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# CONDITIONAL WEAK COMPACTNESS IN VECTOR-VALUED FUNCTION SPACES

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ABSTRACT. Let $E$ be an ideal of $L^0$ over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ and let $E'$ be the Köthe dual of $E$ with $\operatorname{supp} E' = \Omega$. Let $(X, \|\cdot\|_X)$ be a real Banach space, and $X^*$ the topological dual of $X$. Let $E(X)$ be a subspace of the space $L^0(X)$ of equivalence classes of strongly measurable functions $f\colon \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\|f(\cdot)\|_X$ belongs to $E$. For a subset $H$ of $E(X)$ for which the set $\{\|f(\cdot)\|_X\colon f \in H\}$ is $\sigma(E, E')$-bounded the following statement is equivalent to conditional $\sigma(E(X), E'(X^*))$-compactness: the set $\{\|f(\cdot)\|_X\colon f \in H\}$ is conditionally $\sigma(E, E')$-compact and $\{\int_A f(\omega)d\mu\colon f \in H\}$ is a conditionally weakly compact subset of $X$ for each $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$. Applications to Orlicz-Bochner spaces are given.

## 1. Introduction and preliminaries

Given a dual pair $\langle L, K\rangle$, a subset $A$ of $L$ is said to be *conditionally $\sigma(L, K)$-compact* whenever each sequence in $A$ contains a $\sigma(L, K)$-Cauchy subsequence (cf. [MN, p. 100]). The problem of characterizing relatively sequentially $\sigma(L^p(X), L^q(X^*))$-compact subsets of Lebesgue-Bochner spaces $L^p(X)$ (where $1 \leqslant p < \infty$ and $q$ conjugate to $p$) over a finite measure space was considered by F. Bombal [$B_1$] and J. Batt and W. Hiermeyer [BH, Theorem 2.1]. Moreover, F. Bombal characterized relatively sequentially $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$-compact subsets of Orlicz-Bochner spaces $L^\varphi(X)$ [$B_2$, Theorem 3]. C. Abott, E. Bator, R. Bilyeu and P. Lewis [ABBL] obtained the following characterization of conditionally $\sigma(L^1(X), L^\infty(X^*))$-compact subsets of $L^1(X)$.

**Theorem 1.1** (cf. [ABBL, Theorem 2.5]). *Let $(\Omega, \Sigma, \mu)$ be a finite measure space. Then for a norm bounded subset $H$ of $L^1(X)$ the following statements are equivalent:*

(i) *$H$ is conditionally $\sigma(L^1(X), L^\infty(X^*))$-compact.*

(ii) a) *The subset $\{\|f(\cdot)\|_X\colon f \in H\}$ of $L^1$ is uniformly integrable.*

b) *The set $\{\int_A f(\omega)d\mu\colon f \in H\}$ is conditionally weakly compact in $X$ for each $A \in \Sigma$.*

In this paper, by making use of Theorem 1.1 we characterize conditionally $\sigma(E(X), E'(X^*))$-compact subsets of $E(X)$, where $E$ is an ideal of $L^0$ over a $\sigma$-finite measure space.

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Now we establish notation and terminology (see [AB], [KA]).

Let $(\Omega, \Sigma, \mu)$ be a complete $\sigma$-finite measure space and let $L^0$ denote the space of equivalence classes of all $\Sigma$-measurable functions defined and finite a.e. on $\Omega$. Let $\chi_A$ stand for the characteristic function of a set $A$ and let $\mathbb{N}$ denote the set of all natural numbers. Let $E$ be an ideal of $L^0$ with supp $E = \Omega$, and let $E'$ stand for the Köthe dual of $E$, i.e.,

$$E' = \{v \in L^0 \colon \int_\Omega |u(\omega)v(\omega)|d\mu < \infty \quad \text{for all} \quad u \in E\}.$$

We assume that supp $E' = \Omega$.

Let $(X, \|\cdot\|_X)$ be a real Banach space, and let $S_X$ and $B_X$ denote the unit sphere and the closed unit ball in $X$, resp. Let $X^*$ stand for the Banach dual of $X$. By $L^0(X)$ we denote the set of equivalence classes of all strongly $\Sigma$-measurable functions $f \colon \Omega \to X$. For $f \in L^0(X)$ let us set $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$E(X) = \{f \in L^0(X) \colon \widetilde{f} \in E\}.$$

By $\sigma(E(X), E'(X^*))$ we will denote the weak topology on $E(X)$ with respect to the dual system $\langle E(X), E'(X^*)\rangle$ under the natural duality $\langle f, g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle d\mu$ for $f \in E(X),\ g \in E'(X^*)$.

The following characterization of conditional $\sigma(E, E')$-compactness is needed.

**Proposition 1.2** ([$\mathrm{N}_2$, Theorem 1.1])**.** *For a $\sigma(E, E')$-bounded subset $A$ of $E$ the following statements are equivalent:*

(i) *$A$ is conditionally $\sigma(E, E')$-compact.*

(ii) *For each $v \in E'$ the subset $\{uv \colon u \in A\}$ of $L^1$ is uniformly integrable.*

(iii) *The functional $p_A$ on $E'$ defined by $p_A(v) = \sup_{u \in A} \int_\Omega |u(\omega)v(\omega)|d\mu$ is an order continuous Riesz seminorm.*

## 2. Conditionally $\sigma(E(X), E'(X^*))$-compact sets in $E(X)$

Let $\mathrm{ca}(\Omega, \Sigma)$ stand for the Riesz space of countably additive set functions $\nu$ on $\Sigma$. For a sequence $(A_n)$ in $\Sigma$ we write $A_n \searrow_\mu \emptyset$ whenever $A_n \downarrow$ and $\mu\Big(\bigcap_{n=1}^\infty A_n\Big) = 0$ (that is, $A_n \downarrow$ and $\mu(A_n \cap A) \to 0$ for each $A \in \Sigma$ with $\mu(A) < \infty$).

The following well-known result characterizes uniformly $\mu$-continuous sets in $\mathrm{ca}(\Omega, \Sigma)$.

**Lemma 2.1.** *For a subset $\mathcal{K}$ of $ca(\Omega, \Sigma)^+$ the following statements are equivalent:*

(i) *$\mathcal{K}$ is uniformly $\mu$-continuous (i.e., $\lim_n(\sup_{\nu \in \mathcal{K}} \nu(A_n)) = 0$ as $A_n \searrow_\mu \emptyset$).*

(ii) *For each $\eta > 0$ there exist $\delta > 0$ and $A_0 \in \Sigma$ with $\mu(A_0) < \infty$ such that $\nu(A) \leqslant \eta$ and $\nu(\Omega \setminus A_0) \leqslant \eta$ for all $A \in \Sigma$ with $\mu(A) \leqslant \delta$ and all $\nu \in \mathcal{K}$.*

We shall need the following technical result.

**Proposition 2.2.** *Let $\mathcal{K}$ be a subset of $ca(\Omega, \Sigma)^+$ such that each $\nu \in \mathcal{K}$ is $\mu$-continuous. Assume that $\mathcal{K}$ is not uniformly $\mu$-continuous. Then there exist a pairwise disjoint sequence $(B_n)$ in $\Sigma$, a number $\varepsilon_0 > 0$ and a sequence $(\nu_n)$ in $\mathcal{K}$ such that $\nu_n(B_n) > \varepsilon_0$ for all $n \in \mathbb{N}$.*

*Proof.* In view of Lemma 2.1 there exists $\varepsilon_0 > 0$ such that either there exist a sequence $(A_n)$ in $\Sigma$ and a sequence $(\nu_n^1)$ in $\mathcal{K}$ such that

$$\mu(A_n) \to 0 \quad \text{and} \quad \nu_n^1(A_n) > 2\varepsilon_0 \tag{1}$$

or there exists a sequence $(\nu_n^2)$ in $\mathcal{K}$ such that

$$\nu_n^2(\Omega \setminus \Omega_n) > 2\varepsilon_0 \tag{2}$$

whenever $\Omega_n \uparrow \Omega$ and $\mu(\Omega_n) < \infty$ for $n \in \mathbb{N}$.

Assume that condition (1) holds. Then arguing as in [BL, p. 546] one can find a pairwise disjoint sequence $(B_n)$ in $\Sigma$ and a subsequence $(\nu_{k_n}^1)$ of $(\nu_n^1)$ such that $\nu_{k_n}^1(B_n) \geqslant \varepsilon_0$. Let $\nu_n = \nu_{k_n}^1$ for $n \in \mathbb{N}$.

Now assume that condition (2) holds. Let $C_n = \Omega \setminus \Omega_n$ for $n \in \mathbb{N}$. Then $C_n \searrow_\mu \emptyset$, so $\nu(C_n) \to 0$ for each $\nu \in \mathcal{K}$. Let $l_1 = 1$ and choose $l_2 \in \mathbb{N}$ such that $l_2 > l_1$, $\nu_{l_1}^2(C_{l_2}) < \varepsilon_0$. Then choose $l_3 \in \mathbb{N}$ such that $l_3 > l_2$ and $\nu_{l_2}(C_{l_3}) < \varepsilon_0$. Continuing this process inductively we can find an increasing sequence $(l_n)$ in $\mathbb{N}$ such that $\nu_{l_n}^2(C_{l_{n+1}}) < \varepsilon_0$. Let $B_n = \Omega_{l_{n+1}} \setminus \Omega_{l_n}$ for $n \in \mathbb{N}$. Then $(B_n)$ is a disjoint sequence and since $B_n = C_{l_n} \setminus C_{l_{n+1}}$ for $n \in \mathbb{N}$, by making use of (2) we obtain that $\nu_{l_n}^2(B_n) = \nu_{l_n}^2(C_{l_n}) - \nu_{l_n}^2(C_{l_{n+1}}) > 2\varepsilon_0 - \varepsilon_0 = \varepsilon_0$. Put $\nu_n = \nu_{l_n}^2$ for $n \in \mathbb{N}$. $\square$

For a subset $H$ of $E(X)$ let $\widetilde{H} = \{\widetilde{f}\colon f \in H\}$.

Now we are ready to state our main result.

**Theorem 2.3.** *Let $H$ be a subset of $E(X)$ such that the subset $\widetilde{H}$ of $E$ is $\sigma(E, E')$-bounded. Then the following statements are equivalent:*

(i) *$H$ is conditionally $\sigma(E(X), E'(X^*))$-compact.*

(ii) a) *$\widetilde{H}$ is conditionally $\sigma(E, E')$-compact.*

b) *$\left\{\int_A f(\omega)d\mu\colon f \in H\right\}$ is a conditionally weakly compact subset of $X$ for each $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$.*

*Proof.* (i) $\Rightarrow$ (ii) To prove that (a) holds, in view of Proposition 1.2 it is enough to show that for each $0 \leqslant v \in E'$ the subset $\{\widetilde{f}v\colon f \in H\}$ of $L^1$ is uniformly integrable. Assume on the contrary that there exists $0 \leqslant v_0 \in E'$ such that the set $\{\widetilde{f}v_0\colon f \in H\}$ is not uniformly integrable. For each $f \in H$ set $\nu_f(A) = \int_A \widetilde{f}(\omega)v_0(\omega)d\mu$ for $A \in \Sigma$. Then $\nu_f$ is a non-negative $\mu$-continuous countably additive set function on $\Sigma$ but the family $\{\nu_f\colon f \in H\}$ is not uniformly $\mu$-continuous. Hence in view of Proposition 2.2 there exist a pairwise disjoint sequence $(B_n)$ in $\Sigma$, a sequence $(f_n)$ in $H$, and a number $\varepsilon_0 > 0$ such that $\nu_{f_n}(B_n) = \int_{B_n} \widetilde{f}_n(\omega)v_0(\omega)d\mu > \varepsilon_0$ for each $n \in \mathbb{N}$. Clearly $v_0 f_n \in L^1(X)$, so in view of [Bu, Theorem 1.1.(4)]

$$\begin{aligned}\nu_{f_n}(B_n) &= \|\chi_{B_n} v_0 \widetilde{f}_n\|_{L^1} = \|\chi_{B_n} v_0 f_n\|_{L^1(X)} \\ &= \sup\Big\{\Big|\int_{B_n} \langle v_0(\omega)f_n(\omega), g(\omega)\rangle d\mu\Big|\colon g \in L^\infty(X^*),\ \|g\|_{L^\infty(X^*)} \leqslant 1\Big\}.\end{aligned}$$

Hence one can produce a sequence $(g_n)$ in $L^\infty(X^*)$ with $\|g_n\|_{L^\infty(X^*)} \leqslant 1, \chi_{\Omega\setminus B_n} g_n = 0$ and such that

$$\Big|\int_{B_n} \langle v_0(\omega)f_n(\omega), g_n(\omega)\rangle d\mu\Big| > \varepsilon_0. \tag{1}$$

Set $g_0 = \sum_{n=1}^{\infty} g_n$. Then $g_0 \in L^0(X^*)$ and $\|g_0\|_{L^\infty(X^*)} \leqslant 1$. Clearly $v_0 g_0 \in E'(X^*)$, so $\chi_A v_0 g_0 \in E'(X^*)$ for each $A \in \Sigma$. In view of the assumption (i) there exists a $\sigma(E(X), E'(X^*))$-Cauchy subsequence $(f_{k_n})$ of $(f_n)$ so for each $A \in \Sigma$, $\lim_n \int_A \langle f_{k_n}(\omega), v_0(\omega) g_0(\omega)\rangle d\mu$ exists. Setting $\mu_n(A) = \int_A \langle f_{k_n}(\omega), v_0(\omega) g_0(\omega)\rangle d\mu$ for $A \in \Sigma$, in view of Nikodym's convergence theorem (see [D, Chap. 7]), $\{\mu_n \colon n \in \mathbb{N}\}$ is uniformly countably additive on $\Sigma$. Hence there exists $m_0 \in \mathbb{N}$ such that for $m \geqslant m_0$, $\sup_n |\mu_n(B_m)| \leqslant \varepsilon_0$ (see [D, Chap. 7, Theorem 10]). Hence for each $m \geqslant m_0$ we get

$$\begin{aligned} |\mu_m(B_{k_m})| &= \Big| \int_{B_{k_m}} \langle f_{k_m}(\omega), v_0(\omega) g_{k_m}(\omega)\rangle d\mu \Big| \\ &= \Big| \int_{B_{k_m}} \langle v_0(\omega) f_{k_m}(\omega), g_{k_m}(\omega)\rangle d\mu \Big| \leqslant \varepsilon_0 \end{aligned}$$

which contradicts (1). This contradiction establishes that (a) holds.

To show that (b) holds, take $A \in \Sigma$ with $\chi_A \in E'$, and let $(f_n)$ be a sequence in $H$. Set $g = \chi_A x^*$ where $x^* \in S_{X^*}$. Then $g \in E'(X^*)$ and by assumption (i) there exists a subsequence $(f_{k_n})$ of $(f_n)$ such that $\lim_n \int_\Omega \langle f_{k_n}(\omega), g(\omega)\rangle d\mu$ exists. Since $\int_\Omega \langle f_{k_n}(\omega), g(\omega)\rangle d\mu = x^*\Big(\int_A f_{k_n}(\omega) d\mu\Big)$, the set $\Big\{\int_A f(\omega) d\mu \colon f \in H\Big\}$ is conditionally weakly compact in $X$.

(ii) $\Rightarrow$ (i) Let $(f_n)$ be a sequence in $H$. Since supp $E' = \Omega$ there exists a sequence $(\Omega_m)$ in $\Sigma$ such that $\Omega_m \uparrow \Omega$ and $\mu(\Omega_m) < \infty$, $\chi_{\Omega_m} \in E'$ for $m \in \mathbb{N}$ (see [Z, Theorem 86.2]). Setting $A_m = \Omega \setminus \Omega_m$ for $m \in \mathbb{N}$ we see that $A_m \searrow_\mu \emptyset$. Given $m \in \mathbb{N}$ we have $\sup_n \int_{\Omega_m} \widetilde{f}_n(\omega) d\mu = c_m < \infty$, because $\chi_{\Omega_m} \in E'$ and $\widetilde{H}$ is $\sigma(E, E')$-bounded. Hence $\{\chi_{\Omega_m} f_n \colon n \in \mathbb{N}\} \subset L^1_{\Omega_m}(X)$, and by assumption (a), $\{\chi_{\Omega_m} \widetilde{f}_n \colon n \in \mathbb{N}\}$ is a uniformly integrable subset of $L^1_{\Omega_m}$. Combining this observation with (b), in view of Theorem 1.1 we see that $\{\chi_{\Omega_m} f_n \colon n \in \mathbb{N}\}$ is a conditionally $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*))$-compact subset of $L^1_{\Omega_m}(X)$.

In view of the above observation there exists a $\sigma(L^1_{\Omega_1}(X), L^\infty_{\Omega_1}(X^*))$-Cauchy subsequence $(\chi_{\Omega_1} f_{k^1_n})$ of $(\chi_{\Omega_1} f_n)$. Next, there exists a $\sigma(L^1_{\Omega_2}(X), L^\infty_{\Omega_2}(X^*))$-Cauchy subsequence $(\chi_{\Omega_2} f_{k^2_n})$ of $(\chi_{\Omega_1} f_{k^1_n})$. It follows that the diagonal sequence $(f_{k^n_n})$ has the property that for each $m \in \mathbb{N}$ $(\chi_{\Omega_m} f_{k^n_n})$ is a $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*))$-Cauchy sequence. Put $h_n = f_{k^n_n}$ for $n \in \mathbb{N}$.

Let $g \in E'(X^*)$. For $n \in \mathbb{N}$ let us put

$$g_n(\omega) = \begin{cases} g(\omega) & \text{if } \omega \in \Omega_n \text{ and } \|g(\omega)\|_{X^*} \leqslant n, \\ 0 & \text{elsewhere.} \end{cases}$$

Given $\varepsilon > 0$ there exist $m_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\sup_n \int_{\Omega \setminus \Omega_{m_0}} \widetilde{f}_n(\omega)\, \widetilde{g}\,(\omega) d\mu \leqslant \frac{\varepsilon}{4} \quad \text{and} \quad \sup_n \int_A \widetilde{f}_n(\omega)\, \widetilde{g}\,(\omega) d\mu \leqslant \frac{\varepsilon}{4} \tag{2}$$

for each $A \in \Sigma$ with $\mu(A) \leqslant \delta$. For $\eta = \frac{\varepsilon}{4c_{m_0}}$ let

$$B_n = \{\omega \in \Omega_{m_0} \colon \|g(\omega) - g_n(\omega)\|_{X^*} \geqslant \eta\}.$$

It is easy to observe that $B_n \downarrow \emptyset$, so $\mu(B_n) \to 0$. Choose $n_0 \in \mathbb{N}$ with $n_0 \geqslant m_0$ such that $\mu(B_{n_0}) \leqslant \delta$. Then by (2) we get

$$\sup_n \int_{B_{n_0}} \widetilde{h}_n(\omega)\, \widetilde{g}(\omega) d\mu \leqslant \frac{\varepsilon}{4}. \tag{3}$$

Hence, by (3) we have

$$\begin{aligned} &\Big| \int_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{n_0}(\omega)\rangle d\mu \Big| \leqslant \int_{\Omega_{m_0}} \widetilde{h}_n(\omega) \|g(\omega) - g_{n_0}(\omega)\|_{X^*} d\mu \\ &\leqslant \int_{B_{n_0}} \widetilde{h}_n(\omega) \|g(\omega) - g_{n_0}(\omega)\|_{X^*} d\mu + \int_{\Omega_{m_0} \setminus B_{n_0}} \widetilde{h}_n(\omega) \|g(\omega) - g_{n_0}(\omega)\|_{X^*} d\mu \\ &\leqslant \int_{B_{n_0}} \widetilde{h}_n(\omega)\, \widetilde{g}(\omega) d\mu + \eta \int_{\Omega_{m_0}} \widetilde{h}_n(\omega) d\mu \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4c_{m_0}} \cdot c_{m_0} = \frac{\varepsilon}{2}. \end{aligned} \tag{4}$$

Since $\int_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega)\rangle d\mu \underset{n}{\longrightarrow} a$ for some $a \in \mathbb{R}$, we can choose $n_1 \in \mathbb{N}$ such that for $n \geqslant n_1$

$$\Big| \int_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega)\rangle d\mu - a \Big| \leqslant \frac{\varepsilon}{4}. \tag{5}$$

Thus by (2), (4) and (5) for $n \geqslant n_1$ we get

$$\begin{aligned} &\Big| \int_{\Omega} \langle h_n(\omega), g(\omega)\rangle d\mu - a \Big| \\ &\leqslant \Big| \int_{\Omega \setminus \Omega_{m_0}} \langle h_n(\omega), g(\omega)\rangle d\mu \Big| + \Big| \int_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{n_0}(\omega)\rangle d\mu \Big| \\ &\quad + \Big| \int_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega)\rangle d\mu - a \Big| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

This shows that $(h_n)$ is a $\sigma(E(X), E'(X^*))$-Cauchy subsequence of $(f_n)$, so $H$ is conditionally $\sigma(E(X), E'(X^*))$-compact. □

**Corollary 2.4.** *Assume that a Banach space $X$ contains no isomorphic copy of $\ell^1$, and let $H$ be a subset of $E(X)$ such that $\widetilde{H}$ is $\sigma(E, E')$-bounded. Then the following statements are equivalent:*

(i) *$H$ is conditionally $\sigma(E(X), E'(X^*))$-compact.*

(ii) *$\widetilde{H}$ is conditionally $\sigma(E, E')$-compact.*

*Proof.* (i) ⇒ (ii) Obvious.

(ii) ⇒ (i) In view of Theorem 2.3 it is enough to show that $\left\{ \int_A f(\omega) d\mu \colon f \in H \right\}$ is a conditionally weakly compact subset of $X$ for each $A \in \Sigma$ such that $\chi_A \in E'$. In fact, let $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$. Hence $\sup_{f \in H} \| \int_A f(\omega) d\mu \| \leqslant$

$\sup_{f\in H}\int_A \widetilde{f}(\omega)d\mu < \infty$, so in view of the Rosenthal's $\ell^1$-theorem [R] $\Big\{\int_A f(\omega)d\mu\colon\ f\in H\Big\}$ is a conditionally weakly compact subset of $X$, as desired. $\square$

Assume now that $(E, \|\cdot\|_E)$ is a Banach function space. Then the space $E(X)$ provided with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is usually called a Köthe-Bochner space. The associated norm $\|\cdot\|_{E'}$ on the Köthe dual $E'$ can be defined as follows:

$$\|v\|_{E'} = \sup\Big\{\Big|\int_\Omega u(\omega)v(\omega)d\mu\Big|\colon\ u\in E,\ \|u\|_E\leqslant 1\Big\}.$$

Clearly, for a subset $H$ of $E(X)$ the set $\widetilde{H}$ is $\sigma(E,E')$-bounded whenever

$$\sup_{f\in H}\|f\|_{E(X)} < \infty.$$

Combining Corollary 2.4 and Proposition 1.2 we get:

**Corollary 2.5.** *Let $(E, \|\cdot\|_E)$ be a Banach function space, and assume that $X$ contains no isomorphic copy of $\ell^1$. Then the following statements are equivalent:*

(i) *The associated norm $\|\cdot\|_{E'}$ on $E'$ is order continuous.*

(ii) *Every norm bounded set in $E(X)$ is conditionally $\sigma(E(X), E'(X^*))$-compact.*

We now apply the previous results to Orlicz spaces (see [KR], [L] for more details).

By a Young function we mean a mapping $\varphi\colon [0,\infty)\to[0,\infty)$ that is convex, vanishes only at 0 and $\lim_{t\to 0}\frac{\varphi(t)}{t} = 0$, $\lim_{t\to\infty}\frac{\varphi(t)}{t} = \infty$. Let $L^\varphi$ be the Orlicz space associated with $\varphi$ and provided with the Luxemburg norm $\|u\|_\varphi := \inf\{\lambda > 0\colon \int_\Omega \varphi(|u(\omega)|/\lambda)d\mu \leqslant 1\}$. Then $(L^\varphi)' = L^{\varphi^*}$, where $\varphi^*$ denotes the complementary Young function.

We say that a Young function $\psi$ *increases more rapidly than another* $\varphi$, in symbols $\varphi\prec\psi$, if for each $c>0$ there is $d>1$ such that $c\varphi(t)\leqslant \frac{1}{d}\psi(dt)$ for all $t\geqslant 0$ (see [$N_1$]). Note that $\varphi$ satisfies the $\nabla_2$-condition iff $\varphi\prec\varphi$.

As a consequence of Corollary 2.4 and [$N_1$, Theorem 2.5] we get:

**Corollary 2.6.** *Assume that $X$ contains no isomorphic copy of $\ell^1$. Then for a norm bounded subset $H$ of the Orlicz-Bochner space $L^\varphi(X)$ the following statements are equivalent:*

(i) *$H$ is conditionally $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$-compact.*

(ii) *There is a Young function $\psi$ with $\varphi\prec\psi$ and such that $H\subset L^\psi(X)$ and $\sup_{f\in H}\|f\|_{L^\psi(X)} < \infty$.*

**Corollary 2.7.** *Assume that $X$ contains no isomorphic copy of $\ell^1$ and $\varphi$ satisfies the $\nabla_2$-condition. Then every norm bounded subset of $L^\varphi(X)$ is conditionally $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$-compact.*

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