SOME RESULTS RELATED TO THE CORACH-PORTA-RECHT INEQUALITY

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Abstract. Let $L(H)$ be the algebra of all bounded operators on a complex Hilbert space $H$ and let $S$ be an invertible self-adjoint (or skew-symmetric) operator of $L(H)$. Corach-Porta-Recht proved that

$$\forall X \in L(H), \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|.$$  

The problem considered here is that of finding (i) some consequences of the Corach-Porta-Recht Inequality; (ii) a necessary condition (resp. necessary and sufficient condition, when $(P) = (Q)$) for the invertible positive operators $P, Q$ to satisfy the operator-norm inequality $\|PX^{-1} + Q^{-1}XQ\| \geq 2\|X\|$, for all $X$ in $L(H)$; (iii) a necessary and sufficient condition for the invertible operator $S$ in $L(H)$ to satisfy $(*)$.

1. Introduction

All operators considered here are bounded operators on a complex Hilbert space $H$. The collection of operators in $H$ is denoted by $L(H)$.

For $T \in L(H)$, we denote by $\sigma(T)$, $co(\sigma(T))$, $r(T)$, $W_0(T), \{T\}'$ and $\{T\}"$ the spectrum, the convex hull of the spectrum, the spectral radius, the numerical range, the commutant and the bicommutant of $T$, respectively.

If $A = (a_{ij})$ and $B = (b_{ij})$ are two complex $n \times n$ matrices, then define the Schur (or Schur-Hadamard) product of $A$ and $B$ to be the matrix $A \circ B = (a_{ij}b_{ij})$.

In [1], Corach, Porta, and Recht have proved that for any invertible self-adjoint or skew-symmetric operator $S$, the operator-norm inequality

$$\|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|$$

holds for all operators $X$.

It is also clear that, for any invertible operator $S$ and for any two invertible positive operators $P, Q$, we have

(a) $0 < \inf_{\|X\|=1} \|PX^{-1} + Q^{-1}XQ\| \leq 2$,

(b) $0 \leq \inf_{\|X\|=1} \|SXS^{-1} + S^{-1}XS\| \leq 2$.

It may be seen by the Corach-Porta-Recht Inequality that the infimum in (a) is 2, if $P = Q$; and the infimum in (b) is also 2, for $S$ an invertible self-adjoint operator,
or more generally, if \( S \) is of the form \( S = \lambda M \), where \( M \) is an invertible self-adjoint operator and \( \lambda \) is a nonzero scalar.

The purpose of this paper is the following:

1. In \( \S 2 \), we give the following consequences of the Corach-Porta-Recht Inequality. For all invertible positive commuting operators \( P, Q \) and for all operators \( X \), we have

\[
\begin{align*}
(i) & \quad \| PXP^{-1} + Q^{-1}XQ \| \geq 2 \| X \|, \text{ if } \| X \| = r(X), \\
(ii) & \quad \max \{ \| PXP^{-1} + Q^{-1}XQ \|, \| PX^*P^{-1} + Q^{-1}X^*Q \| \} \geq 2 \| X \|, \\
(iii) & \quad \| nX + PXP^{-1} + P^{-1}XP \| \geq (n + 2) \| X \|, \text{ for } n = 0, 1, 2.
\end{align*}
\]

2. If \( X \) is a nonzero scalar.

2. SOME CONSEQUENCES OF THE CORACH-PORTA-RECHT INEQUALITY

**Lemma 2.1 (II).** For an invertible self-adjoint or skew-symmetric operator \( S \), we have \( \forall X \in L(H) : \| SXS^{-1} + S^{-1}XS \| \geq 2 \| X \| \).

**Theorem 2.2.** For any pair \( (P, Q) \) of commuting invertible positive operators and for any \( X \in L(H) \) such that \( \| X \| = r(A) \), we have

\[
\| PXP^{-1} + Q^{-1}XQ \| \geq 2 \| X \|.
\]

**Proof.** Let \( X \in L(H) \) such that \( \| X \| = r(A) \) and put \( Y = P^{1/2}Q^{-1/2}XQ^{1/2}P^{-1/2} \). Then since \( P^{1/2}Q^{1/2} = Q^{1/2}P^{1/2} \) is self-adjoint, we have by Lemma 2.1

\[
\begin{align*}
\| PXP^{-1} + Q^{-1}XQ \| & = \| (P^{1/2}Q^{1/2})Y(P^{1/2}Q^{1/2})^{-1} + (P^{1/2}Q^{1/2})^{-1}Y(P^{1/2}Q^{1/2}) \| \\
& \geq 2 \| Y \| \\
& \geq 2r(X) \\
& \geq 2 \| X \|.
\end{align*}
\]

\( \square \)

**Theorem 2.3.** For any pair \( (P, Q) \) satisfying the condition of Theorem 2.2 and for any operator \( X \), we have

\[
\max \{ \| PXP^{-1} + Q^{-1}XQ \|, \| PX^*P^{-1} + Q^{-1}X^*Q \| \} \geq 2 \| X \|.
\]

**Proof.** For \( X \in L(H) \), let

\[
A = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}, \quad B = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.
\]

The pair \( (A, B) \) satisfies the condition of Theorem 2.2 and \( \| Y \| = r(Y) \) (since \( Y \) is self-adjoint). Then we have

\[
\| AXA^{-1} + B^{-1}XB \| = \| PXP^{-1} + Q^{-1}XQ \| \geq 2 \| Y \| = 2 \| X \|,
\]

i.e.

\[
\max \{ \| PXP^{-1} + Q^{-1}XQ \|, \| PX^*P^{-1} + Q^{-1}X^*Q \| \} \geq 2 \| X \|.
\]

\( \square \)
Theorem 2.4. For any invertible positive operator $P$, and for $n = 0, 1, 2$, we have
\begin{equation}
\forall X \in L(H) : \| nX + PXP^{-1} + P^{-1}XP \| \geq (n + 2) \| X \| .
\end{equation}

Proof. If $n = 0$, (1) follows from Lemma 2.1.

For all $X$, we have
\begin{align*}
\| 2X + PXP^{-1} + P^{-1}XP \| &= \| P^{\frac{1}{2}} \left( P^{\frac{1}{2}} X P^{\frac{1}{2}} + P^{-\frac{1}{2}} X P^{\frac{1}{2}} \right) P^{\frac{1}{2}} \\
&\quad + P^{-\frac{1}{2}} \left( P^{\frac{1}{2}} X P^{\frac{1}{2}} + P^{-\frac{1}{2}} X P^{\frac{1}{2}} \right) P^{\frac{1}{2}} \| \\
&\geq 2 \| P^{\frac{1}{2}} X P^{\frac{1}{2}} + P^{-\frac{1}{2}} X P^{\frac{1}{2}} \| \\
&\geq 4 \| X \| ,
\end{align*}
that is, (1) is true for $n = 2$.

It follows from the case $n = 2$, that for all $X$, we have
\begin{align*}
\| X + PXP^{-1} + P^{-1}XP \| &\geq \| 2X + PXP^{-1} + P^{-1}XP \| - \| X \| \\
&\geq 3 \| X \| .
\end{align*}

Remark 2.1. In the cases $n = 1$ and $n = 2$, the relation (1) is false in general if we replace the condition “positive” by the condition “self-adjoint”; this may be seen by the following example:

\begin{equation}
P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .
\end{equation}

Then
\begin{align*}
\| X + PXP^{-1} + P^{-1}XP \| &= 1 < 3 = 3 \| X \| , \\
\| 2X + PXP^{-1} + P^{-1}XP \| &= 0 < 4 = 4 \| X \| .
\end{align*}

3. Operator-norm inequality and positive operators

Definition 3.1. An operator $A$ in $L(H)$ is called convexoid if $W_0(A) = \cos \alpha$.

Lemma 3.1 (\cite{2}). Let $A \in L(H)$. If $\| A - \alpha \| = r(A - \alpha)$, for all complex $\alpha$, then $A$ is convexoid.

Lemma 3.2. Let $P$ and $Q$ be in $L(H)$ such that $P > 0$ and $Q > 0$. If we have
\begin{equation}
\forall X \in L(H) : \| X \| + \| PXP^{-1} \| \geq 2 \| QXQ^{-1} \| ,
\end{equation}
then $\{ P \} \subset \{ Q \}$.

Proof. (i) Let $X$ be self-adjoint such that $PX = XP$, and let $\alpha$ be a complex number. Then, by (2), $\| X - \alpha \| \geq \| Q(X - \alpha)Q^{-1} \|$, and since $X - \alpha$ is normal, we also have $\| Q(X - \alpha)Q^{-1} \| \geq \| X - \alpha \|$, so that $\| Q(X - \alpha)Q^{-1} \| = \| X - \alpha \|$.

Then, by Lemma 3.1, we have $W_0(QXQ^{-1}) = \cos \alpha$, and since $X$ is self-adjoint, we obtain $QXQ^{-1} = Q^{-1}QX$, and also $QX = XQ$.

(ii) Now let $X = X_1 + iX_2$, where $X_1 = \text{Re}X$ and $X_2 = \text{Im}X$, such that $PX = XP$. Then, we have $PX_1 = X_1P$ and $PX_2 = X_2P$; from (i) it follows that $QX_1 = X_1Q$ and $QX_2 = X_2Q$, and also $QX = XQ$; we conclude that $\{ P \} \subset \{ Q \} . \quad \square
Theorem 3.3. Let $P$ and $Q$ be in $L(H)$ such that $P > 0$ and $Q > 0$. If we have

\begin{equation}
\forall X \in L(H) : \|PX^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|,
\end{equation}

then $\{P\}' = \{Q\}'$.

Proof. From (3), we have

\begin{equation}
\forall X \in L(H) : \|X\| + \|PQXQ^{-1}P^{-1}\| \geq 2\|QXQ^{-1}\|.
\end{equation}

Let $UM$ be the polar decomposition of $PQ$ (U is unitary and $M = (QP^2Q)^{1/2}$). Then, from (4), we obtain

\begin{equation}
\forall X \in L(H) : \|X\| + \|MXM^{-1}\| \geq 2\|QXQ^{-1}\|.
\end{equation}

and, by Lemma 3.2, we have $MQ = QM$; then $PQ = QP$.

Now let $X$ be self-adjoint such that $PX = XP$ and let $\alpha$ be a complex number. Therefore, $QXQ^{-1} \in \{P\}'$ and, from (3), we obtain

\begin{equation}
\forall X \in L(H) : \|Q(X - \alpha)Q^{-1}\| \leq \|X\|.
\end{equation}

It follows that $QX = XQ$, so that $\{P\}' \subset \{Q\}'$.

The symmetric roles of $P, Q$ in (3) also give $\{Q\}' \subset \{P\}'$, and finally we have $\{P\}' = \{Q\}'$. \qed

Corollary 3.4. Let $P$ and $Q$ be in $L(H)$ such that $P > 0$ and $Q > 0$. If we have

\begin{equation}
\forall X \in L(H) : \|PX^{-1} + Q^{-1}XQ\| \geq 2\|X\|,
\end{equation}

then $\{P\}' = \{Q\}'$.

Proof. Since we have $\|PX^{-1}\| + \|Q^{-1}XQ\| \geq \|PX^{-1} + Q^{-1}XQ\|$, for all operators $X$, the result follows immediately by Theorem 3.3. \qed

Lemma 3.5. Let $\varepsilon > 0$ and let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ (for $n \in \mathbb{N}^*$) such that $0 < \alpha_1 < \cdots < \alpha_n \leq 1$, $\{\alpha_1, \ldots, \alpha_n\} = \{\beta_1, \ldots, \beta_n\}$ and $\frac{\alpha_i}{\alpha_j} + \frac{\beta_i}{\beta_j} \geq 2 - \varepsilon$, for all $i, j$.

Then we have $|\alpha_i - \beta_i| < \varepsilon$, for all $i$.

Proof. From the hypothesis, we obtain $\beta_i - \beta_j < \varepsilon$, if $i < j$.

Let $i \in \{1, \ldots, n\}$ such that $\alpha_i \neq \beta_i$ (in the case $\alpha_i = \beta_i$, of course we have $|\alpha_i - \beta_i| = 0 < \varepsilon$).

There are three cases $i = 1$, $i = n$ and $1 < i < n$.

Case 1. $i = 1$. There exists $j \geq 2$, such that $\beta_j = \alpha_1$, so we have $|\beta_1 - \alpha_1| = \beta_1 - \beta_j < \varepsilon$, since $j > 1$.

Case 2. $i = n$. There exists $j < n$, such that $\beta_j = \alpha_n$, so we have $|\beta_n - \alpha_n| = \beta_j - \beta_n < \varepsilon$, since $n > j$.

Case 3. $1 < i < n$. If $\beta_i > \alpha_i$, then there exists $j > i$, such that $\beta_j \leq \alpha_i$, and we have $|\beta_i - \alpha_i| = \beta_i - \beta_j < \varepsilon$, since $j > i$. If $\beta_i < \alpha_i$, then there exists $j < i$, such that $|\beta_i - \alpha_i| = \beta_j - \beta_i < \varepsilon$, since $i > j$. \qed

Theorem 3.6. Let $P$ and $Q$ be in $L(H)$ such that $P > 0$, $Q > 0$ and $\sigma(P) = \sigma(Q)$.

Then the following properties are equivalent:

(i) $\forall X \in L(H), \|PX^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|$.

(ii) $P = Q$. 

Proof. We may assume, without loss of the generality, that \( \|P\| = \|Q\| = 1\). (i) implies (ii). Decompose \( P \) and \( Q \) using the spectral measure
\[
P = \int \lambda dE_{\lambda}, \quad Q = \int \lambda dF_{\lambda}
\]
and consider
\[
P_n = \int h_n(\lambda) \, dE_{\lambda} = h_n(P), \quad Q_n = \int h_n(\lambda) \, dF_{\lambda} = h_n(Q)
\]
where \( h_n(\lambda) \) is a function of the form
\[
h_n(\lambda) = \frac{k}{n}, \quad \text{for} \quad \frac{k}{n} \leq \lambda < \frac{k + 1}{n}, \quad \text{and} \quad k = 0, 1, 2, \ldots.
\]

Then by the spectral theorem and by the form of \( h_n(\lambda) \), we have \( \sigma(P_n) = \sigma(Q_n) = h_n(\sigma(P)) \) is finite, \( P_n \to P, \quad Q_n \to Q \) (uniformly) and \( P_n \in \{P\}'' \), \( Q_n \in \{Q\}'' \) (where \( \{P\}'' = \{Q\}'') \), by Theorem 3.3.

Put \( \sigma(P_n) = \{\alpha_1, \ldots, \alpha_p\} \) such that \( 0 < \alpha_1 < \cdots < \alpha_p \leq 1 \). Then there exist \( p \) orthogonal projections \( E_1, \ldots, E_p \) such that \( E_i E_j = E_j E_i = 0 \) if \( i \neq j \), \( E_1 \oplus \cdots \oplus E_p = I \) and \( P_n = \sum_{i=1}^{p} \alpha_i E_i \).

Since \( \sigma(P_n) = \sigma(Q_n) \), \( P_n Q_n = Q_n P_n \) and \( Q_n \) is normal, there exist \( p \) scalar \( \beta_1, \ldots, \beta_p \) such that \( Q_n = \sum_{i=1}^{p} \beta_i E_i \) and \( \{\alpha_1, \ldots, \alpha_p\} = \{\beta_1, \ldots, \beta_p\} \).

Let \( \varepsilon > 0 \). Then there exists an integer \( N \) such that
\[
\forall n > N, \forall X \in L(H), \quad \|PXP^{-1}\| + \|Q^{-1}XQ\| \geq (2 - \varepsilon) \|X\|.
\]

Let \( n > N \) and \( X_{ij} = E_i X E_j \), for \( X \in L(H) \). Then, by using (\( \ast \)), we have
\[
\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \varepsilon.
\]

By Lemma 3.5, this implies \( |\alpha_i - \beta_i| < \varepsilon \), for all \( i \); therefore
\[
\|P_n - Q_n\| = \max_{1 \leq i \leq p} |\alpha_i - \beta_i| < \varepsilon,
\]
so we obtain \( P = Q \).
(ii) implies (i) is immediate from Lemma 2.1.

Corollary 3.7. Let the pair \((P, Q)\) of operators satisfy the condition of Theorem 3.6. Then the following properties are equivalent:
(i) \( \forall X \in L(H), \quad \|PXP^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\| \).
(ii) \( P = Q \).

4. Characterization of the Corach-Porta-Recht Inequality

Notation. For \( \theta \in [0, \pi] \), we denote by \( D_\theta \) the straight line through the origin in the complex plane.

Lemma 4.1. Let \( \lambda, \mu \in \mathbb{C}^* \) such that \( \frac{1}{n} + \frac{1}{\lambda} \in \mathbb{R} \) and \( \left| \frac{1}{n} + \frac{1}{\lambda} \right| \geq 2 \). Then there exists a scalar \( \theta \in [0, \pi] \) such that \( \lambda, \mu \in D_\theta \).
Lemma 4.3. Let \( S \) be an invertible normal operator. Then the following properties are equivalent:

(i) \( \forall X \in L(H), \|SS^{-1} + S^{-1}XS\| \geq 2\|X\| \).

(ii) \( \sigma(S) \subseteq D_\theta \), for some \( \theta \in [0, \pi] \).

(iii) \( S = \lambda M \), for some nonzero scalar \( \lambda \) and for some invertible self-adjoint operator \( M \).

Proof. (i) implies (ii).

From (i) and Lemma 4.2, \( S \) is normal. Then, by the spectral measure of \( S \), there exists a sequence \( (S_n) \) of invertible normal operators with finite spectrum such that

(a) \( S_n \rightarrow S \) uniformly,

(b) for all \( \lambda \) in \( \sigma(S) \), there exists a sequence \( (\lambda_n) \) such that \( \lambda_n \in \sigma(S_n) \), for all \( n \) and \( \lambda_n \rightarrow \lambda \).

Let \( \lambda, \mu \in \sigma(S) \) and let \( \varepsilon > 0 \). Then by (i), (a) and (b), there exists an integer \( N \) such that

\[
\forall n > N, \forall X \in L(H), \|S_nXS_n^{-1} + S_n^{-1}XS_n\| \geq (2-\varepsilon)\|X\|
\]

and there exist two sequences \( (\lambda_n) \) and \( (\mu_n) \) such that

\[
\forall n, \lambda_n, \mu_n \in \sigma(S_n); \quad \lambda_n \rightarrow \lambda, \mu_n \rightarrow \mu.
\]

Let \( n > N \) and since \( S_n \) is normal with finite spectrum, there exist \( p \) orthogonal projections \( E_1, \ldots, E_p \) such that \( E_kE_j = E_jE_k = 0, \) if \( k \neq j \), \( E_1 \oplus \ldots \oplus E_p = I \) and \( S_n = \sum_{k=1}^{p} \alpha_k E_k \), where \( \sigma(S_n) = \{\alpha_1, \ldots, \alpha_p\} \), \( \alpha_1 = \lambda_n, \alpha_2 = \mu_n \).

Then by (6) and if we put \( A = \begin{bmatrix} 2 & \gamma_n \\ \gamma_n & 2 \end{bmatrix} \), where \( \gamma_n = \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n} \), we obtain

\[
\forall X \in L(C^2), \|A \circ X\| \geq (2-\varepsilon)\|X\|
\]
and if we put $\delta_n = \frac{1}{\gamma_n}$ and $B = \begin{bmatrix} 1 & \delta_n \\ \delta_n & \frac{1}{2} \end{bmatrix}$, then from (7), we also have

\begin{equation} \forall X \in L(\mathbb{C}^2), \quad \|B \circ X\| \leq \frac{\|X\|}{(2 - \varepsilon)}. \end{equation}

From (7), we deduce $\left| \frac{\Delta_n}{\mu_n} + \frac{\rho_n}{\Delta_n} \right| \geq (2 - \varepsilon)$, so we obtain $\left| \frac{\Delta_n}{\mu_n} + \frac{\rho_n}{\Delta_n} \right| \geq 2$.

On the other hand, if in (8) we put $X = \begin{bmatrix} 1 & ia \\ ia & 1 \end{bmatrix}$, where $a > 0$, we obtain $\alpha + a^2 |\gamma_n|^2 + a |\beta_n| \leq \frac{1 + a^2}{(2 - \varepsilon)}$, where $\beta_n = Im \gamma_n$; so that $\frac{\alpha}{\mu} + a^2 |\alpha|^2 + a |\beta| \leq \frac{1 + a^2}{(2 - \varepsilon)}$. Therefore $a |\alpha|^2 + |\beta| \leq \frac{a}{4}$; then $\beta = 0$ and $\frac{\alpha}{\mu} + \frac{\rho_n}{\Delta_n} \in \mathbb{R}$. This implies condition (ii) by Lemma 4.1.

(ii) implies (iii).

If we put $M = e^{-i\theta}S$, then $M$ is an invertible normal operator with real spectrum, so we have $S = e^{i\theta}M$, where $M$ is an invertible self-adjoint operator.

(iii) implies (i) is immediate by Lemma 2.1.

\begin{proof}
\end{proof}

\begin{theorem} The set of all invertible operators $S$, for which
\begin{equation}
\forall X \in L(H), \quad \|SXS^{-1} + S^{-1}XS\| \geq 2 \|X\|
\end{equation}
is the set $\{\lambda M : \lambda \in \mathbb{C}^*, \ M \text{ an invertible self-adjoint operator}\}$.
\end{theorem}

\begin{proof}
This follows immediately by Lemma 4.2 and Lemma 4.3.
\end{proof}

\begin{remark} The extremal class of invertible operators $S$ satisfying the condition
\begin{equation}
\inf_{\|X\|=1} \|SXS^{-1} + S^{-1}XS\| = 2
\end{equation}
has been characterized. So it remains the characterization of the second extremal class of all invertible operators $S$ satisfies the condition
\begin{equation}
\inf_{\|X\|=1} \|SXS^{-1} + S^{-1}XS\| = 0.
\end{equation}
\end{remark}

\textbf{References}


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