CONDITIONAL WEAK COMPACTNESS
IN VECTOR-VALUED FUNCTION SPACES

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ABSTRACT. Let $E$ be an ideal of $L^0$ over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ and let $E'$ be the Köthe dual of $E$ with $\text{supp} E' = \Omega$. Let $(X, \| \cdot \|_X)$ be a real Banach space, and $X^*$ the topological dual of $X$. Let $E(X)$ be a subspace of the space $L^0(X)$ of equivalence classes of strongly measurable functions $f : \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\| f(\cdot) \|_X$ belongs to $E$. For a subset $H$ of $E(X)$ for which the set $\{ \| f(\cdot) \|_X : f \in H \}$ is $\sigma(E, E')$-bounded the following statement is equivalent to conditional $\sigma(E(X), E'(X^*))$-compactness: the set $\{ \| f(\cdot) \|_X : f \in H \}$ is conditionally $\sigma(E, E')$-compact and $\{ \int_A f(\omega) d\mu : f \in H \}$ is a conditionally weakly compact subset of $X$ for each $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$. Applications to Orlicz-Bochner spaces are given.

1. Introduction and preliminaries

Given a dual pair $(L, K)$, a subset $A$ of $L$ is said to be conditionally $\sigma(L, K)$-compact whenever each sequence in $A$ contains a $\sigma(L, K)$-Cauchy subsequence (cf. [MN] p. 100). The problem of characterizing relatively sequentially $\sigma(L^p(X), L^q(X^*))$-compact subsets of Lebesgue-Bochner spaces $L^p(X)$ (where $1 \leq p < \infty$ and $q$ conjugate to $p$) over a finite measure space was considered by F. Bombal [B1] and J. Batt and W. Hiermeyer [BH, Theorem 2.1]. Moreover, F. Bombal characterized relatively sequentially $\sigma(L^p(X), L^{q'}(X^*))$-compact subsets of Orlicz-Bochner spaces $L^{q'}(X)$ [B2, Theorem 3]. C. Abott, E. Bator, R. Bilyeu and P. Lewis [ABBL] obtained the following characterization of conditionally $\sigma(L^1(X), L^{\infty}(X^*))$-compact subsets of $L^1(X)$.

**Theorem 1.1** (cf. [ABBL, Theorem 2.5]). Let $(\Omega, \Sigma, \mu)$ be a finite measure space. Then for a norm bounded subset $H$ of $L^1(X)$ the following statements are equivalent:

(i) $H$ is conditionally $\sigma(L^1(X), L^{\infty}(X^*))$-compact.

(ii) a) The subset $\{ \| f(\cdot) \|_X : f \in H \}$ of $L^1$ is uniformly integrable.

b) The set $\{ \int_A f(\omega) d\mu : f \in H \}$ is conditionally weakly compact in $X$ for each

$A \in \Sigma$.

In this paper, by making use of Theorem 1.1 we characterize conditionally $\sigma(E(X), E'(X^*))$-compact subsets of $E(X)$, where $E$ is an ideal of $L^0$ over a $\sigma$-finite measure space.

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Now we establish notation and terminology (see [AB], [KA]).

Let $(\Omega, \Sigma, \mu)$ be a complete $\sigma$-finite measure space and let $L^0$ denote the space of equivalence classes of all $\Sigma$-measurable functions defined and finite a.e. on $\Omega$. Let $\chi_A$ stand for the characteristic function of a set $A$ and let $\mathbb{N}$ denote the set of all natural numbers. Let $E$ be an ideal of $L^0$ with $\text{supp } E = \Omega$, and let $E'$ stand for the Köthe dual of $E$, i.e.,

$$E' = \{ v \in L^0 : \int_{\Omega} |u(\omega)v(\omega)|d\mu < \infty \text{ for all } u \in E \}.$$ 

We assume that $\text{supp } E' = \Omega$.

Let $(X, \| \cdot \|_X)$ be a real Banach space, and let $S_X$ and $B_X$ denote the unit sphere and the closed unit ball in $X$, resp. Let $X^*$ stand for the Banach dual of $X$. By $L^0(X)$ we denote the set of equivalence classes of all strongly $\Sigma$-measurable functions $f : \Omega \to X$. For $f \in L^0(X)$ let us set $\bar{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$E(X) = \{ f \in L^0(X) : \bar{f} \in E \}.$$ 

By $\sigma(E(X), E'(X^*))$ we will denote the weak topology on $E(X)$ with respect to the dual system $\langle E(X), E'(X^*) \rangle$ under the natural duality $\langle f, g \rangle = \int_{\Omega} \bar{f}(\omega)g(\omega)d\mu$ for $f \in E(X), g \in E'(X^*)$.

The following characterization of conditional $\sigma(E, E')$-compactness is needed.

**Proposition 1.2** ([N2] Theorem 1.1). For a $\sigma(E, E')$-bounded subset $A$ of $E$ the following statements are equivalent:

(i) $A$ is conditionally $\sigma(E, E')$-compact.

(ii) For each $v \in E'$ the subset $\{uv : u \in A\}$ of $L^1$ is uniformly integrable.

(iii) The functional $p_A$ on $E'$ defined by $p_A(v) = \sup_{u \in A} \int_{\Omega} |u(\omega)v(\omega)|d\mu$ is an order continuous Riesz seminorm.

2. CONDITIONALLY $\sigma(E(X), E'(X^*))$-COMPACT SETS IN $E(X)$

Let $\text{ca}(\Omega, \Sigma)$ stand for the Riesz space of countably additive set functions $\nu$ on $\Sigma$. For a sequence $(A_n)$ in $\Sigma$ we write $A_n \searrow_{\mu} \emptyset$ whenever $A_n \downarrow$ and $\mu(\bigcap_{n=1}^{\infty} A_n) = 0$ (that is, $A_n \downarrow$ and $\mu(A_n \cap A) \to 0$ for each $A \in \Sigma$ with $\mu(A) < \infty$).

The following well-known result characterizes uniformly $\mu$-continuous sets in $\text{ca}(\Omega, \Sigma)$.

**Lemma 2.1.** For a subset $K$ of $\text{ca}(\Omega, \Sigma)^+$ the following statements are equivalent:

(i) $K$ is uniformly $\mu$-continuous (i.e., $\lim_{n \to \infty} (\sup_{\nu \in K} \nu(A_n)) = 0$ as $A_n \searrow_{\mu} \emptyset$).

(ii) For each $\eta > 0$ there exist $\delta > 0$ and $A_0 \in \Sigma$ with $\mu(A_0) < \infty$ such that $\nu(A) \leq \eta$ and $\nu(\Omega \setminus A_0) \leq \eta$ for all $A \in \Sigma$ with $\mu(A) \leq \delta$ and all $\nu \in K$.

We shall need the following technical result.

**Proposition 2.2.** Let $K$ be a subset of $\text{ca}(\Omega, \Sigma)^+$ such that each $\nu \in K$ is $\mu$-continuous. Assume that $K$ is not uniformly $\mu$-continuous. Then there exist a pairwise disjoint sequence $(B_n)$ in $\Sigma$, a number $\varepsilon_0 > 0$ and a sequence $(\nu_n)$ in $K$ such that $\nu_n(B_n) > \varepsilon_0$ for all $n \in \mathbb{N}$. 
Proof. In view of Lemma 2.1 there exists \( \varepsilon_0 > 0 \) such that either there exist a sequence \((A_n)\) in \( \Sigma \) and a sequence \((\nu_n^1)\) in \( \mathcal{K} \) such that
\[
(1) \quad \mu(A_n) \to 0 \quad \text{and} \quad \nu_n^1(A_n) > 2\varepsilon_0
\]
or there exists a sequence \((\nu_n^2)\) in \( \mathcal{K} \) such that
\[
(2) \quad \nu_n^2(\Omega \setminus \Omega_n) > 2\varepsilon_0
\]
whenever \( \Omega_n \uparrow \Omega \) and \( \mu(\Omega_n) < \infty \) for \( n \in \mathbb{N} \).

Assume that condition (1) holds. Then arguing as in [BL, p. 546] one can find a pairwise disjoint sequence \((B_n)\) in \( \Sigma \) and a subsequence \((\nu_n^1)\) of \((\nu_n^1)\) such that
\[
\nu_n^1(B_n) \geq \varepsilon_0. \quad \text{Let } \nu_n = \nu_n^1 \quad \text{for } n \in \mathbb{N}.
\]
Now assume that condition (2) holds. Let \( \nu_n = \Omega \setminus \Omega_n \) for \( n \in \mathbb{N} \). Then \( \nu_n = \Omega_n \setminus \Omega_n \) for \( n \in \mathbb{N} \). Then \((B_n)\) is a disjoint sequence and since \(B_n = C_n \setminus C_{n+1}\) for \( n \in \mathbb{N} \), making use of (2) we obtain that \(\nu_n^2(B_n) = \nu_n^2(C_n) - \nu_n^2(C_{n+1}) > 2\varepsilon_0 - \varepsilon_0 = \varepsilon_0\). Put \( \nu_n = \nu_n^2 \) for \( n \in \mathbb{N} \).

For a subset \( H \) of \( E(X) \) let \( \tilde{H} = \{ f : f \in H \} \).

Now we are ready to state our main result.

Theorem 2.3. Let \( H \) be a subset of \( E(X) \) such that the subset \( \tilde{H} \) of \( E \) is \( \sigma(E, E') \)-bounded. Then the following statements are equivalent:
(i) \( H \) is conditionally \( \sigma(E, E') \)-compact.
(ii) a) \( \tilde{H} \) is conditionally \( \sigma(E, E') \)-compact.
b) \( \{ \int_A f(\omega) \nu d\mu : f \in H \} \) is a conditionally weakly compact subset of \( X \) for each \( A \in \Sigma, \mu(A) < \infty \) with \( \chi_A \in E' \).

Proof. (i) \( \Rightarrow \) (ii) To prove that (a) holds, in view of Proposition 1.2 it is enough to show that for each \( 0 \leq v \in E' \) the subset \( \{ \tilde{f} v : f \in H \} \) of \( L^1 \) is uniformly integrable. Assume on the contrary that there exists \( 0 \leq v_0 \in E' \) such that the set \( \{ \tilde{f} v_0 : f \in H \} \) is not uniformly integrable. For each \( f \in H \) set \( \nu_f(A) = \int_A \tilde{f} v_0 d\mu \) for \( A \in \Sigma \). Then \( \nu_f \) is a non-negative \( \mu \)-continuous countably additive set function on \( \Sigma \) but the family \( \{ \nu_f : f \in H \} \) is not uniformly \( \mu \)-continuous. Hence in view of Proposition 2.2 there exist a pairwise disjoint sequence \((B_n)\) in \( \Sigma \), a sequence \((f_n)\) in \( H \), and a number \( \varepsilon_0 > 0 \) such that \(\nu_f(B_n) = \int_{B_n} \tilde{f} v_0 d\mu > \varepsilon_0\) for each \( n \in \mathbb{N} \). Clearly \( v_0 f_n \in L^1(X) \), so in view of [BL, Theorem 1.1, (4)]
\[
\nu_f(B_n) = \| \chi_{B_n} v_0 \tilde{f} n \|_{L^1} = \| \chi_{B_n} v_0 f_n \|_{L^1(X)}
\]
\[
= \sup \left\{ \left| \int_{B_n} (v_0(\omega) f_n(\omega), g(\omega)) d\mu \right| : g \in L^\infty(X^*), \| g \|_{L^\infty(X^*)} \leq 1 \right\}.
\]
Hence one can produce a sequence \((g_n)\) in \( L^\infty(X^*) \) with \( \| g_n \|_{L^\infty(X^*)} \leq 1 \), \( \chi_{\Omega \setminus B_n} g_n = 0 \) and such that
\[
(1) \quad \int_{B_n} (v_0(\omega) f_n(\omega), g_n(\omega)) d\mu > \varepsilon_0.
\]
Set $g_0 = \sum_{n=1}^{\infty} g_n$. Then $g_0 \in L^0(X^*)$ and $\|g_0\|_{L^\infty(X^*)} \leq 1$. Clearly $v_0 g_0 \in E'(X^*)$, so $\chi_A v_0 g_0 \in E'(X^*)$ for each $A \in \Sigma$. In view of the assumption (i) there exists a $\sigma(E(X), E'(X^*))$-Cauchy subsequence $(f_{k_n})$ of $(f_n)$ so for each $A \in \Sigma$, \( \lim_{n} \int_{A} (f_{k_n}(\omega), v_0(\omega)g_0(\omega))d\mu \) exists. Setting \( \mu_n(A) = \int_{A} (f_{k_n}(\omega), v_0(\omega)g_0(\omega))d\mu \) for $A \in \Sigma$, in view of Nikodym’s convergence theorem (see [D, Chap. 7]), $\{\mu_n: n \in \mathbb{N}\}$ is uniformly countably additive on $\Sigma$. Hence there exists $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, $\sup |\mu_n(B_m)| \leq \varepsilon_0$ (see [D, Chap. 7, Theorem 10]). Hence for each $m \geq m_0$ we get

$$|\mu_m(B_{k_m})| = \left| \int_{B_{k_m}} (f_{k_m}(\omega), v_0(\omega)g_{k_m}(\omega))d\mu \right|$$

$$= \left| \int_{B_{k_m}} (v_0(\omega)f_{k_m}(\omega), g_{k_m}(\omega))d\mu \right| \leq \varepsilon_0$$

which contradicts (1). This contradiction establishes that (a) holds.

To show that (b) holds, take $A \in \Sigma$ with $\chi_A \in E'$, and let $(f_n)$ be a sequence in $H$. Set $g = \chi_A x^*$ where $x^* \in S_{X^*}$. Then $g \in E'(X^*)$ and by assumption (i) there exists a subsequence $(f_{k_n})$ of $(f_n)$ such that $\lim_{n} \int_{\Omega} (f_{k_n}(\omega), g(\omega))d\mu$ exists. Since $\int_{\Omega} (f_{k_n}(\omega), g(\omega))d\mu = x^* \left( \int_{A} f_{k_n}(\omega)d\mu \right)$, the set $\left\{ \int_{A} f(\omega)d\mu: f \in H \right\}$ is conditionally weakly compact in $X$.

(ii) $\Rightarrow$ (i) Let $(f_n)$ be a sequence in $H$. Since supp $E' = \Omega$ there exists a sequence $(\Omega_m)$ in $\Sigma$ such that $\Omega_m \uparrow \Omega$ and $\mu(\Omega_m) < \infty$, $\chi_{\Omega_m} \in E'$ for $m \in \mathbb{N}$ (see [Z Theorem 86.2]). Setting $A_m = \Omega \setminus \Omega_m$ for $m \in \mathbb{N}$ we see that $A_m \searrow \emptyset$. Given $m \in \mathbb{N}$ we have $\sup_{\Omega_m} f_n(\omega)d\mu = c_m < \infty$, because $\chi_{\Omega_m} \in E'$ and $H$ is $\sigma(E, E')$-bounded. Hence $\{\chi_{\Omega_m} f_n: n \in \mathbb{N}\} \subset L^1_{\Omega_m}(X)$, and by assumption $\sigma(A, \chi_{\Omega_m} f_n: n \in \mathbb{N})$ is a uniformly integrable subset of $L^1_{\Omega_m}(X)$. Combining this observation with (b), in view of Theorem 1.1 we see that $\{\chi_{\Omega_m} f_n: n \in \mathbb{N}\}$ is a conditionally $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*))$-compact subset of $L^1_{\Omega_m}(X)$.

In view of the above observation there exists a $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*))$-Cauchy subsequence $(\chi_{\Omega_k} f_{k_n})$ of $(\chi_{\Omega_k} f_n)$. Next, there exists a $\sigma(L^1_{\Omega_k}(X), L^\infty_{\Omega_k}(X^*))$-Cauchy subsequence $(\chi_{\Omega_k} f_{k_{n_k}})$ of $(\chi_{\Omega_k} f_{k_n})$. It follows that the diagonal sequence $(f_{k_n})$ has the property that for each $n \in \mathbb{N}$ $(\chi_{\Omega_n} f_{k_n})$ is a $\sigma(L^1_{\Omega_n}(X), L^\infty_{\Omega_n}(X^*))$-Cauchy sequence. Put $h_n = f_{k_n}$ for $n \in \mathbb{N}$.

Let $g \in E'(X^*)$. For $n \in \mathbb{N}$ let us put

$$g_n(\omega) = \begin{cases} g(\omega) & \text{if } \omega \in \Omega_n \text{ and } \|g(\omega)\|_{X^*} \leq n, \\ 0 & \text{elsewhere}. \end{cases}$$

Given $\varepsilon > 0$ there exist $m_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\sup_{n} \int_{\Omega \setminus \Omega_{m_0}} \bar{f}_n(\omega) \bar{g}(\omega)d\mu \leq \frac{\varepsilon}{4} \quad \text{and} \quad \sup_{n} \int_{A} \bar{f}_n(\omega) \bar{g}(\omega)d\mu \leq \frac{\varepsilon}{4}$$

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for each $A \in \Sigma$ with $\mu(A) \leq \delta$. For $\eta = \frac{\varepsilon}{3m_0}$ let

$$B_n = \{ \omega \in \Omega_{m_0} : \|g(\omega) - g_n(\omega)\|_{X^*} \geq \eta \}.$$  

It is easy to observe that $B_n \downarrow \emptyset$, so $\mu(B_n) \to 0$. Choose $n_0 \in \mathbb{N}$ with $n_0 \geq m_0$ such that $\mu(B_{n_0}) \leq \delta$. Then by (2) we get

$$\sup_{n} \int_{B_{n_0}} \tilde{h}_n(\omega) \tilde{g}(\omega) d\mu \leq \frac{\varepsilon}{4}. \tag{3}$$

Hence, by (3) we have

$$\left| \int_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{n_0}(\omega) \rangle d\mu \right| \leq \int_{\Omega_{m_0}} \tilde{h}_n(\omega) \|g(\omega) - g_{n_0}(\omega)\|_{X^*} d\mu$$

$$\leq \int_{B_{n_0}} \tilde{h}_n(\omega) \|g(\omega) - g_{n_0}(\omega)\|_{X^*} d\mu + \int_{\Omega_{m_0} \setminus B_{n_0}} \tilde{h}_n(\omega) \|g(\omega) - g_{n_0}(\omega)\|_{X^*} d\mu$$

$$\leq \int_{B_{n_0}} \tilde{h}_n(\omega) \tilde{g}(\omega) d\mu + \eta \int_{\Omega_{m_0}} \tilde{h}_n(\omega) d\mu \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \tag{4}$$

Since $\int_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega) \rangle d\mu \to a$ for some $a \in \mathbb{R}$, we can choose $n_1 \in \mathbb{N}$ such that for $n \geq n_1$

$$\left| \int_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega) \rangle d\mu - a \right| \leq \frac{\varepsilon}{4}. \tag{5}$$

Thus by (2), (4) and (5) for $n \geq n_1$ we get

$$\left| \int_{\Omega} \langle h_n(\omega), g(\omega) \rangle d\mu - a \right|$$

$$\leq \left| \int_{\Omega \setminus \Omega_{m_0}} \langle h_n(\omega), g(\omega) \rangle d\mu \right| + \left| \int_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{n_0}(\omega) \rangle d\mu \right|$$

$$+ \left| \int_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega) \rangle d\mu - a \right| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

This shows that $(h_n)$ is a $\sigma(E(X), E'(X^*))$-Cauchy subsequence of $(f_n)$, so $H$ is conditionally $\sigma(E(X), E'(X^*))$-compact. \hfill \Box

**Corollary 2.4.** Assume that a Banach space $X$ contains no isomorphic copy of $\ell^1$, and let $H$ be a subset of $E(X)$ such that $H$ is $\sigma(E, E^{'})$-bounded. Then the following statements are equivalent:

1. $H$ is conditionally $\sigma(E(X), E'(X^*))$-compact.
2. $\tilde{H}$ is conditionally $\sigma(E, E^{'})$-compact.

**Proof.** (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) In view of Theorem 2.3 it is enough to show that $\left\{ \int_A f(\omega) d\mu : f \in H \right\}$ is a conditionally weakly compact subset of $X$ for each $A \in \Sigma$ such that $\chi_A \in E'$. In fact, let $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$. Hence $\sup_{f \in H} \| \int_A f(\omega) d\mu \| \leq \varepsilon$.
sup \int f(\omega) d\mu < \infty$, so in view of the Rosenthal’s \(\ell^1\)-theorem [R] \(\{\int f(\omega) d\mu : f \in H\}\) is a conditionally weakly compact subset of \(X\), as desired. \(\square\)

Assume now that \((E, \| \cdot \|_E)\) is a Banach function space. Then the space \(E(X)\) provided with the norm \(\|f\|_{E(X)} := \|f\|_E\) is usually called a Köthe-Bochner space. The associated norm \(\| \cdot \|_{E'}\) on the Köthe dual \(E'\) can be defined as follows:

\[
\|v\|_{E'} = \sup \left\{ \int u(\omega)v(\omega) d\mu : u \in E, \|u\|_E \leq 1 \right\}.
\]

Clearly, for a subset \(H\) of \(E(X)\) the set \(\tilde{H}\) is \(\sigma(E, E')\)-bounded whenever

\[
\sup_{f \in H} \|f\|_{E(X)} < \infty.
\]

Combining Corollary 2.4 and Proposition 1.2 we get:

**Corollary 2.5.** Let \((E, \| \cdot \|_E)\) be a Banach function space, and assume that \(X\) contains no isomorphic copy of \(\ell^1\). Then the following statements are equivalent:

(i) The associated norm \(\| \cdot \|_{E'}\) on \(E'\) is order continuous.

(ii) Every norm bounded set in \(E(X)\) is conditionally \(\sigma(E(X), E'(X^*))\)-compact.

We now apply the previous results to Orlicz spaces (see [KR], [L] for more details).

By a Young function we mean a mapping \(\varphi: [0, \infty) \to [0, \infty)\) that is convex, vanishes only at 0 and \(\varphi(t)/t \to 0\) as \(t \to \infty\). Let \(L^\varphi\) be the Orlicz space associated with \(\varphi\) and provided with the Luxemburg norm \(\|u\|_\varphi := \inf \{\lambda > 0 : \int \varphi(|u(\omega)|/\lambda) d\mu \leq 1\}\). Then \((L^\varphi)' = L^{\varphi^\ast}\), where \(\varphi^\ast\) denotes the complementary Young function.

We say that a Young function \(\varphi\) increases more rapidly than another \(\varphi',\) in symbols \(\varphi \prec \varphi',\) if for each \(c > 0\) there is \(d > 1\) such that \(c \varphi(t) \leq \frac{1}{d} \varphi'(dt)\) for all \(t \geq 0\) (see [N2]). Note that \(\varphi\) satisfies the \(\nabla_2\)-condition iff \(\varphi \prec \varphi\).

As a consequence of Corollary 2.4 and [N2] Theorem 2.5 we get:

**Corollary 2.6.** Assume that \(X\) contains no isomorphic copy of \(\ell^1\). Then for a norm bounded subset \(H\) of the Orlicz-Bochner space \(L^\varphi(X)\) the following statements are equivalent:

(i) \(H\) is conditionally \(\sigma(L^\varphi(X), L^{\varphi^\ast}(X^*))\)-compact.

(ii) There is a Young function \(\psi\) with \(\varphi \prec \psi\) and such that \(H \subseteq L^\psi(X)\) and

\[
\sup_{f \in H} \|f\|_{L^\psi(X)} < \infty.
\]

**Corollary 2.7.** Assume that \(X\) contains no isomorphic copy of \(\ell^1\) and \(\varphi\) satisfies the \(\nabla_2\)-condition. Then every norm bounded subset of \(L^\varphi(X)\) is conditionally \(\sigma(L^\varphi(X), L^{\varphi^\ast}(X^*))\)-compact.

**References**


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