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LINEAR PRESERVERS OF ISOMORPHIC TYPES OF LATTICES OF INVARIANT OPERATOR RANGES

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Abstract. We describe all linear self-mappings of the space of bounded linear operators in an infinite dimensional separable complex Hilbert space which preserve the isomorphism class of the lattice of invariant operator ranges.

1. Main results

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space. Let $\mathcal{L}(\mathcal{H})$ denote the Banach algebra of linear bounded operators on $\mathcal{H}$ with the operator norm. An *operator range* is, by definition, a linear set $\mathcal{M} \subseteq \mathcal{H}$ such that

$$\mathcal{M} = \operatorname{Range} G := \{Gx | x \in \mathcal{H}\}$$

for some $G \in \mathcal{L}(\mathcal{H})$. Equivalently, $\mathcal{M} \subseteq \mathcal{H}$ is an operator range if and only if $\mathcal{M} = \operatorname{Range} G$ for some linear bounded operator $G : \mathcal{H}_0 \to \mathcal{H}$ with zero kernel, where $\mathcal{H}_0$ is a suitable Hilbert space.

If $T \in \mathcal{L}(\mathcal{H})$, we denote by $\mathcal{IOR}(T)$ the set of all operator ranges $\mathcal{M}$ that are T-invariant: $Tx \in \mathcal{M}$ for every $x \in \mathcal{M}$. The set $\mathcal{IOR}(T)$ is a lattice (with respect to addition and intersection); this follows from the general fact that intersection and sum of two operator ranges are again operator ranges. For a proof of this fact and for other fundamental properties of operator ranges see, for example, [3].

In this paper we prove two theorems:

Theorem 1. *For every $T \in \mathcal{L}(\mathcal{H})$, if $\mathcal{M}_1 \subset \mathcal{M}_2$ are two T-invariant operator ranges such that the dimension of the factor linear set $\mathcal{M}_2/\mathcal{M}_1$ exceeds one, then there exists $\mathcal{M} \in \mathcal{IOR}(T)$ with the property that*

$$\mathcal{M}_1 \subset \mathcal{M} \subset \mathcal{M}_2, \qquad \mathcal{M}_1 \neq \mathcal{M} \neq \mathcal{M}_2.$$

Theorem 2. *Let $\phi : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H})$ be a bijective linear map such that for every $T \in \mathcal{L}(\mathcal{H})$, the lattices $\mathcal{IOR}(T)$ and $\mathcal{IOR}(\phi(T))$ are isomorphic. Then there exists a non-zero complex number α, a boundedly invertible $S \in \mathcal{L}(\mathcal{H})$, and a (not necessarily continuous) linear functional $f : \mathcal{L}(\mathcal{H}) \longrightarrow \mathbb{C}$ such that*

$$\phi(T) = \alpha S T S^{-1} + f(T) I \tag{1}$$

for every $T \in \mathcal{L}(\mathcal{H})$.

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Conversely, every mapping of the form (1) *is bijective on* $\mathcal{L}(\mathcal{H})$, *and preserves the isomorphism classes of lattices of invariant operator ranges.*

It was proved in [4] that the same formula (1) describes the bijective linear maps ϕ on $\mathcal{L}(\mathcal{H})$ with the property that the lattice of T-invariant linear sets and the lattice of $\phi(T)$-invariant linear sets are isomorphic, for every $T \in \mathcal{L}(\mathcal{H})$. Combining this result with Theorem 2, we obtain:

Corollary 3. *A bijective linear map* $\phi : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H})$ *has the property that for every* $T \in \mathcal{L}(\mathcal{H})$, *the lattices* $\mathcal{IOR}(T)$ *and* $\mathcal{IOR}(\phi(T))$ *are isomorphic, if and only if* ϕ *has the property that for every* $T \in \mathcal{L}(\mathcal{H})$, *the lattices of* T*-invariant linear sets and of* $\phi(T)$*-invariant linear sets are isomorphic.*

Theorem 1 will be used in the proof of Theorem 2. Perhaps Theorem 1 is independently interesting.

2. Proof of Theorem 1

We start with some preliminaries. Let $\mathcal{N}$ be an operator range. There is a norm $\|\cdot\|_{\mathcal{N}}$ on $\mathcal{N}$ with respect to which $\mathcal{N}$ is a Hilbert space, and in addition,

$$\|x\|_{\mathcal{N}} \geq \|x\|_{\mathcal{H}} \tag{2}$$

for every $x \in \mathcal{N}$, where $\|\cdot\|_{\mathcal{H}}$ is the norm in $\mathcal{H}$ (see Theorem 1.1 of [3]). In fact, if $\mathcal{N} = \operatorname{Range} G$, where $G : \mathcal{H}_0 \to \mathcal{H}$ is a linear bounded operator with zero kernel, then one can choose $\|\cdot\|_{\mathcal{N}}$ so that

$$\|Gy\|_{\mathcal{N}}^2 = \|Gy\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}_0}^2, \quad y \in \mathcal{H}_0. \tag{3}$$

Lemma 4. *If* $T \in \mathcal{L}(\mathcal{H})$, *and if* $\mathcal{N}$ *is a* T*-invariant operator range, then* T *is bounded, as an operator on the Hilbert space* $\mathcal{N}$.

Proof. By the closed graph theorem, we only have to check that the graph of T is closed in the Hilbert space $\mathcal{N} \oplus \mathcal{N}$. Let a sequence $\{(x_n, Tx_n) \in \mathcal{N} \oplus \mathcal{N}\}_{n=1}^{\infty}$ converge to $(y, z) \in \mathcal{N} \oplus \mathcal{N}$. Then $x_n \to y$ and $Tx_n \to z$ in $\mathcal{N}$; therefore also $x_n \to y$ and $Tx_n \to z$ in $\mathcal{H}$. Since $T \in \mathcal{L}(\mathcal{H})$, we must have $z = Ty$, which proves the closedness of the graph of T in $\mathcal{N} \oplus \mathcal{N}$. □

Lemma 5. *The set of operator ranges in the Hilbert space* $\mathcal{N}$ *(in short:* $\mathcal{N}$*-operator ranges) coincides with the set of operator ranges in the Hilbert space* $\mathcal{H}$ *(in short:* $\mathcal{H}$*-operator ranges) that are contained in* $\mathcal{N}$.

Proof. Let $G : \mathcal{H}_0 \to \mathcal{H}$ be a linear bounded operator with zero kernel and range $\mathcal{N}$, and assume that $\|\cdot\|_{\mathcal{N}}$ is given by (3). If $\operatorname{Range} B \subseteq \mathcal{N}$ for some $B \in \mathcal{L}(\mathcal{H})$, then by Douglas' lemma (see [2]), there exists $C \in \mathcal{L}(\mathcal{H}, \mathcal{H}_0)$ such that $B = GC$. Therefore,

$$\|By\|_{\mathcal{N}}^2 = \|GCy\|_{\mathcal{N}}^2 = \|By\|_{\mathcal{H}}^2 + \|Cy\|_{\mathcal{H}_0}^2 \leq (\|B\|^2 + \|C\|^2)\|y\|_{\mathcal{H}}^2,$$

and so B is a bounded operator from $\mathcal{H}$ into $\mathcal{N}$. Hence $\operatorname{Range} B$ is an $\mathcal{N}$-operator range. Conversely, if $\mathcal{M} = \operatorname{Range} B$, $B \in \mathcal{L}(\mathcal{N})$ is an $\mathcal{N}$-operator range, then (2) shows that B is bounded as an operator into $\mathcal{H}$, and so $\mathcal{M}$ is an $\mathcal{H}$-operator range. □

Proof of Theorem 1. Let $T \in \mathcal{L}(\mathcal{H})$, and fix two T-invariant operator ranges $\mathcal{M}_1 \subset \mathcal{M}_2$ satisfying the hypotheses of Theorem 1. In view of Lemmas 4 and 5 (applied for $\mathcal{N} = \mathcal{M}_2$), we can (and do) assume that $\mathcal{M}_2 = \mathcal{H}$.

Let us consider three possibilities:

(i) $\mathcal{M}_1$ is not closed and not dense in $\mathcal{H}$. We are done—take $\mathcal{M}$ to be the closure of $\mathcal{M}_1$.

(ii) $\mathcal{M}_1$ is closed. Note that every $\hat{T} \in \mathcal{L}(\mathcal{H}_0)$, where the dimension of the Hilbert space $\mathcal{H}_0$ exceeds one, has an invariant operator range different from $\{0\}$ and $\mathcal{H}_0$. Indeed, leaving aside the trivial case of a scalar operator $\hat{T}$, since the spectrum of $\hat{T}$ is not empty, for some $\lambda \in \mathbb{C}$ we will have $\operatorname{Ker}(\hat{T} - \lambda I) \neq \{0\}$ or $\operatorname{Range}(\hat{T} - \lambda I) \neq \mathcal{H}_0$. So we may take $\operatorname{Ker}(\hat{T} - \lambda I)$ or $\operatorname{Range}(\hat{T} - \lambda I)$, as appropriate, as the required operator range. Applying the observation to the operator $\hat{T}$ induced by T in the factor space $\mathcal{H}/\mathcal{M}_1$, we complete the proof of Theorem 1 in case $\mathcal{M}_1$ is closed.

(iii) $\mathcal{M}_1$ is dense in $\mathcal{H}$. We have $\mathcal{M}_1 = \operatorname{Range} V$, where V is a bounded positive operator on $\mathcal{H}$ (see [3]). Moreover, by Lemma 4, T is bounded as an operator on the Hilbert space $\mathcal{M}_1$. It is also bounded as an operator on the Hilbert space $\mathcal{H}$. Therefore, by Donoghue's Theorem [1], the operator T maps $\operatorname{Range} \phi(V)$ into itself for every Löwner function ϕ (in fact, it is sufficient to use a much easier result with $\phi(t) = t^{\alpha}$, $0 < \alpha < 1$; see, e.g., [5], Theorem 4.1.10). Using a description of $\operatorname{Range} V^{\alpha}, 0 < \alpha < 1$, in terms of the spectral decomposition of V, one can easily check that these operator ranges are properly contained in $\mathcal{H}$ and properly contain $\mathcal{M}_1$. Thus, we obtain a continuum of required T-invariant operator ranges.

3. Proof of Theorem 2

The converse statement of the theorem is obvious, so we will focus on the proof of the direct statement.

The proof follows the pattern of the proof of Theorem 3.1 in [4]. We need several lemmas, in analogy with the proof given in [4]. In what follows, we denote by lat_n (resp. $\operatorname{lat}_\infty$) the lattice of operator ranges in the n-dimensional ($n < \infty$) (resp. infinite dimensional separable) Hilbert space.

We start with a known result on operator ranges.

Lemma 6. *Let $\mathcal{H}$ be a separable Hilbert space. If $\mathcal{M} \neq \mathcal{H}$ is an operator range in $\mathcal{H}$, then there exists a nonzero operator range $\mathcal{N}$ in $\mathcal{H}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$.*

Proof. The statement is clear if $\mathcal{M}$ is closed. Otherwise, by a result of von Neumann (see [3] for a transparent proof due to Dixmier) there exists a unitary operator U such that $\mathcal{M} \cap U\mathcal{M} = \{0\}$, so we may take $\mathcal{N} = U\mathcal{M}$. □

Lemma 7. *Let $\mathcal{H}$ be a separable Hilbert space, and let $T \in \mathcal{L}(\mathcal{H})$ be such that $\mathcal{IOR}(T)$ is isomorphic as a lattice to* lat_n*, $n < \infty$ (resp.* $\operatorname{lat}_\infty$*). Then T is a scalar multiple of the identity and* $\dim \mathcal{H} = n$ *(resp. $\mathcal{H}$ is infinite dimensional).*

Proof. Assume first that $\mathcal{IOR}(T)$ is isomorphic to lat_n, $n < \infty$. Then every chain

$$\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots \subseteq \mathcal{M}_m, \quad \mathcal{M}_j \in \mathcal{IOR}(T), \quad j = 1, 2, \cdots, m,$$

has at most $n+1$ distinct elements, and there exists such a chain with exactly $n+1$ distinct elements. By Theorem 1, dim $\mathcal{H} = n$. Proposition 2.5 of [4] shows that T has the required form.

Now assume that $\mathcal{IOR}(T)$ is isomorphic to lat_∞. Since every nonzero element of lat_∞ contains a minimal nonzero element, namely, a one-dimensional subspace, the same is true of $\mathcal{IOR}(T)$. By Theorem 1, a minimal nonzero element of $\mathcal{IOR}(T)$ must be a one-dimensional subspace, i.e., the subspace spanned by an eigenvector of T. We obtain that every nonzero T-invariant operator range contains an eigenvector.

Let $\tau : \mathcal{IOR}(T) \to \mathrm{lat}_\infty$ be an isomorphism, where lat_∞ is the lattice of operator ranges in an infinite dimensional separable Hilbert space $\mathcal{H}_0$. Assume that u and v are linearly independent eigenvectors of T corresponding to eigenvalues λ and μ, respectively. The subspace

$$\tau\left((\operatorname{span} u) + (\operatorname{span} v)\right) \subset \mathcal{H}_0$$

is clearly two-dimensional, and therefore contains infinitely many different elements of lat_∞. So the element

$$(\operatorname{span} u) + (\operatorname{span} v) \in \mathcal{IOR}(T) \tag{4}$$

also contains infinitely many different elements of $\mathcal{IOR}(T)$. However, (4) contains infinitely many T-invariant subspaces if and only if $\lambda = \mu$. We obtain that T has only one eigenvalue (perhaps of high multiplicity); call it λ_0.

If $\operatorname{Ker}(T - \lambda_0 I) \neq \mathcal{H}$, then $\tau\left(\operatorname{Ker}(T - \lambda_0 I)\right) \neq \mathcal{H}_0$. By Lemma 6, there exists $\mathcal{M} \in \mathrm{lat}_\infty$, $\mathcal{M} \neq \{0\}$, such that

$$\tau\left(\operatorname{Ker}(T - \lambda_0 I)\right) \cap \mathcal{M} = \{0\}.$$

Then $\tau^{-1}(\mathcal{M})$ is a nonzero T-invariant operator range that has the zero intersection with $\operatorname{Ker}(T - \lambda_0 I)$. On the other hand, we have seen above that $\tau^{-1}(\mathcal{M})$ contains an eigenvector of T corresponding to the eigenvalue λ_0, a contradiction. So we must conclude that $\operatorname{Ker}(T - \lambda_0 I) = \mathcal{H}$. ☐

Lemma 8. *Let $T \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional separable Hilbert space. Then the following are equivalent:*

(a) $T = \alpha P + \beta I$ *with* $\alpha \in \mathbb{C} \setminus \{0\}$, $\beta \in \mathbb{C}$, $P = P^2$, *and* $\operatorname{rank} P = n < \infty$*;*
(b) $\mathcal{IOR}(T)$ *is isomorphic as a lattice to* $\mathrm{lat}_n \oplus \mathrm{lat}_\infty$.

Proof. Assume (a) holds. Clearly, $\mathcal{IOR}(T) = \mathcal{IOR}(P)$. Since every P-invariant operator range $\mathcal{M}$ is of the form $\mathcal{M} = P\mathcal{M} + (I - P)\mathcal{M}$, it follows that $\mathcal{IOR}(P)$ is isomorphic to

$$(P\mathrm{lat}_\infty) \oplus \left((I - P)\mathrm{lat}_\infty\right),$$

where we identify lat_∞ with the lattice of operator ranges in $\mathcal{H}$. By Lemma 5, $(I - P)\mathrm{lat}_\infty$ coincides with the lattice of operator ranges in $\operatorname{Ker} P$, which in turn is isomorphic to lat_∞. Thus (b) holds.

Conversely, assume (b) holds. Fix a lattice isomorphism $\tau : \mathcal{IOR}(T) \to \mathrm{lat}_n \oplus \mathrm{lat}_\infty$. Let $\mathcal{M}_1 = \tau^{-1}(\mathbb{C}^n \oplus \{0\})$ and $\mathcal{M}_2 = \tau^{-1}(\{0\} \oplus \mathcal{H}_0)$. Consider $\mathcal{M}_2$ as a Hilbert space, and T as a linear bounded operator on $\mathcal{M}_2$ (see Lemma 4). Taking into account that the lattice of $T|_{\mathcal{M}_2}$-invariant $\mathcal{M}_2$-operator ranges coincides with the sublattice of those T-invariant $\mathcal{H}$-operator ranges that are contained in $\mathcal{M}_2$ (see Lemma 5), we obtain from Lemma 7 that $T|_{\mathcal{M}_2} = \gamma I$ for some $\gamma \in \mathbb{C}$. Analogously, $T|_{\mathcal{M}_1} = \delta I$ for some $\delta \in \mathbb{C}$.

It turns out that $\gamma \neq \delta$. Indeed, arguing by contradiction, assume that T is a scalar operator. Let $\mathcal{N} \in \mathcal{IOR}(T)$ be any element with the property that every chain

$$\begin{aligned} \{0\} &= \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \cdots \subseteq \mathcal{N}_{m-1} \subseteq \mathcal{N}_m = \mathcal{M}, \\ \{0\} &\neq \mathcal{N}_2 \neq \cdots \neq \mathcal{N}_{m-1} \neq \mathcal{M}, \qquad \mathcal{N}_j \in \mathcal{IOR}(T), \end{aligned} \tag{5}$$

has length 3 (i.e., $m = 3$); in other words, $\dim \mathcal{N} = 2$. Then obviously there exists a continuum of $\mathcal{N}_2 \in \mathcal{IOR}(T)$ that satisfy (5). However, the element $\mathcal{N} = \mathcal{V} \oplus \mathcal{U} \in \operatorname{lat}_n \oplus \operatorname{lat}_\infty$, where $\mathcal{V}$ and $\mathcal{U}$ are one-dimensional subspaces of $\mathbb{C}^n$ and of $\mathcal{H}_0$, respectively, has the property that every chain

$$\begin{aligned} \{0\} &= \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \cdots \subseteq \mathcal{N}_{m-1} \subseteq \mathcal{N}_m = \mathcal{N}, \\ \{0\} &\neq \mathcal{N}_2 \neq \cdots \neq \mathcal{N}_{m-1} \neq \mathcal{N}, \qquad \mathcal{N}_j \in \operatorname{lat}_n \oplus \operatorname{lat}_\infty, \end{aligned} \tag{6}$$

has length 3, but there exist only two elements $\mathcal{N}_2$ that satisfy (6). This contradicts the hypothesis (b).

Once we have ascertained that $\gamma \neq \delta$, (a) follows with $\alpha = \delta - \gamma$, and with P the projection on $\mathcal{M}_1$ along $\mathcal{M}_2$. ☐

If P is assumed to have infinite dimensional rank and kernel, then the analogue of Lemma 8 runs as follows, with essentially the same proof as Lemma 8:

Lemma 9. *Let T be as in Lemma 8. Then the following are equivalent:*

(a) $T = \alpha P + \beta I$ *with* $\alpha \in \mathbb{C} \setminus \{0\}$, $\beta \in \mathbb{C}$, $P = P^2$, *and* $\dim \operatorname{Range} P = \dim \operatorname{Ker} P = \infty$*;*

(b) $\mathcal{IOR}(T)$ *is isomorphic as a lattice to* $\operatorname{lat}_\infty \oplus \operatorname{lat}_\infty$.

Lemma 10. *Let $E = \{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in $\mathcal{H}$. Then there exists $T \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{IOR}(T)$ is not isomorphic to $\mathcal{IOR}(T^t)$, where $T^t \in \mathcal{L}(\mathcal{H})$ is the operator whose infinite matrix with respect to the basis E is the transpose of the infinite matrix representing T (with respect to E).*

Proof. Define T by $Te_j = e_{j+1}$, $j = 1, 2, \cdots$. Clearly, $T^t e_j = e_{j-1}$ for $j = 2, 3, \cdots$, and $T^t e_1 = 0$. The linear span of e_1 is a minimal nonzero element of $\mathcal{IOR}(T^t)$. If $\mathcal{IOR}(T)$ and $\mathcal{IOR}(T^t)$ were isomorphic, then $\mathcal{IOR}(T)$ would also have a minimal nonzero element, which by Theorem 1 would have to be a one-dimensional subspace. However, this is impossible, because $\operatorname{Ker}(\lambda I - T) = \{0\}$ for every $\lambda \in \mathbb{C}$. ☐

Once Lemmas 7 - 10 are established, the proof of Theorem 2 proceeds as that of Theorem 3.1 in [4].

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