternating if and only if $\partial(\eta_\xi) = 0$, hence the homogeneous ideal (η_ξ) generated by η_ξ in $\mathbb{K}(\xi)$ is a subcomplex of $\mathbb{K}(\xi)$. Therefore

$$\overline{\mathbb{K}}(\xi) = \mathbb{K}(\xi)/(\eta_\xi)$$

is a DG-algebra, and we write $\overline{\mathbb{K}}^q(\xi) = \mathbb{K}^q(\xi)/\eta_\xi \mathbb{K}^{q-2}(\xi)$ for the qth graded strand of $\overline{\mathbb{K}}(\xi)$. Since

$$\gamma_i \otimes \gamma_1^* \wedge \cdots \wedge \gamma_g^* = (-1)^{i-1} \eta_\xi (\gamma_1^* \wedge \cdots \wedge \gamma_{i-1}^* \wedge \gamma_{i+1}^* \wedge \cdots \wedge \gamma_g^*),$$

it follows that $\mathrm{S}^k G \otimes \bigwedge^g G^* \subseteq (\eta_\xi)$ for each $k \geq 1$. Thus the complex $\overline{\mathbb{K}}^q(\xi)$ has the form

$$\overline{\mathbb{K}}^q(\xi) = 0 \to K_{q-s,s}(\xi) \to K_{q-s+1,s-1}(\xi) \to \cdots \to K_{q,0}(\xi) \to 0,$$

with $s = g-1$ if $q \geq g+1$, and with $s = q$ otherwise; where

$$K_{i,j}(\xi) = \mathrm{S}^i G \otimes \bigwedge^j G^* / \eta_\xi (\mathrm{S}^{i-1} G \otimes \bigwedge^{j-1} G^*). \tag{1.1}$$

In particular, we have $K_{0,q}(\xi) = \bigwedge^q G^*$ and $K_{q,0}(\xi) = \mathrm{S}^q G$.

The complexes $\mathcal{K}^q(\xi)$. As shown in [6, (2.18)], the differential $\partial : K_{0,g}(\xi) \to K_{1,g-1}(\xi)$ is the zero map, thus we have a direct sum of complexes

$$\overline{\mathbb{K}}^g(\xi) = \mathcal{L}^g(\xi) \oplus \mathcal{K}^g(\xi), \tag{1.2}$$

where $\mathcal{L}^g(\xi)$ is the complex whose only non-zero component is $K_{0,g}(\xi) = \bigwedge^g G^*$ in (homological) degree g, and

$$\mathcal{K}^g(\xi) = 0 \to K_{1,g-1}(\xi) \to \cdots \to K_{g,0}(\xi) \to 0.$$

When $q \neq g$ we set $\mathcal{K}^q(\xi) = \overline{\mathbb{K}}^q(\xi)$.

The homomorphism $d_\xi^{(q)}$. For $q = g - 2k$ define the homomorphism of free R-modules

$$d_\xi^{(q)} : \bigwedge^g G^* \to \bigwedge^q G^*$$

as the composition

$$\bigwedge^g G^* \xrightarrow{\wedge^g \Delta} \bigwedge^g (G^* \oplus G^*) \xrightarrow{\text{proj.}} \bigwedge^{2k} G^* \otimes \bigwedge^q G^* \xrightarrow{\xi^{(k)} \otimes 1} \bigwedge^q G^*,$$

where $\Delta : G^* \to G^* \oplus G^*$ is the diagonal map $\Delta(\gamma^*) = (\gamma^*, \gamma^*)$.

Set $[g] = \{1, \dots, g\}$. For $A = \{a_1 < \cdots < a_{2k}\} \subseteq [g]$ set $\gamma_A^* = \gamma_{a_1}^* \wedge \cdots \wedge \gamma_{a_{2k}}^*$ and write $\sigma(A)$ for the sign of the permutation that arranges the elements of the sequence $(a_1, \dots, a_u, \bar{a}_1, \dots, \bar{a}_{g-2k})$ in increasing order, where $\{\bar{a}_1 < \cdots < \bar{a}_{g-2k}\} = [g] \smallsetminus A$. With this notation we have

$$d_\xi^{(q)}(\gamma_{[g]}^*) = \sum_{\substack{A \subseteq [g] \\ |A| = 2k}} \sigma(A) \xi^{(k)}(\gamma_A^*) \gamma_{[g] \smallsetminus A}^*. \tag{1.3}$$

The complexes $\mathcal{D}^q(\xi)$. We are now ready to give the definition of the finite free complexes $\mathcal{D}^q(\xi)$.

(1.4) Definition ([2, (2.2)], [6, (2.15)])**.** Let $q \geq 0$ be an integer. Set

$$\mathcal{D}^q(\xi) = \begin{cases} 0 \to \bigwedge^g G^* \xrightarrow{d_\xi^{(q)}} K_{0,q}(\xi) \to \cdots \to K_{q,0}(\xi) \to 0 & \text{if } g - q = 2k \geq 2; \\ \mathcal{K}^q(\xi) & \text{otherwise.} \end{cases}$$

When $g - q = 2k \geq 2$ we write also $0 \to G^* \xrightarrow{d_\xi^{(q)}} \mathcal{K}^q(\xi)$ for the complex $\mathcal{D}^q(\xi)$.

It follows easily from the definition above that

$$\operatorname{length}(\mathcal{D}^q(\xi)) = \begin{cases} q + 1 & \text{if } g - q = 2k \geq 4; \\ q & \text{if } g - q = 2k + 1 \geq 3; \\ g - 1 & \text{if } g - q \leq 2. \end{cases} \tag{1.5}$$

2. The Reduction Lemma

Throughout this section we assume that $G = G_1 \oplus G_2$ and $\xi = \xi_1 \oplus \xi_2$, where the maps $\xi_1 : G_1^* \to G_1$ and $\xi_2 : G_2^* \to G_2$ are alternating maps of free modules, ξ_2 is an isomorphism, $\operatorname{rank} G_2 = 2$, and $\operatorname{rank} G_1 = g - 2 \geq 1$.

We also fix a basis $\gamma_1, \dots, \gamma_g$ of G such that $\Gamma_1 = \{\gamma_1, \dots, \gamma_{g-2}\}$ is a basis of G_1, and $\Gamma_2 = \{\gamma_{g-1}, \gamma_g\}$ is a basis of G_2. Let $\{\gamma_1^*, \dots, \gamma_g^*\}$ be the corresponding dual basis of G^*. Thus $\{\gamma_1^*, \dots, \gamma_{g-2}^*\}$ is the basis of G_1^* dual to Γ_1, and $\{\gamma_{g-1}^*, \gamma_g^*\}$ is the basis of G_2^* dual to Γ_2. Write M_2 for the (invertible) 2×2 alternating matrix of ξ_2 in this choice of bases.

Finally, recall that a complex $\mathcal{F}$ is *acyclic* if $\mathrm{H}_i(\mathcal{F}) = 0$ for $i \neq 0$; it is *exact* if $\mathrm{H}_i(\mathcal{F}) = 0$ for all i.

The proof of Theorem A uses induction on the rank of G, and the following technical lemma, that is the main result of this section, allows for the induction step to take place.

(2.1) Reduction Lemma. *If the complex $\mathcal{D}^q(\xi)$ is acyclic, then the complex $\mathcal{D}^q(\xi_1)$ is acyclic.*

The proof requires some preparation.

Let $\overline{\mathcal{C}}$ be the ideal generated in $\overline{\mathbb{K}}(\xi)$ by G_2^* and G_2.

(2.2) Lemma. *There is an isomorphism of DG-algebras* $\overline{\mathbb{K}}(\xi_1) \cong \overline{\mathbb{K}}(\xi)/\overline{\mathcal{C}}$.

Proof. Let (η_ξ, G_2^*, G_2) be the ideal generated in $\mathbb{K}(\xi)$ by η_ξ, G_2^*, and G_2. Since $\eta_\xi = \eta_{\xi_1} + \eta_{\xi_2}$ and $\eta_{\xi_2} \in (G_2^*, G_2)$, we have $(\eta_\xi, G_2^*, G_2) = (\eta_{\xi_1}, G_2^*, G_2)$. Therefore we obtain isomorphisms of DG-algebras

$$\overline{\mathbb{K}}(\xi_1) \cong \mathbb{K}(\xi_1)/(\eta_{\xi_1}) \cong \mathbb{K}(\xi)/(\eta_{\xi_1}, G_2^*, G_2) = \mathbb{K}(\xi)/(\eta_\xi, G_2^*, G_2) \cong \overline{\mathbb{K}}(\xi)/(G_2^*, G_2),$$

which completes the proof of the lemma. □

Let $\overline{\mathcal{C}}^q \subset \overline{\mathbb{K}}^q(\xi)$ be the qth graded strand of $\overline{\mathcal{C}}$. Since $\bigwedge^g G^* = \bigwedge^{g-2} G_1^* \otimes \bigwedge^2 G_2^*$, we have $\mathcal{L}^g(\xi) \subset \overline{\mathcal{C}}^g$, and therefore (1.2) yields

$$\overline{\mathcal{C}}^g = \mathcal{L}^g(\xi) \oplus \mathcal{C}^g,$$

where $\mathcal{C}^g = \overline{\mathcal{C}}^g \cap \mathcal{K}^g(\xi)$. For $q \neq g$ we set $\mathcal{C}^q = \overline{\mathcal{C}}^q$.

Thus for each q we have inclusions of complexes

$$\mathcal{C}^q \subseteq \mathcal{K}^q(\xi) \subseteq \mathcal{D}^q(\xi). \tag{2.3}$$

The next two lemmas contain the main steps in the proof of (2.1).

(2.4) Lemma. *There is an exact sequence of complexes*

$$0 \to \mathcal{C}^q \to \mathcal{D}^q(\xi) \to \overline{\mathcal{D}}^q(\xi_1) \to 0,$$

where $\overline{\mathcal{D}}^q(\xi_1) \cong \mathcal{D}^q(\xi_1)$ *for* $q \neq g-2$, *and* $\overline{\mathcal{D}}^{g-2}(\xi_1) \cong$ (*exact complex*) $\oplus\, \mathcal{D}^{g-2}(\xi_1)$.

Proof. In order to establish the lemma, we consider the following cases:

(1) $g - q \leq 1$ or $g - q = 2k+1$;
(2) $g - q = 2k \geq 4$; and
(3) $g - q = 2$.

Case (1). In this case $\mathcal{D}^q(\xi) = \mathcal{K}^q(\xi)$ and $\mathcal{D}^q(\xi_1) = \mathcal{K}^q(\xi_1) = \overline{\mathbb{K}}^q(\xi_1)$. Furthermore $\mathcal{K}^q(\xi)/\mathcal{C}^q \cong \overline{\mathbb{K}}^q(\xi)/\overline{\mathcal{C}}^q$, hence by Lemma (2.2) we obtain

$$\mathcal{D}^q(\xi)/\mathcal{C}^q \cong \overline{\mathbb{K}}^q(\xi)/\overline{\mathcal{C}}^q \cong \overline{\mathbb{K}}^q(\xi_1) = \mathcal{D}^q(\xi_1),$$

which is what we needed.

Case (2). We have $\mathcal{K}^q(\xi) \cong \overline{\mathbb{K}}^q(\xi)$ and $\mathcal{C}^q \cong \overline{\mathcal{C}}^q$. Furthermore, $\mathcal{D}^q(\xi)$ has the form $0 \to \bigwedge^g G^* \to^{d_\xi^{(q)}} \overline{\mathbb{K}}^q(\xi)$. Since $\mathcal{K}^q(\xi_1) = \overline{\mathbb{K}}^q(\xi_1)$, by (2.2) the complex $\mathcal{D}^q(\xi)/\mathcal{C}^q$ has the form

$$0 \to \bigwedge^g G^* \xrightarrow{\bar{d}} \mathcal{K}^q(\xi_1),$$

where the map $\bar{d}$ is the composition

$$\bigwedge^g G^* \xrightarrow{d_\xi^{(q)}} \bigwedge^q G^* \xrightarrow{\text{proj}} \bigwedge^q G_1^*. \tag{2.5}$$

It is immediate from (1.3) and (2.5) that

$$\begin{aligned} \bar{d}(\gamma_{[g]}^*) &= \sum_{\substack{A \subseteq [g-2] \\ |A| = 2k-2}} \sigma(A) \xi_2^{(1)}(\gamma_{g-1}^* \wedge \gamma_g^*) \xi_1^{(k-1)}(\gamma_A^*) \gamma_{[g-2] \smallsetminus A}^* \\ &= \mathrm{pf}_2(M_2) d_{\xi_1}^{(q)}(\gamma_{[g-2]}^*). \end{aligned} \tag{2.6}$$

Since ξ_2 is an isomorphism, $\mathrm{pf}_2(M_2)$ is invertible in R. Therefore, the map

$$\bigwedge^g G^* = \bigwedge^{g-2} G_1^* \otimes \bigwedge^2 G_2^* \xrightarrow{1\otimes\xi_2^{(1)}} \bigwedge^{g-2} G_1^*$$

yields an isomorphism $\phi : \bigwedge^g G^* \to \bigwedge^{g-2} G_1^*$ and an isomorphism of complexes

$$\begin{array}{ccccc} 0 & \longrightarrow & \bigwedge^g G^* & \xrightarrow{\bar d} & \mathcal{K}^q(\xi_1) \\ & & \phi\big\downarrow & & \big\| \\ 0 & \longrightarrow & \bigwedge^{g-2} G_1^* & \xrightarrow[d_{\xi_1}^{(q)}]{} & \mathcal{K}^q(\xi_1). \end{array}$$

This completes the proof in case (2).

Case (3). We have $\mathcal{K}^{g-2}(\xi) \cong \overline{\mathbb{K}}^{g-2}(\xi)$ and $\mathcal{C}^{g-2} \cong \overline{\mathcal{C}}^{g-2}$. Furthermore, the complex $\mathcal{D}^{g-2}(\xi)$ has the form

$$0 \longrightarrow \bigwedge^g G^* \xrightarrow{d_\xi^{(g-2)}} \overline{\mathbb{K}}^{g-2}(\xi),$$

hence by (2.2) the complex $\mathcal{D}^{g-2}(\xi)/\mathcal{C}^{g-2}$ has the form

$$0 \longrightarrow \bigwedge^{g-2} G^* \xrightarrow{\bar d} \overline{\mathbb{K}}^{g-2}(\xi_1),$$

where $\bar d$ is defined in (2.5). As $q = g-2$, formula (2.6) becomes

$$\bar d(\gamma^*_{[g]}) = \xi_2^{(1)}(\gamma^*_{g-1} \wedge \gamma^*_g)\gamma^*_{[g-2]},$$

therefore $\bar d = \phi : \bigwedge^g G^* \to \bigwedge^{g-2} G_1^*$ is an isomorphism. From (1.2) we obtain

$$\overline{\mathbb{K}}^{g-2}(\xi_1) = 0 \longrightarrow \bigwedge^{g-2} G_1^* \xrightarrow{0} \mathcal{K}^{g-2}(\xi_1),$$

and hence an isomorphism of complexes

$$\mathcal{D}^q(\xi)/\mathcal{C}^q \cong (0 \longrightarrow \bigwedge^g G^* \xrightarrow{\bar d} \bigwedge^{g-2} G_1^* \longrightarrow 0) \oplus \mathcal{K}^{g-2}(\xi_1).$$

Since $\mathcal{D}^{g-2}(\xi_1) = \mathcal{K}^{g-2}(\xi_1)$, we obtain the desired conclusion also in Case (3).

The proof of Lemma (2.4) is now complete. □

(2.7) Lemma. *If the complex $\mathcal{D}^q(\xi)$ is acyclic, then the complex $\mathcal{C}^q$ is exact.*

Proof. Let $c \in \mathcal{C}^q$ be a cycle in degree i; thus $c \in K_{q-i,i}(\xi)$ and $\partial(c) = 0$. We will show that c is a boundary in $\mathcal{C}^q$.

Assume first that $i = 0$. Then $K_{q,0}(\xi) = \mathrm{S}^q G = \bigoplus_{b=0}^q \mathrm{S}^{q-b}G_1 \otimes \mathrm{S}^b G_2$, hence

$$c = \sum_{b\geq 1} c_{q-b,b} \quad \text{with} \quad c_{q-b,b} \in \mathrm{S}^{q-b}G_1 \otimes \mathrm{S}^b G_2. \tag{2.8}$$

If $q = 0$ we must have $c = 0$ and we are done; therefore we may assume $q \geq 1$. In view of (1.2) and (2.3) we consider (2.8) as a decomposition in $\overline{\mathcal{C}}^q \subseteq \overline{\mathbb{K}}^q(\xi)$. Since ξ_2 is an isomorphism, the complexes $\mathbb{K}^b(\xi_2)$, $b \geq 1$, are split exact, hence the complexes $\mathbb{K}^{q-b}(\xi_1) \otimes \mathbb{K}^b(\xi_2)$ are exact for $b \geq 1$. As each $c_{q-b,b}$ is a cycle when considered as an element of $\mathbb{K}^{q-b}(\xi_1) \otimes \mathbb{K}^b(\xi_2)$, there exist elements $a_{q-b,b} \in \mathbb{K}^{q-b}(\xi_1) \otimes \mathbb{K}^b(\xi_2)$ such that $\partial(a_{q-b,b}) = c_{q-b,b}$ for each $b \geq 1$. Set

$$a = \sum_{b\geq 1} a_{q-b,b} \in \mathbb{K}^q(\xi) = \bigoplus_{b=0}^q \mathbb{K}^{q-b}(\xi_1) \otimes \mathbb{K}^b(\xi_2).$$

Thus the image $\bar{a}$ of a in $\overline{\mathbb{K}}^q(\xi)$ satisfies $\bar{a} \in \overline{\mathcal{C}}^q$ and $\partial(\bar{a}) = c$, therefore c is a boundary in $\overline{\mathcal{C}}^q$, hence also in $\mathcal{C}^q$.

We assume for the rest of the proof that $i \geq 1$. As $\mathcal{D}^q(\xi)$ is acyclic, c is a boundary in $\mathcal{D}^q(\xi)$, hence there exists $a \in \mathcal{D}^q(\xi)$ such that $\partial(a) = c$.

There are two possible cases: (1) $i = q$ with $g - q = 2k \geq 2$; and (2) otherwise.

In case (1) we have $a = x\gamma^*_{[g]} \in \bigwedge^g G^*$ for some $x \in R$. Since $c = d^{(q)}_\xi(a) \in \mathcal{C}^q$, we obtain $\bar{d}(a) = 0$ for the map $\bar{d}$ from (2.5). It follows from (2.6) that $x\,\mathrm{Pf}_{g-2-q}(\xi_1) = 0$. Since ξ_2 is an isomorphism we have $\mathrm{Pf}_{g-2-q}(\xi_1) = \mathrm{Pf}_{g-q}(\xi)$, therefore $x\,\mathrm{Pf}_{g-q}(\xi) = 0$. Thus $c = d^{(q)}_\xi(a) = 0$, which completes the proof of the lemma in case (1).

In case (2) we necessarily have $a \in \mathcal{K}^q(\xi)$. Thus $c = \partial(a)$ is a boundary also in $\overline{\mathbb{K}}^q(\xi)$, and it suffices to show that we can choose $a' \in \overline{\mathcal{C}}^q$ such that $\partial(a') = c$.

Let $f \in \mathbb{K}^q(\xi) = \bigoplus_{b=0}^q \mathbb{K}^{q-b}(\xi_1) \otimes \mathbb{K}^b(\xi_2)$ be a preimage of a, and write f_1 for the component of f in $\mathbb{K}^q(\xi_1)$. Thus $f - f_1 \in (G_2^*, G_2) \subset \mathbb{K}(\xi)$.

If $f_1 \in \eta_{\xi_1}\mathbb{K}^{q-2}(\xi_1)$, then $a \in \overline{\mathcal{C}}^q$ and we are done, thus we may assume that $f_1 \notin \eta_{\xi_1}\mathbb{K}^{q-2}(\xi_1)$. Since $\partial(a) \in \overline{\mathcal{C}}^q$, we get $\partial(f) \in (\eta_{\xi_1}, G_2^*, G_2)$. Therefore, $\partial(f_1) \in \eta_{\xi_1}\mathbb{K}^{q-2}(\xi_1)$, i.e. $\partial(f_1) = \eta_{\xi_1}c_1$ for some $c_1 \in \mathbb{K}^{q-2}(\xi_1)$. This yields $0 = \partial(\eta_{\xi_1}c_1) = \eta_{\xi_1}\partial(c_1)$, hence [6, (6.16)] implies $\partial(c_1) = \eta_{\xi_1}c_2$ for some $c_2 \in \mathbb{K}^{q-4}(\xi_1)$.

Mapping onto $\overline{\mathbb{K}}^q$ we obtain (where $^-$ denotes image in $\overline{\mathbb{K}}^q$) $\partial(\bar{f}_1) = \bar{\eta}_{\xi_1}\bar{c}_1 = -\bar{\eta}_{\xi_2}\bar{c}_1$. Observe that $\partial(e) = -\eta_{\xi_2}$ for $e = -\,\mathrm{pf}(M_2)^{-1}\gamma^*_{g-1} \wedge \gamma^*_g \in \bigwedge^2 G_2^*$; since $e\eta_{\xi_2} \in G_2 \otimes \bigwedge^3 G_2^* = 0$, we have

$$\partial(\bar{e}\bar{c}_1) = \partial(\bar{e})\bar{c}_1 + \bar{e}\partial(\bar{c}_1) = -\bar{\eta}_{\xi_2}\bar{c}_1 + \bar{e}\bar{\eta}_{\xi_1}\bar{c}_2 = -\bar{\eta}_{\xi_2}\bar{c}_1 - \bar{e}\bar{\eta}_{\xi_2}\bar{c}_2 = -\bar{\eta}_{\xi_2}\bar{c}_1 = \partial(\bar{f}_1).$$

It follows that for $a' = a - \bar{f}_1 + \bar{e}\bar{c}_1$ we have $a' = \overline{f - f_1} + \bar{e}\bar{c}_1 \in \overline{\mathcal{C}}^q$ and

$$\partial(a') = \partial(a) = c,$$

therefore a' is the desired element of $\overline{\mathcal{C}}^q$. The proof of (2.7) is now complete. □

The proof of the Reduction Lemma is now straightforward.

Proof of Reduction Lemma (2.1). By (2.7) the complex $\mathcal{C}^q$ is exact. The acyclicity of the complex $\mathcal{D}^q(\xi_1)$ now follows from the homology exact sequence for the exact sequence of complexes from Lemma (2.4). □

3. Proofs of the main theorems

If ϕ is a map of finite free R-modules, we write $I_s(\phi)$ for the ideal generated in R by the $s \times s$ minors of ϕ.

Also, if J is an ideal in R and $u \leq v$ are integers, then as an easy consequence of the elementary properties of the notion of grade (see e.g. [3, §1.2]) we have that the inequality $\mathrm{grade}_R\, J \geq u$ holds if and only if $\mathrm{grade}_{R_\mathfrak{p}}\, J_\mathfrak{p} \geq u$ for every prime ideal $\mathfrak{p}$ with $\mathrm{depth}\, R_\mathfrak{p} < v$.

We proceed with the proof of Theorem A.

Proof of Theorem A. (iii) $\Longrightarrow$ (ii). This is [6, (6.2)].

(ii) $\Longrightarrow$ (i). This is trivial.

(i) $\Longrightarrow$ (iii). We induce on g. Note that the case $g = 1$ is trivial, so assume that $g \geq 2$. The possible cases are: (1) $g - q = 2k \geq 2$; (2) $g - q = 2k + 1 \geq 3$; and (3) $g - q \leq 1$.

Case (1). In this case $r = 2k-1 = g-q-1$, the complex $\mathcal{D}^q(\xi)$ has the form $0 \to \bigwedge^g G^* \xrightarrow{d_\xi^{(q)}} \mathcal{K}^q(\xi)$, and we need to prove that

$$\operatorname{grade}\operatorname{Pf}_{2t}(\xi) \geq g-2t+1 \quad \text{for} \quad g-q \leq 2t \leq g.$$

Note that $I_1(d_\xi^{(q)}) = \operatorname{Pf}_{g-q}(\xi)$, hence the acyclicity of $\mathcal{D}^q(\xi)$ and (1.5) yield (by the acyclicity criterion of Buchsbaum and Eisenbud [4]) that

$$\operatorname{grade}\operatorname{Pf}_{g-q}(\xi) \geq \operatorname{length}(\mathcal{D}^q(\xi)) = q+1,$$

which is the first of the inequalities we need to prove. If $g \leq 3$ we are done; otherwise (i.e. when $g \geq 4$) it is enough (see the remarks in the beginning of this section) to establish that the remaining inequalities hold in $R_\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth} R_\mathfrak{p} < q-1$. Thus we may assume that R is local of depth $< q-1$. Then $R = \operatorname{Pf}_{g-q}(\xi) \subseteq \operatorname{Pf}_2(\xi)$, hence $I_1(\xi) = \operatorname{Pf}_2(\xi) = R$ and the map ξ decomposes as $\xi = \xi_1 \oplus \xi_2$ where $\xi_1 : G_1^* \to G_1$ and $\xi_2 : G_2^* \to G_2$ are alternating maps, ξ_2 is an isomorphism, $G = G_1 \oplus G_2$, and $\operatorname{rank} G_2 = 2$. By the Reduction Lemma the complex $\mathcal{D}^q(\xi_1)$ is acyclic, hence our induction hypothesis yields

$$(*) \qquad \operatorname{grade}\operatorname{Pf}_{2t}(\xi_1) \geq (g-2)-2t+1 \quad \text{for} \quad r'+1 \leq 2t \leq g-2,$$

where $r' = 2\max\{0, \lfloor (g-2-q-1)/2 \rfloor\}+1$. If $q = g-2$ we get $r' = 1$, and the inequalities (*) become

$$\operatorname{grade}\operatorname{Pf}_{2t+2}(\xi) \geq g-(2t+2)+1 \quad \text{for} \quad 4 \leq 2t+2 \leq g,$$

which is what we needed. If $q \leq g-4$, then $r' = g-q-3$, and the inequalities (*) become

$$\operatorname{grade}\operatorname{Pf}_{2t+2}(\xi) \geq g-(2t+2)+1 \quad \text{for} \quad g-q \leq 2t+2 \leq g,$$

which is again exactly what we needed. This completes the proof of Case (1).

Case (2). In this case $r = g-q$, the complex $\mathcal{D}^q(\xi) = \mathcal{K}^q(\xi)$, and we need to prove that

$$\operatorname{grade}\operatorname{Pf}_{2t}(\xi) \geq g-2t+1 \quad \text{for} \quad g-q+1 \leq 2t \leq g.$$

Since the differential of $\mathcal{D}^q(\xi) = \mathcal{K}^q(\xi)$ is induced from the differential of the Koszul complex, the acyclicity criterion [4] and (1.5) yield

$$\operatorname{grade} I_1(\xi) \geq \operatorname{length}(\mathcal{D}^q(\xi)) = q.$$

Since $g-2t+1 \leq q$ for $g-q+1 \leq 2t$, it suffices to show that the desired inequalities hold in $R_\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth} R_\mathfrak{p} < q$. Thus we may assume that R is local of depth $< q$. Then $I_1(\xi) = R$, hence ξ decomposes into $\xi = \xi_1 \oplus \xi_2$ as in Case (1). Since $g-2 \geq q+1 \geq 2$, by the Reduction Lemma the complex $\mathcal{D}^q(\xi_1)$ is acyclic, and our induction hypothesis yields

$$\operatorname{grade}\operatorname{Pf}_{2t}(\xi_1) \geq (g-2)-2t+1 \quad \text{for} \quad r'+1 \leq 2t \leq g-2,$$

where $r' = 2\max\{0, \lfloor (g-2-q-1)/2 \rfloor\}+1 = g-2-q$. Rewriting, we obtain

$$\operatorname{grade}\operatorname{Pf}_{2t+2}(\xi) \geq g-(2t+2)+1 \quad \text{for} \quad g-q+1 \leq 2t+2 \leq g,$$

which is exactly what we need. This completes the proof of Case (2).

Case (3). In this case $r = 1$, the complex $\mathcal{D}^q(\xi) = \mathcal{K}^q(\xi)$, and we need to prove that

$$\operatorname{grade}\operatorname{Pf}_{2t}(\xi) \geq g-2t+1 \quad \text{for} \quad 2 \leq 2t \leq g.$$

As in Case (2) we obtain $\operatorname{grade}\operatorname{Pf}_2(\xi) = \operatorname{grade} I_1(\xi) \geq \operatorname{length}(\mathcal{D}^q(\xi)) = g-1$, which is the first inequality we need to prove. If $g \leq 3$, then the proof of Theorem A is complete. Otherwise (i.e. when $g \geq 4$), it is enough to show that the remaining inequalities hold in $R_\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth} R_\mathfrak{p} < g-1$, thus we may assume that R is local of depth $< g-1$. Then as in Case (1) ξ decomposes into $\xi = \xi_1 \oplus \xi_2$ and by the Reduction Lemma the complex $\mathcal{D}^q(\xi_1)$ is acyclic. Therefore, the induction hypothesis yields

$$\operatorname{grade}\operatorname{Pf}_{2t}(\xi_1) \geq (g-2) - 2t + 1 \quad \text{for} \quad r' + 1 \leq 2t \leq g-2,$$

where $r' = 2\max\{0, \lfloor (g-2-q-1)/2 \rfloor\} + 1 = 1$. Rewriting the inequalities above we obtain

$$\operatorname{grade}\operatorname{Pf}_{2t+2}(\xi) \geq g - (2t+2) + 1 \quad \text{for} \quad 4 \leq 2t+2 \leq g,$$

which is what we needed. This completes the proof of Case (3), hence also the proof of Theorem A. □

With Theorem A available, the proof of Theorem B is not hard.

Proof of Theorem B. (iii) $\Longrightarrow$ (ii). First note that g must be odd. Indeed, otherwise $I = \operatorname{Pf}_g(\xi)$ is a principal ideal and $I \neq R$ because $M \neq 0$, hence $\operatorname{grade} I \leq 1$ which contradicts (iii). Therefore, g is odd, and (ii) follows from [6, (6.17)].

(ii) $\Longrightarrow$ (i). This is a triviality.

(i) $\Longrightarrow$ (iii). Since $\mathcal{D}^q(\xi)$ is acyclic, Theorem A yields $\operatorname{grade} I \geq 1$. Let $x \in I$ be a regular element. Since $\mathrm{S}^q M$ is torsion free, the complex $\mathcal{D}^q(\xi) \otimes R/xR = \mathcal{D}^q(\xi \otimes R/xR)$ is acyclic; see e.g. [3, (1.1.5)]. Thus Theorem A yields also

$$\operatorname{grade}_{R/xR}\operatorname{Pf}_{2i}(\xi \otimes R/xR) \geq g - 2i + 1 \quad \text{for} \quad r+1 \leq 2i \leq g.$$

Since $x \in \operatorname{Pf}_{2i}(\xi)$ for $r+1 \leq 2i \leq g$, lifting back to R gives the desired inequalities.

The proof of Theorem B is now complete. □

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