SOME RESULTS RELATED TO THE CORACH-PORTA-RECHT INEQUALITY

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Abstract. Let $L(H)$ be the algebra of all bounded operators on a complex Hilbert space $H$ and let $S$ be an invertible self-adjoint (or skew-symmetric) operator of $L(H)$. Corach-Porta-Recht proved that

$$\forall X \in L(H), \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|.$$ (*)&

The problem considered here is that of finding (i) some consequences of the Corach-Porta-Recht Inequality; (ii) a necessary condition (resp. necessary and sufficient condition, when $P = Q$) for the invertible positive operators $P, Q$ to satisfy the operator-norm inequality $\|PX^{-1} + Q^{-1}XQ\| \geq 2\|X\|$, for all $X$ in $L(H)$; (iii) a necessary and sufficient condition for the invertible operator $S$ in $L(H)$ to satisfy (*)

1. Introduction

All operators considered here are bounded operators on a complex Hilbert space $H$. The collection of operators in $H$ is denoted by $L(H)$.

For $T \in L(H)$, we denote by $\sigma(T), co(\sigma(T)), r(T), W_0(T), \{T\}$ and $\{T\}$ the spectrum, the convex hull of the spectrum, the spectral radius, the numerical range, the commutant and the bicommutant of $T$, respectively.

If $A = (a_{ij})$ and $B = (b_{ij})$ are two complex $n \times n$ matrices, then define the Schur (or Schur-Hadamard) product of $A$ and $B$ to be the matrix $A \circ B = (a_{ij}b_{ij})$.

In [1], Corach, Porta, and Recht have proved that for any invertible self-adjoint or skew-symmetric operator $S$, the operator-norm inequality

$$\|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|$$

holds for all operators $X$.

It is also clear that, for any invertible operator $S$ and for any two invertible positive operators $P, Q$, we have

(a) $0 < \inf \frac{\|PX^{-1} + Q^{-1}XQ\|}{\|X\|} \leq 2$,

(b) $0 \leq \inf \frac{\|SXS^{-1} + S^{-1}XS\|}{\|X\|} \leq 2$.

It may be seen by the Corach-Porta-Recht Inequality that the infimum in (a) is 2, if $P = Q$; and the infimum in (b) is also 2, for $S$ an invertible self-adjoint operator,

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or more generally, if $S$ is of the form $S = \lambda M$, where $M$ is an invertible self-adjoint operator and $\lambda$ is a nonzero scalar.

The purpose of this paper is the following:

(1) In §2, we give the following consequences of the Corach-Porta-Recht Inequality. For all invertible positive commuting operators $P, Q$ and for all operators $X$, we have

\begin{enumerate}
\item $\|PX^{-1} + Q^{-1}X\| \geq 2 \|X\|$, if $\|X\| = r(X)$,
\item $\max\{\|PX^{-1} + Q^{-1}X\|, \|PX^{-1} + P^{-1}Q^{-1}XQ\|\} \geq 2 \|X\|$,
\item $\|nX + PX^{-1} + P^{-1}XP\| \geq (n + 2) \|X\|$, for $n = 0, 1, 2$.
\end{enumerate}

(2) In §3, we show that the infimum in (a) is 2 only if $\{P\}' = \{Q\}'$; on the other hand, if $\sigma(P) = \sigma(Q)$, then the infimum in (a) is 2 if and only if $P = Q$.

(3) In §4, we show that the only operators $S$ for which the infimum in (b) is 2 are those of the form $S = \lambda M$, where $M$ is an invertible self-adjoint operator and $\lambda$ is a nonzero scalar.

2. Some consequences of the Corach-Porta-Recht Inequality

Lemma 2.1 ([II]). For an invertible self-adjoint or skew-symmetric operator $S$, we have $\forall X \in L(H) : \|SX^{-1}S^{-1} + S^{-1}X\| \geq 2 \|X\|$.

Theorem 2.2. For any pair $(P, Q)$ of commuting invertible positive operators and for any $X \in L(H)$ such that $\|X\| = r(A)$, we have

$$\|PX^{-1} + Q^{-1}X\| \geq 2 \|X\|.$$  

Proof. Let $X \in L(H)$ such that $\|X\| = r(A)$ and put $Y = P^{\frac{1}{2}}Q\frac{1}{2}XQ\frac{1}{2}P^{\frac{1}{2}}$. Then since $P^{\frac{1}{2}}Q\frac{1}{2} = Q\frac{1}{2}P^{\frac{1}{2}}$ is self-adjoint, we have by Lemma 2.1

$$\|PX^{-1} + Q^{-1}X\| = \left\|\left(P^{\frac{1}{2}}Q\frac{1}{2}\right)Y(P^{\frac{1}{2}}Q\frac{1}{2})^{-1} + (P^{\frac{1}{2}}Q\frac{1}{2})^{-1}Y(P^{\frac{1}{2}}Q\frac{1}{2})\right\|$$

$$\geq 2 \|Y\|$$

$$\geq 2r(X)$$

$$\geq 2 \|X\|. \quad \square$$

Theorem 2.3. For any pair $(P, Q)$ satisfying the condition of Theorem 2.2 and for any operator $X$, we have

$$\max\{\|PX^{-1} + Q^{-1}X\|, \|PX^{-1} + Q^{-1}X^{-1}Q\|\} \geq 2 \|X\|.$$  

Proof. For $X \in L(H)$, let

$$A = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}, B = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.$$  

The pair $(A, B)$ satisfies the condition of Theorem 2.2 and $\|Y\| = r(Y)$ (since $Y$ is self-adjoint). Then we have

$$\|AX^{-1} + B^{-1}X\| = \left\|\begin{bmatrix} 0 & PX^{-1} + Q^{-1}X \| \end{bmatrix} \begin{bmatrix} P \|X\| \end{bmatrix} \right\|$$

$$\geq 2 \|Y\| = 2 \|X\|,$$

i.e.

$$\max\{\|PX^{-1} + Q^{-1}X\|, \|PX^{-1} + Q^{-1}X^{-1}Q\|\} \geq 2 \|X\|. \quad \square$$
Lemma 3.2. Let $A$ be convexoid. Then, by Lemma 3.1, we have (2)

$$\|2X + PXP^{-1} + P^{-1}XP\| = \left\| P^\frac{1}{2} \left( P^\frac{1}{2} X P^{-\frac{1}{2}} + P^{-\frac{1}{2}} X P^\frac{1}{2} \right) P^{-\frac{1}{2}} \right\|
\geq 2 \left\| P^\frac{1}{2} X P^{-\frac{1}{2}} + P^{-\frac{1}{2}} X P^\frac{1}{2} \right\|
\geq 4 \|X\|,$$

that is, (1) is true for $n = 2$.

It follows from the case $n = 2$, that for all $X$, we have

$$\|X + PXP^{-1} + P^{-1}XP\| \geq \|2X + PXP^{-1} + P^{-1}XP\| - \|X\| \geq 3 \|X\|.$$

\[ \square \]

Remark 2.1. In the cases $n = 1$ and $n = 2$, the relation (1) is false in general if we replace the condition “positive” by the condition “self-adjoint”; this may be seen by the following example:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\begin{cases}
\|X + PXP^{-1} + P^{-1}XP\| = 1 < 3 = 3 \|X\|, \\
\|2X + PXP^{-1} + P^{-1}XP\| = 0 < 4 = 4 \|X\|.
\end{cases}$$

### Results Related to the Corach-Porta-Recht Inequality

#### Theorem 2.4.
For any invertible positive operator $P$, and for $n = 0, 1, 2$, we have

$$\forall X \in L(H) : \|nX + PXP^{-1} + P^{-1}XP\| \geq (n + 2) \|X\|.$$

Proof. If $n = 0$, (1) follows from Lemma 2.1.

For all $X$, we have

$$\left\| 2X + PXP^{-1} + P^{-1}XP \right\| = \left\| P^\frac{1}{2} \left( P^\frac{1}{2} X P^{-\frac{1}{2}} + P^{-\frac{1}{2}} X P^\frac{1}{2} \right) P^{-\frac{1}{2}} \right\|
\geq 2 \left\| P^\frac{1}{2} X P^{-\frac{1}{2}} + P^{-\frac{1}{2}} X P^\frac{1}{2} \right\|
\geq 4 \|X\|,$$

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$$\|X + PXP^{-1} + P^{-1}XP\| \geq \|2X + PXP^{-1} + P^{-1}XP\| - \|X\| \geq 3 \|X\|.$$

### 3. Operator-norm Inequality and Positive Operators

#### Definition 3.1.
An operator $A$ in $L(H)$ is called convexoid if $W_0(A) = \cos(\alpha)$.

#### Lemma 3.1 (2).
Let $A \in L(H)$. If $\|A - \alpha\| = r(A - \alpha)$, for all complex $\alpha$, then $A$ is convexoid.

#### Lemma 3.2.
Let $P$ and $Q$ be in $L(H)$ such that $P > 0$ and $Q > 0$. If we have

$$\forall X \in L(H) : \|X\| + \|PXP^{-1}\| \geq 2 \|QXQ^{-1}\|,$$

then $\{P\}' \subset \{Q\}'$.

Proof. (i) Let $X$ be self-adjoint such that $PX = XP$, and let $\alpha$ be a complex number. Then, by (2), $\|X - \alpha\| \geq \|Q(X - \alpha)Q^{-1}\|$, and since $X - \alpha$ is normal, we also have $\|Q(X - \alpha)Q^{-1}\| \geq \|X - \alpha\|$, so that $\|Q(X - \alpha)Q^{-1}\| = \|X - \alpha\|$. Then, by Lemma 3.1, we have $W_0(QXQ^{-1}) = \cos(\alpha)$, and since $X$ is self-adjoint, we obtain $QXQ^{-1} = Q^{-1}XQ$, and also $QX = XQ$.

(ii) Now let $X = X_1 + iX_2$, where $X_1 = ReX$ and $X_2 = ImX$, such that $PX = XP$. Then, we have $PX_1 = X_1P$ and $PX_2 = X_2P$; from (i) it follows that $QX_1 = X_1Q$ and $QX_2 = X_2Q$, and also $QX = XQ$; we conclude that $\{P\}' \subset \{Q\}'$. \[ \square \]
Theorem 3.3. Let $P$ and $Q$ be in $L(H)$ such that $P > 0$ and $Q > 0$. If we have
\[(3) \quad \forall X \in L(H) : \|PX P^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|,
\]
then \(\{P\}' = \{Q\}'\).
Proof. From (3), we have
\[(4) \quad \forall X \in L(H) : \|X\| + \|PQXQ^{-1}P^{-1}\| \geq 2\|QXQ^{-1}\|.
\]
Let $UM$ be the polar decomposition of $PQ$ ($U$ is unitary and $M = (QP^2Q)^{\frac{1}{2}}$). Then, from (4), we obtain
\[\forall X \in L(H) : \|X\| + \|MXM^{-1}\| \geq 2\|QXQ^{-1}\|\]
and, by Lemma 3.2, we have $MQ = QM$; then $PQ = QP$. Now let $X$ be self-adjoint such that $PX = XP$ and let $\alpha$ be a complex number. Therefore, $QXQ^{-1} \in \{P\}'$ and, from (3), we obtain
\[\forall X \in L(H) : \|(Q(X - \alpha)Q^{-1}\| \leq \|X\|.
\]
It follows that $QX = XQ$, so that \(\{P\}' \subset \{Q\}'\).

The symmetric roles of $P, Q$ in (3) also give \(\{Q\}' \subset \{P\}'\), and finally we have \(\{P\}' = \{Q\}'\).
\[
\square
\]
Corollary 3.4. Let $P$ and $Q$ be in $\mathcal{L}(H)$ such that $P > 0$ and $Q > 0$. If we have
\[\forall X \in \mathcal{L}(H) : \|PX P^{-1} + Q^{-1}XQ\| \geq 2\|X\|,
\]
then \(\{P\}' = \{Q\}'\).
Proof. Since we have \(\|PX P^{-1}\| + \|Q^{-1}XQ\| \geq \|PX P^{-1} + Q^{-1}XQ\|\), for all operators $X$, the result follows immediately by Theorem 3.3.
\[
\square
\]
Lemma 3.5. Let $\epsilon > 0$ and let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ (for $n \in \mathbb{N}^*$) such that $0 < \alpha_1 < \cdots < \alpha_n \leq 1$, $\{\alpha_1, \ldots, \alpha_n\} = \{\beta_1, \ldots, \beta_n\}$ and $\frac{\alpha_i}{\alpha_j} + \frac{\beta_i}{\beta_j} \geq 2 - \epsilon$, for all $i, j$.
Then we have $|\alpha_i - \beta_i| < \epsilon$, for all $i$.
Proof. From the hypothesis, we obtain $\beta_i - \beta_j < \epsilon$, if $i < j$.

Let $i \in \{1, \ldots, n\}$ such that $\alpha_i \neq \beta_i$ (in the case $\alpha_i = \beta_i$, of course we have $|\alpha_i - \beta_i| = 0 < \epsilon$).

There are three cases $i = 1$, $i = n$ and $1 < i < n$.

Case 1. $i = 1$. There exists $j \geq 2$, such that $\beta_j = \alpha_1$, so we have $|\beta_1 - \alpha_1| = \beta_1 - \beta_j < \epsilon$, since $j > 1$.

Case 2. $i = n$. There exists $j < n$, such that $\beta_j = \alpha_n$, so we have $|\beta_n - \alpha_n| = \beta_j - \beta_n < \epsilon$, since $n > j$.

Case 3. $1 < i < n$. If $\beta_i > \alpha_i$, then there exists $j > i$, such that $\beta_j \leq \alpha_i$, and we have $|\beta_i - \alpha_i| = \beta_i - \beta_j < \epsilon$, since $j > i$. If $\beta_i < \alpha_i$, then there exists $j < i$, such that $|\beta_i - \alpha_i| = \beta_j - \beta_i < \epsilon$, since $i > j$.
\[
\square
\]
Theorem 3.6. Let $P$ and $Q$ be in $L(H)$ such that $P > 0$, $Q > 0$ and $\sigma(P) = \sigma(Q)$. Then the following properties are equivalent:

(i) $\forall X \in L(H), \|PX P^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|$.

(ii) $P = Q$.
Proof. We may assume, without loss of the generality, that \( \|P\| = \|Q\| = 1 \).

(i) implies (ii). Decompose \( P \) and \( Q \) using the spectral measure
\[
P = \int \lambda dE_\lambda, \quad Q = \int \lambda dF_\lambda
\]
and consider
\[
P_n = \int h_n(\lambda) \, dE_\lambda = h_n(P), \quad Q_n = \int h_n(\lambda) \, dF_\lambda = h_n(Q)
\]
where \( h_n(\lambda) \) is a function of the form
\[
h_n(\lambda) = \frac{k}{n}, \text{ for } \frac{k}{n} \leq \lambda < \frac{k + 1}{n}, \text{ and } k = 0, 1, 2, \ldots.
\]

Then by the spectral theorem and by the form of \( h_n(\lambda) \), we have \( \sigma(P_n) = \sigma(Q_n) = h_n(\sigma(P)) \) is finite, \( P_n \rightarrow P, Q_n \rightarrow Q \) (uniformly) and \( P_n \in \{P\}^n, Q_n \in \{Q\}^n \) (where \( \{P\}^n = \{Q\}^n \), by Theorem 3.3).

Put \( \sigma(P_n) = \{\alpha_1, \ldots, \alpha_p\} \) such that \( 0 < \alpha_1 < \cdots < \alpha_p \leq 1 \). Then there exist \( p \) orthogonal projections \( E_1, \ldots, E_p \) such that \( E_i E_j = E_j E_i = 0 \) if \( i \neq j \), \( E_1 \oplus \cdots \oplus E_p = I \) and \( P_n = \sum_{i=1}^{p} \alpha_i E_i \).

Since \( \sigma(P_n) = \sigma(Q_n) \), \( P_n Q_n = Q_n P_n \) and \( Q_n \) is normal, there exist \( p \) scalar \( \beta_1, \ldots, \beta_p \) such that \( Q_n = \sum_{i=1}^{p} \beta_i E_i \) and \( \{\alpha_1, \ldots, \alpha_p\} = \{\beta_1, \ldots, \beta_p\} \).

Let \( \varepsilon > 0 \). Then there exists an integer \( N \) such that
\[
(\ast) \quad \forall n > N, \forall X \in L(H), \quad \|P XP^{-1}\| + \|Q^{-1} XQ\| \geq (2 - \varepsilon) \|X\|.
\]

Let \( n > N \) and \( X_{ij} = E_i X E_j \), for \( X \in L(H) \). Then, by using (\ast), we have
\[
\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \varepsilon.
\]

By Lemma 3.5, this implies \( |\alpha_i - \beta_i| < \varepsilon \), for all \( i \); therefore
\[
\|P_n - Q_n\| = \max_{1 \leq i \leq p} |\alpha_i - \beta_i| < \varepsilon,
\]
so we obtain \( P = Q \).

(ii) implies (i) is immediate from Lemma 2.1. \(\square\)

**Corollary 3.7.** Let the pair \((P, Q)\) of operators satisfy the condition of Theorem 3.6. Then the following properties are equivalent:

(i) \( \forall X \in L(H), \|P XP^{-1} + Q^{-1} XQ\| \geq 2 \|X\| \).

(ii) \( P = Q \).

4. Characterization of the Corach-Porta-Recht Inequality

**Notation.** For \( \theta \in [0, \pi] \), we denote by \( D_\theta \) the straight line through the origin in the complex plane.

**Lemma 4.1.** Let \( \lambda, \mu \in \mathbb{C}^* \) such that \( \frac{\lambda}{n} + \frac{\mu}{\lambda} \in \mathbb{R} \) and \( \left| \frac{\lambda}{n} + \frac{\mu}{\lambda} \right| \geq 2 \). Then there exists a scalar \( \theta \in [0, \pi] \) such that \( \lambda, \mu \in D_\theta \).
Lemma 4.2. All invertible operators $S$ satisfying the condition

$$\forall X \in L(H), \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|$$

are normal.

Let $S$ be an invertible operator satisfying (5) and let $UP$, $VQ$ be the polar decompositions respectively of $S$ and $S^*$. Then, by (5), we obtain

$$\forall X \in L(H), \|PX^{-1}XP^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|.$$  

Since $P^2 = S^*S$ and $Q^2 = SS^*$, then $\sigma(P^2) = \sigma(Q^2)$, and by the spectral theorem, we obtain $\sigma(P) = \sigma(Q)$; so we have, by Theorem 3.6, $P = Q$, and also $S^*S = SS^*$. Therefore $S$ is normal.

Lemma 4.3. Let $S$ be an invertible normal operator. Then the following properties are equivalent:

(i) $\forall X \in L(H), \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|.$

(ii) $\sigma(S) \subset D_\theta, \text{ for some } \theta \in [0, \pi].$

(iii) $S = \lambda M$, for some nonzero scalar $\lambda$ and for some invertible self-adjoint operator $M$.

Proof. (i) implies (ii).

From (i) and Lemma 4.2, $S$ is normal. Then, by the spectral measure of $S$, there exists a sequence $(S_n)$ of invertible normal operators with finite spectrum such that

(a) $S_n \to S$ uniformly,

(b) for all $\lambda \in \sigma(S)$, there exists a sequence $(\lambda_n)$ such that $\lambda_n \in \sigma(S_n)$, for all $n$ and $\lambda_n \to \lambda$.

Let $\lambda, \mu \in \sigma(S)$ and let $\varepsilon > 0$. Then by (i), (a) and (b), there exists an integer $N$ such that

$$\forall n > N, \forall X \in L(H), \|S_nXS_n^{-1} + S_n^{-1}XS_n\| \geq (2 - \varepsilon)\|X\|$$

and there exist two sequences $(\lambda_n)$ and $(\mu_n)$ such that

$$\forall n, \lambda_n, \mu_n \in \sigma(S_n); \lambda_n \to \lambda, \mu_n \to \mu.$$  

Let $n > N$ and since $S_n$ is normal with finite spectrum, there exist $p$ orthogonal projections $E_1, \ldots, E_p$ such that $E_kE_j = E_jE_k = 0$, if $k \neq j$, $E_1 \oplus \ldots \oplus E_p = I$ and $S_n = \sum_{k=1}^{p} \alpha_k E_k$, where $\sigma(S_n) = \{\alpha_1, \ldots, \alpha_p\}$, $\alpha_1 = \lambda_n, \alpha_2 = \mu_n$.

Then by (6) and if we put $A = \begin{bmatrix} 2 & \gamma_n \\ \gamma_n & 2 \end{bmatrix}$, where $\gamma_n = \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}$, we obtain

$$\forall X \in L(\mathbb{C}^2), \|A \circ X\| \geq (2 - \varepsilon)\|X\|,$$
and if we put $\delta_n = \frac{1}{\gamma_n}$ and $B = \begin{bmatrix} \frac{1}{\delta_n} & \delta_n \frac{1}{2} \end{bmatrix}$, then from (7), we also have

(8) \hspace{1cm} \forall X \in L(C^2), \| B \circ X \| \leq \frac{\| X \|}{(2 - \varepsilon)}.

From (7), we deduce $\left| \frac{\Delta_n}{\mu_n} + \frac{\Delta_i}{n} \right| \geq (2 - \varepsilon)$, so we obtain $\left| \frac{\Delta_n}{\mu} + \frac{\Delta_i}{X} \right| \geq 2$.

On the other hand, if in (8) we put $X = [\frac{1}{ia} \frac{ia}{1}]$, where $a > 0$, we obtain $\frac{1}{\mu} + a^2 |\gamma_n|^2 + a |\beta_n| \leq \frac{1 + a^2}{2 - \varepsilon}$, where $\beta_n = Im \gamma_n$; so that $\frac{1}{\mu} + a^2 |\alpha|^2 + a |\beta| \leq \frac{1 + a^2}{2 - \varepsilon}$. Therefore $a |\alpha|^2 + |\beta| \leq \frac{a}{4}$; then $\beta = 0$ and $\frac{\Delta_n}{\mu} + \frac{\Delta_i}{X} \in \mathbb{R}$. This implies condition (ii) by Lemma 4.1.

(ii) implies (iii).

If we put $M = e^{-i\phi}S$, then $M$ is an invertible normal operator with real spectrum, so we have $S = e^{i\phi}M$, where $M$ is an invertible self-adjoint operator.

(iii) implies (i) is immediate by Lemma 2.1. \qed

**Theorem 4.4.** The set of all invertible operators $S$, for which

\[ \forall X \in L(H), \| SXS^{-1} + S^{-1}XS \| \geq 2 \| X \| \]

is the set $\{ \lambda M : \lambda \in \mathbb{C}^*, M \text{ an invertible self-adjoint operator} \}$.

**Proof.** This follows immediately by Lemma 4.2 and Lemma 4.3. \qed

**Remark 4.1.** The extremal class of invertible operators $S$ satisfying the condition

\[ \inf_{\| X \| = 1} \| SXS^{-1} + S^{-1}XS \| = 2 \]

has been characterized. So it remains the characterization of the second extremal class of all invertible operators $S$ satisfies the condition

\[ \inf_{\| X \| = 1} \| SXS^{-1} + S^{-1}XS \| = 0. \]

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