SOME CLASSES OF TOPOLOGICAL QUASI *-ALGEBRAS

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Abstract. The completion $\mathcal{A}[\tau]$ of a locally convex *-algebra $\mathcal{A}[\tau]$ with not jointly continuous multiplication is a *-vector space with partial multiplication $xy$ defined only for $x$ or $y \in \mathcal{A}_0$, and it is called a topological quasi *-algebra. In this paper two classes of topological quasi *-algebras called strict CQ*-algebras and HCQ*-algebras are studied. Roughly speaking, a strict CQ*-algebra (resp. HCQ*-algebra) is a Banach (resp. Hilbert) quasi *-algebra containing a C*-algebra endowed with another involution $\#$ and C*-norm $\|\cdot\#$. HCQ*-algebras are closely related to left Hilbert algebras. We shall show that a Hilbert space is a HCQ*-algebra if and only if it contains a left Hilbert algebra with unit as a dense subspace. Further, we shall give a necessary and sufficient condition under which a strict CQ*-algebra is embedded in a HCQ*-algebra.

1. Introduction

Topological quasi *-algebras were first introduced by Lassner [6] for the mathematical description of some quantum physical models, and after that, they have been studied by Lassner [6, 7], Trapani [10] and Bagarello-Trapani [8, 9], etc. In this paper we shall study two classes of topological quasi *-algebras called strict CQ*-algebras and HCQ*-algebras from a mathematical point of view but also in the perspective of possible physical applications. Let $\mathcal{A}$ be a *-algebra with two involutions $*$ and $\#$ and two norms $\|\cdot\|$ and $\|\cdot\#|$ satisfying $\|x^*\| = \|x\|$, $\|x\| \leq \|x\#\|$ and $\|x\#x\#\| = \|x\|^2$ for each $x, y \in \mathcal{A}$. Then the completion $\mathcal{A}[[\tau]]$ of $\mathcal{A}[[\tau]]$ is a topological quasi *-algebra containing (under natural assumptions) two C*-algebras $\mathcal{A}[[\tau]]$ and $\mathcal{A}[[\tau]]$, with different involutions $\#$ and $\bar{\#}$, respectively, which are connected by the isometric involution $J : x \rightarrow x^*$. This is called a pseudo CQ*-algebra. If $\|x\|_{\#} = \sup\{\|xy\|; y \in \mathcal{A} \}$, then $\mathcal{A}[[\tau]]$ is a particular kind of CQ*-algebra as defined and studied in [1, 2], and it is called a strict CQ*-algebra, and denoted by $(\mathcal{A}[[\tau]], \#)$. Let $\mathcal{A}[[\tau]]$ be a topological quasi *-algebra with isometric involution $J : x \rightarrow x^*$ and Hilbertian norm $\|\cdot\|$. If $\mathcal{A}$ has another involution $\#$ satisfying $\|x\| \leq \|L_x\|$ and $L_x^* = L_{x\#}$ for each $x \in \mathcal{A}$, where $L_x$ is the bounded linear operator on the Hilbert space $\mathcal{A}[[\tau]]$ defined by $L_xy = xy, y \in \mathcal{A}$, then $\mathcal{A}[[\tau]]$ is a strict CQ*-algebra with involution $\#$ and C*-norm $\|x\|_{\#} \equiv \|L_x\|, x \in \mathcal{A}$, and it is called a HCQ*-algebra and denoted by $(\mathcal{A}[[\tau]], \#)$. HCQ*-algebras are closely related to left Hilbert algebras. Let $(\mathcal{A}[[\tau]], \#)$ be an HCQ*-algebra. Then $\mathcal{A}$ is
a left Hilbert algebra in the Hilbert space \( \mathcal{A}\| \| \) with involution \# , and the full left Hilbert algebra \( \mathcal{A}' \) of \( \mathcal{A} \) has unit. But, the isometric involution \( J \) does not necessarily coincide with the modular conjugation operator \( J_\mathcal{A} \) of the left Hilbert algebra \( \mathcal{A} \). If \( J_\mathcal{A} = J \), then the HCQ*-algebra \( (\mathcal{A}\| \| ,\#) \) is said to be \textit{standard}. Suppose that \( (\mathcal{A}\| \| ,\#) \) is standard. Then \( \mathcal{A} \) is contained in the maximal Tomita algebra \( (\mathcal{A}' \) of \( \mathcal{A}' \) and \( (\mathcal{A}'\) of \( \mathcal{A}' \) is a standard HCQ*-algebra with the one-parameter group \( \{\Delta_\mathcal{A}^t\}_{t \in \mathbb{R}} \) of \( \ast \)-automorphisms, where \( \Delta_\mathcal{A} \) is the modular operator of \( \mathcal{A} \). From these results, it is shown that a Hilbert space is a standard HCQ*-algebra if and only if it contains a left Hilbert algebra as dense subspace. Finally, we give a necessary and sufficient condition under which a strict CQ*-algebra is embedded into a standard HCQ*-algebra using the GNS-construction of positive sesquilinear form on the strict CQ*-algebra \( \mathcal{A}\| \| \).

2. \textbf{Strict CQ*-algebras and HCQ*-algebras}

Let \( \mathcal{A}\| \| \) be a normed \( \ast \)-algebra with isometric involution \( \ast \) and separately (but not jointly) continuous multiplication. Then the completion, \( \mathcal{A}\| \| \), of \( \mathcal{A}\| \| \) is a topological quasi \( \ast \)-algebra that we call, as is natural, a \textit{Banach quasi \( \ast \)-algebra}. In particular, if \( \| \| \) is a Hilbertian norm, then \( \mathcal{A}\| \| \) is called a \textit{Hilbert quasi \( \ast \)-algebra}. For any \( a \in \mathcal{A}\| \| \) we put

\[
L_\alpha x = ax \quad \text{and} \quad R_\alpha x = xa, \quad x \in \mathcal{A}.
\]

Then \( L_\alpha \) and \( R_\alpha \) are linear maps of \( \mathcal{A} \) into \( \mathcal{A}\| \| \). In particular, if \( a \in \mathcal{A} \), then \( L_\alpha \) and \( R_\alpha \) can be extended to bounded linear operators on the Banach space \( \mathcal{A}\| \| \) and they are denoted by the same symbols \( L_\alpha \) and \( R_\alpha \).

Let \( \mathcal{A}\| \| \) be a Banach quasi \( \ast \)-algebra and assume that the \( \ast \)-algebra \( \mathcal{A} \) has another norm \( \| \# \) and another involution \( \# \) satisfying the following conditions:

(\textit{a.1}) \( \| x^\ast x \# \| = \| x \|^2 \| \ast \), \quad \forall x \in \mathcal{A}.
(\textit{a.2}) \( \| x \| \leq \| x \# \| \), \quad \forall x \in \mathcal{A}.
(\textit{a.3}) \( \| xy \| \leq \| x \# \| \| y \| \), \quad \forall x, y \in \mathcal{A}.

Then by (\textit{a.2}), the identity map \( \hat{i} : \mathcal{A}\| \| \rightarrow \mathcal{A}\| \| \) has a continuous extension \( \hat{i} \) from the completion \( \mathcal{A}_\# \) of \( \mathcal{A}\| \| \) into \( \mathcal{A}\| \| \). If \( \hat{i} \) is injective, then \( \mathcal{A}_\# \) is (identified with) a dense subspace of \( \mathcal{A} \). This happens if, and only if,

(\textit{a.4}) \( \| \# \) and \( \| \# \) are compatible in the following sense \[5\]: for any sequence \( \{x_n\} \subset \mathcal{A} \) such that \( \| x_n \| \rightarrow 0 \) and \( x_n \rightarrow x \) in \( \mathcal{A}_\# \), \( x = 0 \) results, i.e. if \( \hat{i}^{-1} : \mathcal{A}\| \| \rightarrow \mathcal{A}_\# \) is closable.

\textbf{Definition 2.1.} A Banach quasi \( \ast \)-algebra \( \mathcal{A}\| \| \) is said to be a \textit{pseudo CQ*-algebra} if the \( \ast \)-algebra \( \mathcal{A} \) has a norm \( \| \# \) and another involution \( \# \) satisfying the conditions (\textit{a.1})–(\textit{a.4}) above. Furthermore, if \( \| x \# \| = \| L_x \| = \sup \{\| xy \| ; y \in \mathcal{A} \} \) for each \( x \in \mathcal{A} \), then \( \mathcal{A}\| \| \) is said to be a \textit{strict CQ*-algebra}. A pseudo CQ*-algebra \( \mathcal{A}\| \| \) is fully determined by the involution \( \# \) and the \( C^* \)-norm \( \| \# \), and so it will often be denoted by \( (\mathcal{A}\| \| ,\#) \). On the other hand, a strict CQ*-algebra is fully determined when the new involution \( \# \) is known; so it can be simply denoted as \( (\mathcal{A}\| \| ,\#) \), making lighter in this way the notation introduced by two of us in \[12\]. Let \( (\mathcal{A}\| \| ,\#) \) be a pseudo CQ*-algebra and, as above, let \( \mathcal{A}_\# \) be the \( C^* \)-algebra obtained by completing the \( \# \)-algebra \( \mathcal{A} \) with respect to the \( C^* \)-norm \( \| \# \). Let \( J \) be the involution \( \ast \) of the Banach quasi
\textbf{SOME CLASSES OF TOPOLOGICAL QUASI $*$-ALGEBRAS} 2975

$*$-algebra $\overline{A}$ with involution $\overline{A}$. Then $A_0 \equiv JA_0$ is a $C^*$-algebra equipped with the operations $x^* + y^* = (x + y)^*$, $\lambda x^* = (\overline{\lambda}x)^*$, $xy^* = \overline{yx}$, the involution $(x^*)^* = x^*$ and the $C^*$-norm $\|x^\ast\| = \|x\|_*$, $\forall x, y \in A_0$.

\textbf{Proposition 2.2.} A pseudo $CQ^*$-algebra $(\overline{A}, \#, ||||)$ contains two $C^*$-algebras $A_\#$ and $A_0 \equiv JA_0$ with different involutions $\#$ and $\ast$, respectively, as dense subalgebras. In particular, if $(\overline{A}, \#, ||||, \#)$ is a strict $CQ^*$-algebra, then $L_{A_\#}$ and $R_{A_0}$ are $C^*$-algebras, $L_x R_y = R_y L_x$ for each $x \in A_\#$ and $y \in A_0$ and $R_{A_\#} = JL_{A_\#}$.

By Proposition 2.2, every strict $CQ^*$-algebra is a $CQ^*$-algebra in the sense of [1,2] but the converse is not true in general ($\langle A_\# \cap A_0 \rangle$ is not required to be $\#$-invariant).

We summarize the situation with the following scheme:

\begin{center}
\begin{tabular}{ccc}
\textsf{normed }$*$\textsf{-algebra} & $\subseteq$ & $\textsf{C}^*\text{-algebras}$
\hline
$A$ & & $\overline{A}$
$A_\#$ & & $\overline{A}_\#$
\end{tabular}
\end{center}

which summarizes the situation: the $*$-algebra $A$ is contained in its closures, $A_\# = \overline{A}_\#$ and $A_0 = \overline{A}_0$. These $C^*$-algebras, moreover, are both contained in $\overline{A}$.

\textbf{Definition 2.3.} A Hilbert quasi $*$-algebra $\overline{A}$ is said to be a $HCQ^*$-algebra if there is another involution $\#$ of $A$ such that $L_x^\ast = L_x$ and $\|x\| \leq \|L_x\|$ for each $x \in A$. Here we denote it by $(\overline{A}, ||||, \#)$.

$HCQ^*$-algebras are closely related to left Hilbert algebras. Before going forth, for the reader’s convenience, we briefly review the definitions and the basic properties of left Hilbert algebras. A $*$-algebra $A$ with involution $\#$ is said to be a left Hilbert algebra if it is a dense subspace in a Hilbert space $\mathcal{H}$ with inner product $(\cdot | \cdot)$, satisfying the following conditions:

(i) For any $x \in A$ the map $y \in A \rightarrow xy \in A$ is continuous.
(ii) $(xy)z = (y(xz), \forall x, y, z \in A$.
(iii) $A^2 = \{xy, x, y \in A\}$ is total in $\mathcal{H}$.
(iv) The involution $x \rightarrow x^\#$ is closable in $\mathcal{H}$.

By (i), for any $x \in A$ we denote by $\pi_A(x)$ the unique continuous linear extension to $\mathcal{H}$ of the map $y \in A \rightarrow xy \in A$; then $\pi_A$ is a $*$-representation of $A$ on $\mathcal{H}$.

We denote by $S_A$ the closure of the involution $\#$. Let $S_A = J_A \Delta_A^\perp$ be the polar decomposition of $S_A$. Then $J_A$ is an isometric involution on $\mathcal{H}$ and $\Delta_A$ is a non-singular positive self-adjoint operator in $\mathcal{H}$ such that $S_A = J_A \Delta_A^\perp = \Delta_A^{-\frac{1}{2}} J_A$, and $S_A^* = J_A \Delta_A^{-\frac{1}{2}} = \Delta_A^\perp J_A$, and $J_A$ is called the modular conjugation operator of $A$ and $\Delta_A$ is called the modular operator of $A$. We define the commutant $\mathfrak{A}'$ of $A$ as follows: For any $y \in D(S_A)$ we put $\pi_A(x)y = \pi_A(x)y, x \in A'$ and put $A' = \{y \in D(S_A''): \pi_A'(y) \text{ is bounded}\}$. Then $\mathfrak{A}'$ is a left Hilbert algebra in $\mathcal{H}$ with involution $S_A$ and multiplication $y_1 y_2 = \pi_A'(y_2)y_1$. Similarly, the commutant $\mathfrak{A}''$ of $\mathfrak{A}'$ is defined by $\mathfrak{A}'' = \{x \in D(S_A): y \in \mathfrak{A}' \rightarrow xy \text{ is continuous}\}$. For any $x \in \mathfrak{A}''$ we denote by $\pi_A(x)$ the unique continuous linear operator on $\mathcal{H}$ such that $\pi_A(x)y = \pi_A'(y)x, y \in \mathfrak{A}'$. Then $\mathfrak{A}''$ is a left Hilbert algebra in $\mathcal{H}$ with involution $S_A$ and multiplication $x_1 x_2 = \pi_A(x_1)x_2$ containing $A$. A left Hilbert algebra $\mathfrak{A}$ is said to be full if $\mathfrak{A} = \mathfrak{A}''$. It is well-known as the Tomita fundamental theorem that
Suppose that $(\mathcal{A}|| \ |, \#)$ is a HCQ*-algebra. Then the following statements hold:

(i) $(\mathcal{A}|| \ |, \#)$ is a strict CQ*-algebra with the C*-norm $\|x\|_# = \|L_x\|, x \in \mathcal{A}$.

(ii) $\mathcal{A}$ is a left Hilbert algebra in the Hilbert space $\mathcal{H} \equiv \mathcal{A}|| \ |$ whose full left Hilbert algebra $\mathcal{A}'$ has a unit $u$.

Proof. (i) The proof is mostly trivial. We prove only that the condition (a.4) is satisfied in this case. Indeed, if $\{x_n\} \subset \mathcal{A}$ is a sequence such that $\|x_n\| \to 0$ and $x_n \to x$ in $\mathcal{A}|| \ |$, then by the assumption $L_{x_n} \to L_x$ with respect to the operator norm. The continuity of the multiplication in $\mathcal{A}|| \ |$ easily implies that $L_x = 0$, thus $\|x\|_# = 0$ and $x = 0$.

(ii) We first show that $\mathcal{A}$ is a left Hilbert algebra in $\mathcal{H}$ with involution #. Since the C*-algebra $\mathcal{A}_#$ has an approximate identity $\{u_n\}$, $\mathcal{A}$ is dense in the C*-algebra $\mathcal{A}_#$ and $\|x\| \leq \|x\|_#$ for each $x \in \mathcal{A}$, and then it follows that $\mathcal{A}^2$ is total in $\mathcal{A}|| \ |$. The assumption $L_x = L_x^#(\forall x \in \mathcal{A})$ implies that $(xy)z = (y|x|^#z)$ for each $x, y, z \in \mathcal{A}$, where $\langle \ | \ \rangle$ is the inner product defined by the Hilbertian norm $\| \ |$. Further, we have $\pi_{\mathcal{A}}(x) = L_x$, $\forall x \in \mathcal{A}$ and $\pi_{\mathcal{A}}(x)$ is bounded. Take any sequence $\{x_n\} \subset \mathcal{A}$ such that $\lim_{n \to \infty} \|x_n\| = 0$ and $\lim_{n \to \infty} \|x_n^# - y\| = 0$. Then it follows that $(y|x_1x_2^#) = \lim_{n \to \infty} (x_n^# | x_1x_2^#) = \lim_{n \to \infty} (x_2x_1^# | x_n) = 0$ for each $x_1, x_2 \in \mathcal{A}$, which implies that $x \in \mathcal{A} \to x^# \in \mathcal{A}$ is closable. Thus $\mathcal{A}$ is a left Hilbert algebra in $\mathcal{H}$ with the involution #. We next show that the full left Hilbert algebra $\mathcal{A}'$ has a unit $u$. For any $\varepsilon > 0$ and for any finite subsets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ of $\mathcal{A}$, we define the set

$$K(\varepsilon; \{x_1, \ldots, x_m\}, \{y_1, \ldots, y_m\}) = \{a \in \mathcal{H}; \|a\| \leq 1, |(ax_k - x_k)y_k| \leq \varepsilon$$

and $|(x_k a - x_k y_k)| \leq \varepsilon, k = 1, \ldots, m\}.$

Since the C*-algebra $\mathcal{A}_#$ has an approximate identity and $\|x\| \leq \|x\|_#$ for each $x \in \mathcal{A}_#$, it follows that $K(\varepsilon; \{x_1, \ldots, x_m\}, \{y_1, \ldots, y_m\}) \neq \emptyset$. Now let $K$ be the family of all subsets $K(\varepsilon; \{x_1, \ldots, x_m\}, \{y_1, \ldots, y_m\})$ where $\varepsilon > 0$ and $\{x_1, \ldots, x_m\}$, $\{y_1, \ldots, y_m\}$ are finite subsets. Then $K$ is a family of non-empty weakly closed subsets of the weakly compact set $\mathcal{H}_1 = \{a \in \mathcal{H}; \|a\| \leq 1\}$. Hence, the intersection of all the sets in $K$ is non-empty. Hence, an element $u$ of this intersection is such that $u$ is a quasi-unit of the topological quasi *-algebra $\mathcal{A}|| \ |$, that is, $u \in \mathcal{A}|| \ |$ and $ux = xu = x$ for each $x \in \mathcal{A}$. Since

$$\langle S_\mathcal{A}x | u \rangle = (x^# | u) = (u | L_x u) = (u | x)$$

for each $x \in \mathcal{A}$, it follows that $u \in D(S_\mathcal{A}^*)$ and $\pi_\mathcal{A}(u) = I$. Hence, $u \in \mathcal{A}'$ and $S_\mathcal{A}^* u = u$, which implies that

$$\langle S_\mathcal{A}^* y | u \rangle = (\pi_\mathcal{A}'(S_\mathcal{A}^*)u | u) = (u | \pi_\mathcal{A}'(y)u) = (u | y)$$

for each $y \in \mathcal{A}$. Therefore, $u$ is a unit of $\mathcal{A}'$. The proof is complete.
for each $y \in \mathcal{A}'$. Hence, we have $u \in \mathcal{A}''$ and $S_Au = u$. This completes the proof. 

By Proposition 2.4, the situation of HCQ*-algebras can be schematized with the following diagram:

$$
\begin{array}{c}
\mathcal{A}' \subset \mathcal{A}' = \mathcal{A}' & \subset \mathcal{D}(S_A) & \subset \\
\mathcal{A} & \uparrow J & \uparrow J_A & \uparrow J_A & \mathcal{A} \\
\subset \mathcal{A}_b \subset \mathcal{A}' = \mathcal{A}_b' & \subset \mathcal{D}(S'_A) & \subset \\
\end{array}
$$

We now look for conditions under which $J = J_A$.

Lemma 2.5. Let $(\mathcal{A} \| | \|, \#)$ be a HCQ*-algebra. Then the following statements are equivalent:

(i) $J = J_A$.

(ii) $(x^\# | x^*) \geq 0$ for each $x \in \mathcal{A}$.

Proof. (i) $\Rightarrow$ (ii) This follows from $(x^\# | x^*) = (J_A \Delta_A^\frac{1}{2} x | J_Ax) = (x | \Delta_A^\frac{1}{2} x) \geq 0$, $\forall x \in \mathcal{A}$.

(ii) $\Rightarrow$ (i) By the assumption (ii) we have $S_A = J(JJ_A \Delta_A^\frac{1}{2})$ and $JJ_A \Delta_A^\frac{1}{2} \geq 0$. The uniqueness of the polar decomposition of $S_A$ implies $J = J_A$.

If any one of the two equivalent statements of Lemma 2.5 holds, we say that the HCQ*-algebra $(\mathcal{A} \| | \|, \#)$ is standard.

Remark 2.6. Let $(\mathcal{A} \| | \|, \#)$ be a HCQ*-algebra. If it is standard, then $R'_A = L'_A$. Conversely, if $R'_A = L'_A$, then $JJ_A = J_AJ$, but we don’t know whether $J = J_A$.

Since two HCQ*-algebras $(\mathcal{A} \| | \|, \#)$ with $(\mathcal{B} \| | \|, \#), (\mathcal{A} \| | \|) = (\mathcal{B} \| | \|)$ as Hilbert spaces, need not coincide as HCQ*-algebras, we introduce the following notion:

Definition 2.7. A HCQ*-algebra $(\mathcal{A} \| | \|)$ is said to be an extension of a HCQ*-algebra $(\mathcal{B} \| | \|)$ if $\mathcal{B}$ is a dense $*$-subalgebra of $\mathcal{A}$ and $S_A = S_B$.

Proposition 2.8. Let $(\mathcal{A} \| | \|, \#)$ be a standard HCQ*-algebra, and $\mathcal{B} \equiv (\mathcal{A}'')_0$ the maximal Tomita algebra of the full left Hilbert algebra $\mathcal{A}''$. Then $(\mathcal{B} \| | \|, S_A)$ is a standard HCQ*-algebra and it is an extension of $(\mathcal{A} \| | \|, S_A)$. Further, $\{\Delta^\mu_A\}_{t \in \mathbb{R}}$ is a one-parameter group of $*$-automorphisms of the Hilbert quasi-$*$-algebra $\mathcal{B} \| | \|$, that is, $\Delta^\mu_B = B$, $\Delta^\mu_A(a^*) = \Delta^\mu_B(a^*)$, $\Delta^\mu_A(ax) = (\Delta^\mu_B(a))(\Delta^\mu_B(x))$ and $\Delta^\mu_A(ax) = (\Delta^\mu_A(x))(\Delta^\mu_A(a))$ for all $a \in \mathcal{B} \| | \|, x \in \mathcal{B}$ and $t \in \mathbb{R}$.

Proof. It is almost clear that $\mathcal{B} \| | \|$ is a Hilbert quasi-$*$-algebra with the involution $J_A = J_B$ and further $(\mathcal{B} \| | \|, S_A)$ is a standard HCQ*-algebra. Since $\{\Delta^\mu_A\}_{t \in \mathbb{R}}$ is a one-parameter group of $*$-automorphisms of the Hilbert quasi-$*$-algebra $\mathcal{B} \| | \|$, it follows that $\{\Delta^\mu_A\}_{t \in \mathbb{R}}$ is also a one-parameter group of $*$-automorphisms of the Hilbert quasi-$*$-algebra $\mathcal{B} \| | \|$. 

Finally, we consider the question of when a Hilbert space can be regarded as a standard HCQ*-algebra. By Proposition 2.4, 2.8 and [14], Theorem 13.1, we have the following:
Theorem 2.9. Let $\mathcal{H}$ be a Hilbert space. The following statements are equivalent:
(i) $\mathcal{H}$ is a standard HCQ*-algebra.
(ii) $\mathcal{H}$ contains a left Hilbert algebra with unit as dense subspace.
(iii) There exists a von Neumann algebra on $\mathcal{H}$ with a cyclic and separating vector.

It is worth noticing, in particular, that the implication (iii) $\Rightarrow$ (i) shows that the class of standard HCQ*-algebras is rather rich.

3. The structure of strict CQ*-algebras

In this section we study when a strict CQ*-algebra is embedded in a standard HCQ*-algebra. For that, we need a GNS-like construction for a class of positive sesquilinear forms on strict CQ*-algebras $(\mathcal{A}|| |, \# , || |)$.

A sesquilinear form $\varphi$ on $\mathcal{A}|| |$ is said to be positive if $\varphi(a,a) \geq 0$ for all $a \in \mathcal{A}|| |$, and $\varphi$ is said to be faithful if $\varphi(a,a) = 0, a \in \mathcal{A}|| |$, implies $a = 0$. Further, we need the following notion:

Definition 3.1. Let $(\mathcal{A}|| |, \# , || |)$ and $(\mathcal{B}|| |, \# , || |)$ be strict CQ*-algebras. A linear map $\Phi : \mathcal{A}|| | \to \mathcal{B}|| |$ is said to be a $\ast$-homomorphism of $(\mathcal{A}|| |, \# , || |)$ into $(\mathcal{B}|| |, \# , || |)$ if (i) $\Phi$ is a $\ast$-homomorphism of the quasi $\ast$-algebra $\mathcal{A}|| |$ into the quasi $\ast$-algebra $\mathcal{B}|| |$, that is, $\Phi(\mathcal{A}) \subset \mathcal{B}$ and $\Phi(a^*) = \Phi(a^*)$, $\Phi(\mathcal{a}) = \mathcal{\Phi(a)}\mathcal{\Phi(x)}$ and $\Phi(\mathcal{xa}) = \mathcal{\Phi(x)}\mathcal{\Phi(a)}$ for all $a \in \mathcal{A}|| |$ and $x \in \mathcal{A}$; (ii) $\Phi(\mathcal{A})$ is a $\ast$-homomorphism of the $C^*$-algebra $\mathcal{A}|| |$ into the $C^*$-algebra $\mathcal{B}|| |$. A bijective (resp. injective) $\ast$-homomorphism $\Phi$ such that $\Phi(\mathcal{A}) = \mathcal{B}$ and $\Phi(\mathcal{A}) = \mathcal{B}$, is called a $\ast$-isomorphism of $(\mathcal{A}|| |, \# , || |)$ onto (resp. into) $(\mathcal{B}|| |, \# , || |)$. A $\ast$-isomorphism $\Phi$ is said to be contractive if $\|\Phi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}|| |$. A contractive $\ast$-isomorphism whose inverse is also contractive is called an isometric $\ast$-isomorphism.

Theorem 3.2. Let $(\mathcal{A}|| |, \# , || |)$ be a strict CQ*-algebra with quasi-unit $u$. Then the following statements are equivalent:

(i) There exists a contractive $\ast$-homomorphism (resp. $\ast$-isomorphism) of the strict CQ*-algebra $(\mathcal{A}|| |, \# , || |)$ into a HCQ*-algebra $(\mathcal{B}|| |, \# , || |)$.

(ii) There exists a (resp. faithful) positive sesquilinear form $\varphi$ on $\mathcal{A}|| | \times \mathcal{A}|| |$ satisfying

(ii) $\varphi(x,y) = \varphi(u, x^\# y)$, $\forall x, y \in \mathcal{A}$;
(ii) $|\varphi(x,y)| \leq \|x\|\|y\|$, $\forall x, y \in \mathcal{A}$;
(ii) $\varphi(x,y) = \varphi(y^*, x^*)$, $\forall x, y \in \mathcal{A}$.

Further, $(\mathcal{B}|| |, \# , || |)$ is standard if and only if

(ii) $\varphi(x^*, x^\#) \geq 0$, $\forall x \in \mathcal{A}$.

Proof. (i) $\Rightarrow$ (ii) We put

$$\varphi(a,b) = (\Phi(a)|\Phi(b)),$$

where $( | )$ is the inner product defined by the Hilbertian norm $\| |$ on $\mathcal{B}|| |$. Then it is easily shown that $\varphi$ is a positive sesquilinear form on $\mathcal{A}|| | \times \mathcal{A}|| |$ satisfying the condition (ii) $\sim$ (ii). If $(\mathcal{B}|| |, \# , || |)$ is standard, then (ii) $\Rightarrow$ follows from Lemma 2.5.

(ii) $\Rightarrow$ (i) We put $\mathcal{N}_\varphi = \{a \in \mathcal{A}|| |; \varphi(a,a) = 0\}$. Then $\mathcal{N}_\varphi$ is a subspace of $\mathcal{A}|| |$ and, due to the positivity of $\varphi$, which implies $\varphi(a,b) = \varphi(b,a)$ for each $a, b \in \mathcal{A}|| |$,
it follows that the quotient space $\lambda_\varphi(\mathcal{A}(|| \cdot ||)) \equiv \mathcal{A}(|| \cdot ||)/\mathcal{N}_\varphi = \{\lambda_\varphi(a) \equiv a + \mathcal{N}_\varphi; a \in \mathcal{A}(|| \cdot ||)\}$ is a pre-Hilbert space with inner product $\langle \lambda_\varphi(a), \lambda_\varphi(b) \rangle = \varphi(a, b), a, b \in \mathcal{A}(|| \cdot ||)$. We denote by $\| \cdot \|_\varphi$ the norm defined by the inner product $\langle \cdot , \cdot \rangle_\varphi$ and by $\mathcal{H}_\varphi$ the completion of $\lambda_\varphi(\mathcal{A}(|| \cdot ||))$. Since $\mathcal{A}$ is $\| \cdot \|$-dense in $\mathcal{A}(|| \cdot ||)$, it follows that
\[
\text{(ii)}_{\varphi} \| \varphi(a, b) \| \leq \| a \| \| b \|, \forall a, b \in \mathcal{A}(|| \cdot ||);
\]
\[
\text{(ii)}_{\varphi} \varphi(a, b) = \varphi(b^*, a^*), \forall a, b \in \mathcal{A}(|| \cdot ||),
\]
and since (ii)’ and $\| x \| \leq \| x \|_\varphi, \forall x \in \mathcal{A}$, it follows that
\[
\text{(ii)}_{\varphi} \varphi(x, y) = \varphi(u, x^# y), \forall x, y \in \mathcal{A}_#.
\]
By (ii)’ $A_{#} \equiv \lambda_\varphi(A)$ is a dense subspace of the Hilbert space $\mathcal{H}_\varphi$, and further, it is a $*$-algebra equipped with the multiplication $\lambda_\varphi(x) \lambda_\varphi(y) = L_{\lambda_\varphi(x)} \lambda_\varphi(y) = \lambda_\varphi(xy)$ and the involution $\lambda_\varphi(x)^* \equiv \lambda_\varphi(x^*)$. By (ii)’ the involution $\lambda_\varphi(x) \rightarrow \lambda_\varphi(x^*)$ can be extended to the isometric involution $I_\varphi$ on $\mathcal{H}_\varphi$. By (ii)’ the linear functional on the C*-algebra $A_{#} : x \rightarrow \varphi(x, u)$ is positive, and so $\varphi(y^#(x^# y)u) \leq \| x \|^2 \varphi(y, y)$ for each $x, y \in \mathcal{A}$. Hence it follows from (ii)’ that
\[
\| \lambda_\varphi(x) \lambda_\varphi(y) \|^2_\varphi = \varphi(x, xy) = \varphi(y^# x^# xy, u) \leq \| x \|^2 \| \lambda_\varphi(y) \|^2_\varphi
\]
for each $x, y \in \mathcal{A}$, so that $L_{\lambda_\varphi(x)}$ is bounded and $\| L_{\lambda_\varphi(x)} \| \leq \| x \|_\varphi$ for each $x \in \mathcal{A}$. Thus $\mathcal{H}_\varphi = \mathcal{A}(|| \cdot ||)_\varphi$ is a Hilbert quasi $*$-algebra. Further, the map $\lambda_\varphi(x) \rightarrow \lambda_\varphi(x)^*$ is an involution of $A_{#}$ and by (ii) $L_{\lambda_\varphi(x)} = L_{\lambda_\varphi(x)^*}$, for each $x \in \mathcal{A}$. Hence, $(\mathcal{A}(|| \cdot ||)_\varphi, \#_1)$ is a HCQ*-algebra. Here we put $\Phi(a) = \lambda_\varphi(a), a \in \mathcal{A}(|| \cdot ||)$. Then it is easily shown that $\Phi$ is a $*$-homomorphism of the strict CQ*-algebra into the HCQ*-algebra $(\mathcal{A}(|| \cdot ||)_\varphi, \#_1)$ satisfying $\Phi(A_0) = A_{#}$, and by (ii)’ it is contractive. Suppose that $\varphi$ is faithful. Then the $*$-representation of the C*-algebra $A_{#}$ on $\mathcal{H}_\varphi$ defined by $x \rightarrow L_{\lambda_\varphi(x)} x \in A_{#}$ is faithful, which implies that $\| L_{\lambda_\varphi(x)} \| = \| x \|_\varphi$ for each $x \in \mathcal{A}_{#}$. Further, since $\Phi(A_0) = A_{#}$, it follows that $\Phi(A_{#}) = (A_{#})_{#_1}$, and $\Phi(A_{#})$ is a $*$-isomorphism of the C*-algebra $A_{#}$ onto the C*-algebra $(\mathcal{A}(|| \cdot ||)_\varphi, \#_1)$. Hence $\Phi$ is a $*$-isomorphism of $(\mathcal{A}(|| \cdot ||), || \cdot ||) \rightarrow (\mathcal{A}(|| \cdot ||)_\varphi, || \cdot ||)$. By Lemma 2.5, the HCQ*-algebra $(\mathcal{A}(|| \cdot ||)_\varphi, || \cdot ||, \#_1)$ is standard if and only if (ii)’ holds. This completes the proof.

Now the question arises as to whether positive sesquilinear forms as described in (ii) do really exist. The answer is certainly positive due to the existence of standard HCQ*-algebras stated in Theorem 2.9. Indeed, the inner product $\langle \cdot , \cdot \rangle$ of a left Hilbert algebra satisfies conditions (ii)1–(ii)4.

Furthermore, Theorem 3.2 answers the main question in this section: any form $\varphi$ over a strict CQ*-algebra $(\mathcal{A}(|| \cdot ||), || \cdot ||)$ with quasi-unit, can be used to construct a HCQ*-algebra where $\mathcal{A}$ is contractively embedded.

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**References**


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