

## SOME RESULTS RELATED TO THE CORACH-PORTA-RECHT INEQUALITY

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ABSTRACT. Let  $L(H)$  be the algebra of all bounded operators on a complex Hilbert space  $H$  and let  $S$  be an invertible self-adjoint (or skew-symmetric) operator of  $L(H)$ . Corach-Porta-Recht proved that

$$(*) \quad \forall X \in L(H), \quad \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|.$$

The problem considered here is that of finding (i) some consequences of the Corach-Porta-Recht Inequality; (ii) a necessary condition (resp. necessary and sufficient condition, when  $\sigma(P) = \sigma(Q)$ ) for the invertible positive operators  $P, Q$  to satisfy the operator-norm inequality  $\|PXP^{-1} + Q^{-1}XQ\| \geq 2\|X\|$ , for all  $X$  in  $L(H)$ ; (iii) a necessary and sufficient condition for the invertible operator  $S$  in  $L(H)$  to satisfy (\*).

### 1. INTRODUCTION

All operators considered here are bounded operators on a complex Hilbert space  $H$ . The collection of operators in  $H$  is denoted by  $L(H)$ .

For  $T \in L(H)$ , we denote by  $\sigma(T)$ ,  $co(\sigma(T))$ ,  $r(T)$ ,  $W_0(T)$ ,  $\{T\}'$  and  $\{T\}''$  the spectrum, the convex hull of the spectrum, the spectral radius, the numerical range, the commutant and the bicommutant of  $T$ , respectively.

If  $A = (a_{ij})$  and  $B = (b_{ij})$  are two complex  $n \times n$  matrices, then define the Schur (or Schur-Hadamard) product of  $A$  and  $B$  to be the matrix  $A \circ B = (a_{ij}b_{ij})$ .

In [1], Corach, Porta, and Recht have proved that for any invertible self-adjoint or skew-symmetric operator  $S$ , the operator-norm inequality

$$\|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|$$

holds for all operators  $X$ .

It is also clear that, for any invertible operator  $S$  and for any two invertible positive operators  $P, Q$ , we have

$$(a) \quad 0 < \inf_{\|X\|=1} \|PXP^{-1} + Q^{-1}XQ\| \leq 2,$$

$$(b) \quad 0 \leq \inf_{\|X\|=1} \|SXS^{-1} + S^{-1}XS\| \leq 2.$$

It may be seen by the Corach-Porta-Recht Inequality that the infimum in (a) is 2, if  $P = Q$ ; and the infimum in (b) is also 2, for  $S$  an invertible self-adjoint operator,

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or more generally, if  $S$  is of the form  $S = \lambda M$ , where  $M$  is an invertible self-adjoint operator and  $\lambda$  is a nonzero scalar.

The purpose of this paper is the following:

(1) In §2, we give the following consequences of the Corach-Porta-Recht Inequality. For all invertible positive commuting operators  $P, Q$  and for all operators  $X$ , we have

- (i)  $\|PXP^{-1} + Q^{-1}XQ\| \geq 2\|X\|$ , if  $\|X\| = r(X)$ ,
- (ii)  $\max\{\|PXP^{-1} + Q^{-1}XQ\|, \|PX^*P^{-1} + Q^{-1}X^*Q\|\} \geq 2\|X\|$ ,
- (iii)  $\|nX + PXP^{-1} + P^{-1}XP\| \geq (n + 2)\|X\|$ , for  $n = 0, 1, 2$ .

(2) In §3, we show that the infimum in (a) is 2 only if  $\{P\}' = \{Q\}'$ ; on the other hand, if  $\sigma(P) = \sigma(Q)$ , then the infimum in (a) is 2 if and only if  $P = Q$ .

(3) In §4, we show that the only operators  $S$  for which the infimum in (b) is 2 are those of the form  $S = \lambda M$ , where  $M$  is an invertible self-adjoint operator and  $\lambda$  is a nonzero scalar.

## 2. SOME CONSEQUENCES OF THE CORACH-PORTA-RECHT INEQUALITY

**Lemma 2.1** ([1]). *For an invertible self-adjoint or skew-symmetric operator  $S$ , we have  $\forall X \in L(H) : \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|$ .*

**Theorem 2.2.** *For any pair  $(P, Q)$  of commuting invertible positive operators and for any  $X \in L(H)$  such that  $\|X\| = r(A)$ , we have*

$$\|PXP^{-1} + Q^{-1}XQ\| \geq 2\|X\|.$$

*Proof.* Let  $X \in L(H)$  such that  $\|X\| = r(A)$  and put  $Y = P^{\frac{1}{2}}Q^{-\frac{1}{2}}XQ^{\frac{1}{2}}P^{-\frac{1}{2}}$ . Then since  $P^{\frac{1}{2}}Q^{\frac{1}{2}} = Q^{\frac{1}{2}}P^{\frac{1}{2}}$  is self-adjoint, we have by Lemma 2.1

$$\begin{aligned} \|PXP^{-1} + Q^{-1}XQ\| &= \left\| (P^{\frac{1}{2}}Q^{\frac{1}{2}})Y(P^{\frac{1}{2}}Q^{\frac{1}{2}})^{-1} + (P^{\frac{1}{2}}Q^{\frac{1}{2}})^{-1}Y(P^{\frac{1}{2}}Q^{\frac{1}{2}}) \right\| \\ &\geq 2\|Y\| \\ &\geq 2r(X) \\ &\geq 2\|X\|. \end{aligned} \quad \square$$

**Theorem 2.3.** *For any pair  $(P, Q)$  satisfying the condition of Theorem 2.2 and for any operator  $X$ , we have*

$$\max\{\|PXP^{-1} + Q^{-1}XQ\|, \|PX^*P^{-1} + Q^{-1}X^*Q\|\} \geq 2\|X\|.$$

*Proof.* For  $X \in L(H)$ , let

$$A = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}, B = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.$$

The pair  $(A, B)$  satisfies the condition of Theorem 2.2 and  $\|Y\| = r(Y)$  (since  $Y$  is self-adjoint). Then we have

$$\begin{aligned} \|AXA^{-1} + B^{-1}XB\| &= \left\| \begin{array}{cc} 0 & PXP^{-1} + Q^{-1}XQ \\ PX^*P^{-1} + Q^{-1}X^*Q & 0 \end{array} \right\| \\ &\geq 2\|Y\| = 2\|X\|, \end{aligned}$$

i.e.

$$\max\{\|PXP^{-1} + Q^{-1}XQ\|, \|PX^*P^{-1} + Q^{-1}X^*Q\|\} \geq 2\|X\|. \quad \square$$

**Theorem 2.4.** For any invertible positive operator  $P$ , and for  $n = 0, 1, 2$ , we have

$$(1) \quad \forall X \in L(H) : \|nX + PXP^{-1} + P^{-1}XP\| \geq (n + 2) \|X\|.$$

*Proof.* If  $n = 0$ , (1) follows from Lemma 2.1.

For all  $X$ , we have

$$\begin{aligned} \|2X + PXP^{-1} + P^{-1}XP\| &= \left\| P^{\frac{1}{2}} \left( P^{\frac{1}{2}}XP^{-\frac{1}{2}} + P^{-\frac{1}{2}}XP^{\frac{1}{2}} \right) P^{-\frac{1}{2}} \right. \\ &\quad \left. + P^{-\frac{1}{2}} \left( P^{\frac{1}{2}}XP^{-\frac{1}{2}} + P^{-\frac{1}{2}}XP^{\frac{1}{2}} \right) P^{\frac{1}{2}} \right\| \\ &\geq 2 \left\| P^{\frac{1}{2}}XP^{-\frac{1}{2}} + P^{-\frac{1}{2}}XP^{\frac{1}{2}} \right\| \\ &\geq 4 \|X\|, \end{aligned}$$

that is, (1) is true for  $n = 2$ .

It follows from the case  $n = 2$ , that for all  $X$ , we have

$$\begin{aligned} \|X + PXP^{-1} + P^{-1}XP\| &\geq \|2X + PXP^{-1} + P^{-1}XP\| - \|X\| \\ &\geq 3 \|X\|. \end{aligned}$$

□

*Remark 2.1.* In the cases  $n = 1$  and  $n = 2$ , the relation (1) is false in general if we replace the condition “positive” by the condition “self-adjoint”; this may be seen by the following example:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\begin{cases} \|X + PXP^{-1} + P^{-1}XP\| = 1 < 3 = 3 \|X\|, \\ \|2X + PXP^{-1} + P^{-1}XP\| = 0 < 4 = 4 \|X\|. \end{cases}$$

### 3. OPERATOR-NORM INEQUALITY AND POSITIVE OPERATORS

**Definition 3.1.** An operator  $A$  in  $L(H)$  is called convexoid if  $W_0(A) = \text{co}\sigma(A)$ .

**Lemma 3.1** ([2]). Let  $A \in L(H)$ . If  $\|A - \alpha\| = r(A - \alpha)$ , for all complex  $\alpha$ , then  $A$  is convexoid.

**Lemma 3.2.** Let  $P$  and  $Q$  be in  $L(H)$  such that  $P > 0$  and  $Q > 0$ . If we have

$$(2) \quad \forall X \in L(H) : \|X\| + \|PXP^{-1}\| \geq 2 \|QXQ^{-1}\|,$$

then  $\{P\}' \subset \{Q\}'$ .

*Proof.* (i) Let  $X$  be self-adjoint such that  $PX = XP$ , and let  $\alpha$  be a complex number. Then, by (2),  $\|X - \alpha\| \geq \|Q(X - \alpha)Q^{-1}\|$ , and since  $X - \alpha$  is normal, we also have  $\|Q(X - \alpha)Q^{-1}\| \geq \|X - \alpha\|$ , so that  $\|Q(X - \alpha)Q^{-1}\| = \|X - \alpha\|$ . Then, by Lemma 3.1, we have  $W_0(QXQ^{-1}) = \text{co}\sigma(X)$ , and since  $X$  is self-adjoint, we obtain  $QXQ^{-1} = Q^{-1}XQ$ , and also  $QX = XQ$ .

(ii) Now let  $X = X_1 + iX_2$ , where  $X_1 = \text{Re}X$  and  $X_2 = \text{Im}X$ , such that  $PX = XP$ . Then, we have  $PX_1 = X_1P$  and  $PX_2 = X_2P$ ; from (i) it follows that  $QX_1 = X_1Q$  and  $QX_2 = X_2Q$ , and also  $QX = XQ$ ; we conclude that  $\{P\}' \subset \{Q\}'$ . □

**Theorem 3.3.** Let  $P$  and  $Q$  be in  $L(H)$  such that  $P > 0$  and  $Q > 0$ . If we have

$$(3) \quad \forall X \in L(H) : \|PXP^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|,$$

then  $\{P\}' = \{Q\}'$ .

*Proof.* From (3), we have

$$(4) \quad \forall X \in L(H) : \|X\| + \|PQXQ^{-1}P^{-1}\| \geq 2\|QXQ^{-1}\|.$$

Let  $UM$  be the polar decomposition of  $PQ$  ( $U$  is unitary and  $M = (QP^2Q)^{\frac{1}{2}}$ ). Then, from (4), we obtain

$$\forall X \in L(H) : \|X\| + \|MXM^{-1}\| \geq 2\|QXQ^{-1}\|$$

and, by Lemma 3.2, we have  $MQ = QM$ ; then  $PQ = QP$ .

Now let  $X$  be self-adjoint such that  $PX = XP$  and let  $\alpha$  be a complex number. Therefore,  $QXQ^{-1} \in \{P\}'$  and, from (3), we obtain

$$\forall X \in L(H) : \|Q(X - \alpha)Q^{-1}\| \leq \|X\|.$$

It follows that  $QX = XQ$ , so that  $\{P\}' \subset \{Q\}'$ .

The symmetric roles of  $P, Q$  in (3) also give  $\{Q\}' \subset \{P\}'$ , and finally we have  $\{P\}' = \{Q\}'$ .  $\square$

**Corollary 3.4.** Let  $P$  and  $Q$  be in  $\mathcal{L}(H)$  such that  $P > 0$  and  $Q > 0$ . If we have

$$\forall X \in \mathcal{L}(H) : \|PXP^{-1} + Q^{-1}XQ\| \geq 2\|X\|,$$

then  $\{P\}' = \{Q\}'$ .

*Proof.* Since we have  $\|PXP^{-1}\| + \|Q^{-1}XQ\| \geq \|PXP^{-1} + Q^{-1}XQ\|$ , for all operators  $X$ , the result follows immediately by Theorem 3.3.  $\square$

**Lemma 3.5.** Let  $\varepsilon > 0$  and let  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$  (for  $n \in \mathbb{N}^*$ ) such that  $0 < \alpha_1 < \dots < \alpha_n \leq 1$ ,  $\{\alpha_1, \dots, \alpha_n\} = \{\beta_1, \dots, \beta_n\}$  and  $\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \varepsilon$ , for all  $i, j$ . Then we have  $|\alpha_i - \beta_i| < \varepsilon$ , for all  $i$ .

*Proof.* From the hypothesis, we obtain  $\beta_i - \beta_j < \varepsilon$ , if  $i < j$ .

Let  $i \in \{1, \dots, n\}$  such that  $\alpha_i \neq \beta_i$  (in the case  $\alpha_i = \beta_i$ , of course we have  $|\alpha_i - \beta_i| = 0 < \varepsilon$ ).

There are three cases  $i = 1$ ,  $i = n$  and  $1 < i < n$ .

*Case 1.*  $i = 1$ . There exists  $j \geq 2$ , such that  $\beta_j = \alpha_1$ , so we have  $|\beta_1 - \alpha_1| = \beta_1 - \beta_j < \varepsilon$ , since  $j > 1$ .

*Case 2.*  $i = n$ . There exists  $j < n$ , such that  $\beta_j = \alpha_n$ , so we have  $|\beta_n - \alpha_n| = \beta_j - \beta_n < \varepsilon$ , since  $n > j$ .

*Case 3.*  $1 < i < n$ . If  $\beta_i > \alpha_i$ , then there exists  $j > i$ , such that  $\beta_j \leq \alpha_i$ , and we have  $|\beta_i - \alpha_i| = \beta_i - \beta_j < \varepsilon$ , since  $j > i$ . If  $\beta_i < \alpha_i$ , then there exists  $j < i$ , such that  $|\beta_i - \alpha_i| = \beta_j - \beta_i < \varepsilon$ , since  $i > j$ .  $\square$

**Theorem 3.6.** Let  $P$  and  $Q$  be in  $L(H)$  such that  $P > 0$ ,  $Q > 0$  and  $\sigma(P) = \sigma(Q)$ . Then the following properties are equivalent:

- (i)  $\forall X \in L(H), \|PXP^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|$ .
- (ii)  $P = Q$ .

*Proof.* We may assume, without loss of the generality, that  $\|P\| = \|Q\| = 1$ .

(i) implies (ii). Decompose  $P$  and  $Q$  using the spectral measure

$$P = \int \lambda dE_\lambda, \quad Q = \int \lambda dF_\lambda$$

and consider

$$P_n = \int h_n(\lambda) dE_\lambda = h_n(P), \quad Q_n = \int h_n(\lambda) dF_\lambda = h_n(Q)$$

where  $h_n(\lambda)$  is a function of the form

$$h_n(\lambda) = \frac{k}{n}, \text{ for } \frac{k}{n} \leq \lambda < \frac{k+1}{n}, \text{ and } k = 0, 1, 2, \dots$$

Then by the spectral theorem and by the form of  $h_n(\lambda)$ , we have  $\sigma(P_n) = \sigma(Q_n) = h_n(\sigma(P))$  is finite,  $P_n \rightarrow P$ ,  $Q_n \rightarrow Q$  (uniformly) and  $P_n \in \{P\}''$ ,  $Q_n \in \{Q\}''$  (where  $\{P\}'' = \{Q\}''$ , by Theorem 3.3).

Put  $\sigma(P_n) = \{\alpha_1, \dots, \alpha_p\}$  such that  $0 < \alpha_1 < \dots < \alpha_p \leq 1$ . Then there exist  $p$  orthogonal projections  $E_1, \dots, E_p$  such that  $E_i E_j = E_j E_i = 0$  if  $i \neq j$ ,  $E_1 \oplus \dots \oplus E_p = I$  and  $P_n = \sum_{i=1}^p \alpha_i E_i$ .

Since  $\sigma(P_n) = \sigma(Q_n)$ ,  $P_n Q_n = Q_n P_n$  and  $Q_n$  is normal, there exist  $p$  scalar  $\beta_1, \dots, \beta_p$  such that  $Q_n = \sum_{i=1}^p \beta_i E_i$  and  $\{\alpha_1, \dots, \alpha_p\} = \{\beta_1, \dots, \beta_p\}$ .

Let  $\varepsilon > 0$ . Then there exists an integer  $N$  such that

$$(*) \quad \forall n > N, \forall X \in L(H), \|PXP^{-1}\| + \|Q^{-1}XQ\| \geq (2 - \varepsilon)\|X\|.$$

Let  $n > N$  and  $X_{ij} = E_i X E_j$ , for  $X \in L(H)$ . Then, by using  $(*)$ , we have

$$\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \varepsilon.$$

By Lemma 3.5, this implies  $|\alpha_i - \beta_i| < \varepsilon$ , for all  $i$ ; therefore

$$\|P_n - Q_n\| = \max_{1 \leq i \leq p} |\alpha_i - \beta_i| < \varepsilon,$$

so we obtain  $P = Q$ .

(ii) implies (i) is immediate from Lemma 2.1. □

**Corollary 3.7.** *Let the pair  $(P, Q)$  of operators satisfy the condition of Theorem 3.6. Then the following properties are equivalent:*

- (i)  $\forall X \in L(H), \|PXP^{-1} + Q^{-1}XQ\| \geq 2\|X\|$ .
- (ii)  $P = Q$ .

#### 4. CHARACTERIZATION OF THE CORACH-PORTA-RECHT INEQUALITY

*Notation.* For  $\theta \in [0, \pi[$ , we denote by  $D_\theta$  the straight line through the origin in the complex plane.

**Lemma 4.1.** *Let  $\lambda, \mu \in \mathbb{C}^*$  such that  $\frac{\lambda}{\mu} + \frac{\mu}{\lambda} \in \mathbb{R}$  and  $\left| \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right| \geq 2$ . Then there exists a scalar  $\theta \in [0, \pi[$  such that  $\lambda, \mu \in D_\theta$ .*

*Proof.* Let  $\lambda = r_1 e^{i\theta_1}$  and  $\mu = r_2 e^{i\theta_2}$  be the polar decompositions respectively of  $\lambda$  and  $\mu$ . Then we have

$$\frac{\lambda}{\mu} + \frac{\mu}{\lambda} = \left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right) \cos(\theta_1 - \theta_2) + i \left(\frac{r_1}{r_2} - \frac{r_2}{r_1}\right) \sin(\theta_1 - \theta_2),$$

so we obtain  $r_1 = r_2$  or  $\theta_1 - \theta_2 \equiv 0 \pmod{\pi}$ .

The case  $r_1 = r_2$  also gives  $\theta_1 - \theta_2 \equiv 0 \pmod{\pi}$ , therefore  $\lambda, \mu \in D_\theta$ , for some  $\theta \in [0, \pi[$ . □

**Lemma 4.2.** *All invertible operators  $S$  satisfying the condition*

$$(5) \quad \forall X \in L(H), \quad \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|$$

*are normal.*

Let  $S$  be an invertible operator satisfying (5) and let  $UP, VQ$  be the polar decompositions respectively of  $S$  and  $S^*$ . Then, by (5), we obtain

$$\forall X \in L(H), \quad \|PXP^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|.$$

Since  $P^2 = S^*S$  and  $Q^2 = SS^*$ , then  $\sigma(P^2) = \sigma(Q^2)$ , and by the spectral theorem, we obtain  $\sigma(P) = \sigma(Q)$ ; so we have, by Theorem 3.6,  $P = Q$ , and also  $S^*S = SS^*$ . Therefore  $S$  is normal.

**Lemma 4.3.** *Let  $S$  be an invertible normal operator. Then the following properties are equivalent:*

- (i)  $\forall X \in L(H), \quad \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|.$
- (ii)  $\sigma(S) \subset D_\theta$ , for some  $\theta \in [0, \pi[$ .
- (iii)  $S = \lambda M$ , for some nonzero scalar  $\lambda$  and for some invertible self-adjoint operator  $M$ .

*Proof.* (i) implies (ii).

From (i) and Lemma 4.2,  $S$  is normal. Then, by the spectral measure of  $S$ , there exists a sequence  $(S_n)$  of invertible normal operators with finite spectrum such that

- (a)  $S_n \rightarrow S$  uniformly,
- (b) for all  $\lambda$  in  $\sigma(S)$ , there exists a sequence  $(\lambda_n)$  such that  $\lambda_n \in \sigma(S_n)$ , for all  $n$  and  $\lambda_n \rightarrow \lambda$ .

Let  $\lambda, \mu \in \sigma(S)$  and let  $\varepsilon > 0$ . Then by (i), (a) and (b), there exists an integer  $N$  such that

$$(6) \quad \forall n > N, \forall X \in L(H) \quad \|S_nXS_n^{-1} + S_n^{-1}XS_n\| \geq (2 - \varepsilon)\|X\|$$

and there exist two sequences  $(\lambda_n)$  and  $(\mu_n)$  such that

$$\forall n, \lambda_n, \mu_n \in \sigma(S_n); \quad \lambda_n \rightarrow \lambda, \quad \mu_n \rightarrow \mu.$$

Let  $n > N$  and since  $S_n$  is normal with finite spectrum, there exist  $p$  orthogonal projections  $E_1, \dots, E_p$  such that  $E_kE_j = E_jE_k = 0$ , if  $k \neq j$ ,  $E_1 \oplus \dots \oplus E_p = I$  and  $S_n = \sum_{k=1}^p \alpha_k E_k$ , where  $\sigma(S_n) = \{\alpha_1, \dots, \alpha_p\}$ ,  $\alpha_1 = \lambda_n, \alpha_2 = \mu_n$ .

Then by (6) and if we put  $A = \begin{bmatrix} 2 & \gamma_n \\ \gamma_n & 2 \end{bmatrix}$ , where  $\gamma_n = \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}$ , we obtain

$$(7) \quad \forall X \in L(\mathbb{C}^2), \quad \|A \circ X\| \geq (2 - \varepsilon)\|X\|,$$

and if we put  $\delta_n = \frac{1}{\gamma_n}$  and  $B = \begin{bmatrix} \frac{1}{2} & \delta_n \\ \delta_n & \frac{1}{2} \end{bmatrix}$ , then from (7), we also have

$$(8) \quad \forall X \in L(\mathbb{C}^2), \|B \circ X\| \leq \frac{\|X\|}{(2-\varepsilon)}.$$

From (7), we deduce  $\left| \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n} \right| \geq (2-\varepsilon)$ , so we obtain  $\left| \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right| \geq 2$ .

On the other hand, if in (8) we put  $X = \begin{bmatrix} 1 & ia \\ ia & 1 \end{bmatrix}$ , where  $a > 0$ , we obtain  $\frac{1}{4} + a^2 |\gamma_n|^2 + a |\beta_n| \leq \frac{1+a^2}{(2-\varepsilon)^2}$ , where  $\beta_n = \text{Im } \gamma_n$ ; so that  $\frac{1}{4} + a^2 |\alpha|^2 + a |\beta| \leq \frac{1+a^2}{(2-\varepsilon)^2}$ . Therefore  $a |\alpha|^2 + |\beta| \leq \frac{a}{4}$ ; then  $\beta = 0$  and  $\frac{\lambda}{\mu} + \frac{\mu}{\lambda} \in \mathbb{R}$ . This implies condition (ii) by Lemma 4.1.

(ii) implies (iii).

If we put  $M = e^{-i\theta} S$ , then  $M$  is an invertible normal operator with real spectrum, so we have  $S = e^{i\theta} M$ , where  $M$  is an invertible self-adjoint operator.

(iii) implies (i) is immediate by Lemma 2.1.  $\square$

**Theorem 4.4.** *The set of all invertible operators  $S$ , for which*

$$\forall X \in L(H), \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|$$

*is the set  $\{\lambda M : \lambda \in \mathbb{C}^*, M \text{ an invertible self-adjoint operator}\}$ .*

*Proof.* This follows immediately by Lemma 4.2 and Lemma 4.3.  $\square$

*Remark 4.1.* The extremal class of invertible operators  $S$  satisfying the condition

$$\inf_{\|X\|=1} \|SXS^{-1} + S^{-1}XS\| = 2$$

has been characterized. So it remains the characterization of the second extremal class of all invertible operators  $S$  satisfies the condition

$$\inf_{\|X\|=1} \|SXS^{-1} + S^{-1}XS\| = 0.$$

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