AN EXAMPLE OF AN ASYMPTOTICALLY HILBERTIAN SPACE WHICH FAILS THE APPROXIMATION PROPERTY

P. G. CASAZZA, C. L. GARCÍA, AND W. B. JOHNSON

(Communicated by N. Tomczak-Jaegermann)

Abstract. Following Davie’s example of a Banach space failing the approximation property (1973), we show how to construct a Banach space $E$ which is asymptotically Hilbertian and fails the approximation property. Moreover, the space $E$ is shown to be a subspace of a space with an unconditional basis which is “almost” a weak Hilbert space and which can be written as the direct sum of two subspaces all of whose subspaces have the approximation property.

1. Introduction

This paper is concerned with the relationship between the approximation property and notions about Banach spaces which are in some sense close to Hilbert space, namely, the notions of asymptotically Hilbertian space and of weak Hilbert space.

The spaces we discuss are of the form $Z = \left( \sum_{n=0}^{\infty} \ell_{p_n}^{k_n} \right)_2$ with $p_n \uparrow 2$ and $k_n \uparrow \infty$. It is easy to check that any space of this form is an asymptotically Hilbertian space (see below for definitions). For particular sequences $(p_n)$ and $(k_n)$ we show that such a $Z$ has a subspace $E$ failing the approximation property. Moreover, we can choose a subsequence of $(p_n)$, such that if $N_1 = \{j | p_{n_2(k+1)} \leq j < p_{n_2(k+1)}, k \geq 0\}$ and $N_2 = N - N_1$, then for $Z_i = \left( \sum_{j \in N_i} \ell_{p_j}^{k_j} \right)_2$, $i = 1, 2$, we have that $Z = Z_1 \oplus Z_2$ and that all subspaces of $Z_1$ and of $Z_2$ have the approximation property (\cite{J}).

The construction of $E$ provides quantitative estimates which show that $Z$, and hence also $E$, is surprisingly close to being a weak Hilbert space (note that weak Hilbert spaces enjoy the approximation property \cite{P}).

First we recall the notion of asymptotically Hilbertian space. Given integers $n \geq 0$, $m \geq 1$ and a constant $K$, say that $X$ satisfies $H(n, m, K)$ provided there is an $n$-codimensional subspace $X_m$ of $X$ so that every $m$-dimensional subspace of $X_m$ is $K$-isomorphic to $\ell_2^m$. A Banach space $X$ is said to be asymptotically Hilbertian provided there is a constant $K$ so that for every $m$ there exists $n$ so that $X$ satisfies $H(n, m, K)$. Since here we are interested in good estimates, we...
denote by $H_X(m, K)$ the smallest $n$ for which $X$ has $H(n, m, K)$. Thus if $X$ is
$K$-isomorphic to a Hilbert space, then $H_X(m, K) = 0$ for all $m$. The growth rate
of $H_X(m, K)$ for a fixed $K$ as $m \to \infty$ is one measurement of the closeness of $X$ to
a Hilbert space.

A Banach space is called a weak Hilbert space provided that there are positive
constants $\delta$ and $K$ so that for every $n$, every $n$-dimensional subspace of $X$ contains
a further subspace $E$ of dimension at least $\delta n$ so that $E$ is $K$-isomorphic to a Hilbert
space and $E$ is $K$-complemented in $X$ (that is, there is a projection having norm
at most $K$ from $X$ onto $E$).

The definition of the property $H(n, m, K)$ was made in [J], although the nomen-
clature “asymptotically Hilbertian” was coined by Pisier [P]. Weak Hilbert spaces
were introduced by Pisier [P], who gave many equivalences to the property of being
weak Hilbert; we chose the one most relevant for this paper as the definition of
weak Hilbert.

Relations between the weak Hilbert property and the asymptotically Hilbertian
property are given in [J] and [P]. First, a weak Hilbert space must be asymptotically
Hilbertian ([P, Section 4]). It seems likely that if $X$ is weak Hilbert, then for some
$K$, $H_X(m, K) \leq K m$, but in fact no reasonable estimates are known for $H_X(m, K)$
when $X$ is a weak Hilbert space. It is known (see [CJT, NTJ]) that if $X$ is a weak
Hilbert space which has an unconditional basis and there is a $K$ so that $H_X(m, K)$
is dominated by $f(m)$ for some iterate $f$ of exp, then for any iterate $g$ of log there
is another constant $K'$ so that $H_X(m, K') \leq K' g(m)$. In the other direction, it
follows from [J] that if for some $K$ the sequence $H_X(m, K)$ grows sufficiently slowly
as $m \to \infty$ (say, like $\log \log m$), then $X$ is a weak Hilbert space. In this paper we
are interested in examples of spaces which are of type 2. We refer to Chapter 11
of [DJT] or Section 1.4 of [T-J] for the definitions and basic theory of type $p$ and
cotype $p$ as well and the type $p$ and cotype $p$ constants $T_p(X)$, $C_p(X)$ of a Banach
space $X$. Relevant for us is that if $X$ is a type 2 space and $E$ is a subspace of $X$
which is $K$-isomorphic to a Hilbert space, then, by Maurey’s extension theorem, $E$
is $T_2(X)K$-complemented in $X$ ([DJT Corollary 12.24]). Thus it is clear that if $X$
is of type 2 and for some $K$, $H_X(m, K) \leq K m$, then $X$ is weak Hilbert. Here
we should mention that by [FLM], polynomial growth of $H_X(m, K)$ (as $m \to \infty$)
implies linear growth of $H_X(m, K)$ for some $K'$.

Our main interest here is the linkage among the weak Hilbert property, the
asymptotically Hilbertian property, and the approximation property. The arguments in
[J] show that if $X$ has type 2 and for some $K$, $H_X(m, K) \leq K \log m$ for
infinitely many $m$, then all subspaces of $X$ (even all subspaces of every
quotient of $X$) have the approximation property. In [P] it is shown that all weak
Hilbert spaces have the approximation property. Thus if $X$ is of type 2 and for
some $K$, $H_X(m, K) \leq K m$ for all $m$, then all subspaces of every quotient of $X$
have the approximation property. It is easy to build examples of a type 2 space
$X$ for which there is a constant $K$ so that for any iterate $f$ of the log function,$H_X(m, K) \leq K f(m)$ for infinitely many $m$ and yet $X$ is not a weak Hilbert space.
Now such a space $X$ is in some sense close to Hilbert space and, in particular, every
subspace of $X$ has the approximation property. In this paper we show that there
are two such spaces, call them $Z_1$ and $Z_2$, so that $Z := Z_1 \oplus Z_2$ has a subspace
which fails the approximation property. Moreover, $Z$ has an unconditional basis
($Z = \sum_{n=0}^{\infty} p_n k_n$ for appropriate $p_n \downarrow 2$ and $k_n \uparrow \infty$) and is nearly a weak Hilbert
space in the sense that for some $K$, the growth rate of $H_Z(m, K)$ as $m \to \infty$ is
close to being polynomial in \( m \) \( (H_Z(m, K) \leq m^{\log \log m} \) is what we get; recall that polynomial growth of \( H_X(m, K) \) gives linear growth of \( H_X(m, K') \) for some \( K' \).

2. The Example

We closely follow A. M. Davie’s construction of a Banach space failing the approximation property \([D]\). Davie constructed for \( p > 2 \) a subspace of \( \ell_p = \left( \sum k_n^p \right)_p \) which fails the approximation property. He could as well have used \( \left( \sum k_n^p \right)_r \) for any \( 1 \leq r \leq \infty \). Here we use instead \( Z := \left( \sum_{n=0}^{\infty} \ell_{p_n} \right)_2 \) where \( p_n \downarrow 2 \) appropriately and \( k_n \) as in \([D]\). Basically we compute how fast \( p_n \) can go to 2 so that Davie’s argument yields a subspace of \( Z \) which fails the a.p. The obvious condition is that \( k_n^{1/2 - 1/p_n} \) cannot be bounded, for if \( k_n^{1/2 - 1/p_n} \) is bounded, then \( \left( \sum_{n=0}^{\infty} \ell_{p_n} \right)_2 \) is isomorphic to \( \ell_2 \).

For any integer \( n \geq 0 \) consider an Abelian group \( G_n \) of order \( k_n = 3 \cdot 2^n \), and let \( \sigma_1^n, \ldots, \sigma_{2^n}^n, \tau_1^n, \ldots, \tau_{2^n+1}^n \) be the characters of \( G_n \). Lemma (b) in \([D]\) shows that this enumeration of the characters of \( G_n \) can be chosen so that there exists an absolute constant \( A > 0 \) such that for all \( g \in G_n \),

\[
|2 \sum_{j=1}^{2^n} \sigma_j^n(g) - \sum_{j=1}^{2^n+1} \tau_j^n(g)| \leq A(n + 1)^{1/2} 2^{n/2}.
\]

Let \( G \) be the disjoint union of the sets \( G_n \) and for each \( n \geq 0 \) and \( 1 \leq j \leq 2^n \) define \( e_j^n : G \to \mathbb{C} \) via

\[
e_j^n(g) = \begin{cases} 
\tau_j^{n-1}(g), & \text{if } g \in G_{n-1}, n \geq 1, \\
\varepsilon_j^n \sigma_j^n(g), & \text{if } g \in G_n, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \varepsilon_j^n = \pm 1 \) is a choice of signs for which the inequality (2.5) below is satisfied.

To define \( E \) let, as above, \( k_n = 3 \cdot 2^n \) and let \( (p_n)_{n=0}^{\infty} \), \( 2 < p_n < 3 \), be a decreasing sequence converging to 2. The appropriate rate of decrease of the sequence \( (p_n)_{n=0}^{\infty} \) will be chosen later.

Let \( Z = \left( \sum_{n=0}^{\infty} \ell_{p_n} \right)_2 \) which in our setting is

\[
Z = \left\{ f : G \to \mathbb{C} \mid \sum_{n=0}^{\infty} \left( \sum_{g \in G_n} |f(g)|^{p_n} \right)^{2/p_n} < \infty \right\}.
\]

Define \( E \) to be the closed linear span in \( Z \) of \( \{ e_j^n | n \geq 0, 1 \leq j \leq 2^n \} \). To show that \( E \) fails the approximation property one proceeds as follows:

For \( n \geq 0 \) and \( 1 \leq j \leq 2^n \) define \( \alpha_j^n \in E^* \) by

\[
\alpha_j^n(f) = 3^{-1} 2^{-n} \sum_{g \in G_n} \varepsilon_j^n \sigma_j^n(g^{-1}) f(g).
\]

When \( n \geq 1 \) the expression above equals

\[
\alpha_j^n(f) = 3^{-1} 2^{-n} \sum_{g \in G_{n-1}} \tau_j^{n-1}(g^{-1}) f(g).
\]

This follows from the fact that \( \alpha_j^n(e_k^n) = \delta_{ij} \cdot \delta_{kn} \) (because of the orthogonality of the characters of a group) and then a linearity and continuity argument shows that (2.2) and (2.3) agree on \( E \).
Now let $B(E)$ be the space of bounded, linear operators on $E$, and for each $n \geq 0$ define $\beta^n$ in the dual space $B(E)^*$ as

$$\beta^n(T) = 2^{-n} \sum_{j=1}^{2^n} \alpha^n_j(T(e^n_j)), \quad T \in B(E).$$

Using (2.2) we can rewrite $\beta^n$ as

$$\beta^n(T) = 3^{-1}4^{-n} \sum_{g \in G_n} T\left(\sum_{j=1}^{2^n} \varepsilon^n_j \sigma^n_j (g^{-1})e^n_j\right)(g),$$

and from (2.3) we get

$$\beta^{n+1}(T) = 6^{-1}4^{-n} \sum_{g \in G_n} T\left(\sum_{j=1}^{2^n} \tau^n_j (g^{-1})e^{n+1}_j\right)(g).$$

Hence,

$$\beta^{n+1}(T) - \beta^n(T) = 3^{-1}2^{-n} \sum_{g \in G_n} T(\Phi^n_g)(g)$$

where

$$\Phi^n_g = 2^{-n-1} \sum_{j=1}^{2^{n+1}} \tau^n_j (g^{-1})e^{n+1}_j - 2^{-n} \sum_{j=1}^{2^n} \varepsilon^n_j \sigma^n_j (g^{-1})e^n_j, \quad g \in G_n.$$

Note that $\Phi^n_g \in E$ for every $g \in G_n$ and $n \geq 1$. Now we estimate the right-hand side of (2.4). If $n \geq 1$ and $g \in G_n$, then

$$3^{-1}2^{-n} \sum_{g \in G_n} |T(\Phi^n_g)(g)| \leq \sup_{g \in G_n} \{||T(\Phi^n_g)||_{\infty}\} \leq \sup_{g \in G_n} \{||T(\Phi^n_g)||_Z\}.$$

Therefore,

$$|\beta^{n+1}(T) - \beta^n(T)| \leq \sup_{g \in G_n} \{||T(\Phi^n_g)||_Z\} \text{ for every } T \in B(E).$$

Note that from (2.1) we have that $|\Phi^n_g(h)| \leq A(n+1)^{1/2}2^{-n/2}$ for $g, h \in G_n$. By applying lemma (a) of [12], the signs $\varepsilon^n_j$, $1 \leq j \leq 2^n$, can be chosen so that

$$|\Phi^n_g(h)| \leq A_2(n+1)^{1/2}2^{-n/2} \text{ for } g \in G_n, h \in G_{n-1} \quad (n \geq 1)$$

where $A_2$ is some absolute constant. An algebraic argument shows that a similar estimate can be obtained for $g \in G_n$ and $h \in G_{n+1}$. In brief, we have that there is an absolute constant, say $A$, such that

$$|\Phi^n_g(h)| \leq A(n+1)^{1/2}2^{-n/2} \quad \text{for } g \in G_n \text{ and } h \in G_{n-1} \sqcup G_n \sqcup G_{n+1}.$$

Now, if $n \geq 1$ and $g \in G_n$, then

$$\|\Phi^n_g\|_Z^2 = \sum_{j=1}^{2^n} \left(\sum_{h \in G_j} |\Phi^n_g(h)|^p_j\right)^{2/p_j} \leq A^2(n+1)^2 \sum_{j=1}^{2^n} \left(\frac{2^n}{p_j} + \frac{2^{n+1}}{p_{j+1}}\right)^{1/2} \leq 3A^2(n+1)2^{-n} \left(\sum_{j=1}^{2^n} \frac{2^{n+1}}{p_j} + \frac{2^{n+1}}{p_{j+1}}\right) \leq 18A^2(n+1)2^{2n(1/2n+1)/2}.$$
Thus,
\[
\|\Phi^n_g\|_Z \leq 3\sqrt{2} A(n + 1)^{1/2} 2^{n(1/p_{n+1}-1/2)}.
\]

Consider the set
\[
C = \{e^0_i\} \cup \{(n + 1)^2 \Phi^n_g | g \in G_n, n \geq 1\}.
\]

The estimate in (2.7) clearly shows that when
\[
(n + 1)^{5/2} 2^{n(1/p_{n+1}-1/2)} \to 0
\]
the set \(C\) becomes a relatively compact subset of \(E\). Obviously there are many choices for \((p_n)_{n=0}^\infty, p_n \downarrow 2\), that satisfy (2.8); in particular, \(1/p_n = 1/2 - 1/(n+1)^\alpha\) for any \(\alpha < 1\) gives a sequence satisfying (2.8). When
\[
n(1/p_{n+1} - 1/2) = -3 \log_2(n + 1),
\]
the sequence \((p_n)\) is the one with the slowest (up to a constant) possible rate of decrease for this construction. This makes the space \(Z\) “almost” a weak Hilbert space in the sense that for some \(K\), \(H_Z(m, K) \leq m \log \log m\) for large \(m\). Indeed, consider \(F\), a subspace of \(\left(\sum_{j=n+1}^{\infty} \ell^2_{\jmath}\right)_2\), of dimension \(m = m(k_2 + \cdots + k_n)\), where \(m\) is the largest integer such that \(0 < 1/2 - 1/p_{n+1} < 1/\log_2(m)\). Then, \(d(F, \ell^2_{\jmath}) \leq T_2(F)C_2(F)\). The type 2 constant of \(F\) is bounded by an absolute constant independent of \(m\), say \(c_1\). The cotype 2 constant of \(F\) can be estimated, using Tomczak’s lemma, by the cotype 2 constant \(C_{2, m}(\cdot)\) on \(m\) vectors (see Section 5.25 in [1-J]):
\[
C_2(F) \leq \sqrt{2} C_{2, m}(F) \leq \sqrt{2} C_{p_{n+1}}(F)m^{1/2-1/p_{n+1}} \leq \sqrt{2} C_{2m}^{1/2-1/p_{n+1}} \leq 2\sqrt{2} c_2 \quad \text{(by the choice of \(m\))}
\]

Hence, for \(K := 2\sqrt{2} c_1 c_2\) we obtain that \(d(F, \ell^2_{\jmath}) \leq K\).

Finally, to show that \(E\) fails the approximation property the argument in [1] finishes as follows: for every \(T \in B(E)\),
\[
|\beta^{n+1}(T) - \beta^n(T)| \leq \sup_{g \in G_n} \{\|T(\Phi^n_g)\|_Z\} \leq (n + 1)^{-2} \sup_{x \in C} \|Tx\|_Z.
\]

Also,
\[
|\beta^0(T)| \leq \|Te^1_0\| \leq \sup_{x \in C} \|Tx\|_Z.
\]

Hence \(\beta(T) = \lim_{n \to \infty} \beta^n(T)\) exists for all \(T \in B(E)\) and satisfies
\[
|\beta(T)| \leq 3 \sup_{x \in C} \|Tx\|_Z.
\]

In particular, when \(C\) is compact, \(\beta\) is a continuous linear functional on \(B(E)\) when \(B(E)\) is given the topology of uniform convergence on compact sets.

If \(I_E\) is the identity map on \(E\), it follows from the definition of \(\beta^n\) that \(\beta^n(I_E) = 1\) for all \(n\), so \(\beta(I_E) = 1\). On the other hand it is easy to see that \(\beta\) vanishes on the set of finite rank operators on \(E\), thus \(E\) cannot have the approximation property.
Remark 2.1. For \((p_n, k_n)_{n=0}^\infty\) as above, we obtained an asymptotically Hilbertian space \(Z\) which has a subspace failing the approximation property. The space \(Z\) can be decomposed as the direct sum of two subspaces, say \(Z_1\) and \(Z_2\), all of whose subspaces have the approximation property. Indeed, as in example 1.g.7 in [LT], it is enough to construct a subsequence \((p_{n_j})\) of \((p_n)\) as follows: set \(p_{n_1} = p_0\) and \(k_{n_1} = k_0\). Having chosen \(p_{n_1} \cdots p_{n_j}\) (and their respective \(k_{n_1} \cdots k_{n_j}\)), choose \(p_{n_{j+1}}\) such that if \(F \subseteq \ell_p^m (2 < p < p_{n_{j+1}})\) has dimension \(m \leq 2 \cdot 5^j k_{n_j}\), then \(m^{1/2-1/p_{n_{j+1}}} \leq 2\). Now set \(N_j = \{i \mid p_{n_{j+1}} \leq j < p_{n_{j+k+1}}, k \geq 0\}\) and \(N_2 = \mathbb{N} - N_1\). Let \(Z_i = \left(\sum_{j \in N_i} \ell_{p_j}^{k_j}\right)/2\), \(i = 1, 2\).

Our example is the best possible in light of current theory and the current wisdom in the field. First, it follows from the arguments in [J] that the spaces \(Z_1, Z_2\) have the property that every subspace of every quotient space of \(Z_i\), \(i = 1, 2\), has a decomposition of the form \(Z_i = \left(\sum_{k=1}^\infty E_k\right)_{\ell_2}\), where \(\dim E_k < \infty\) for each \(k = 1, 2, 3, \ldots\). Also, the argument of Szarek [S] shows that the spaces \(Z_i\) have subspaces without bases. One might try to refine this example to produce \(Z_i\)'s for which every subspace has a basis. However, this may not be possible since it is an open question of whether Banach spaces for which every subspace has a basis must be weak Hilbert. Since the direct sum of weak Hilbert spaces is weak Hilbert, and every subspace of a weak Hilbert space has the approximation property, a positive answer to this question would show that our construction cannot be improved to produce \(Z_i\)'s for which every subspace has a basis. It was shown by Maurey and Pisier (see [M]) that every separable weak Hilbert space \(X\) has a finite-dimensional decomposition. That is, there is a sequence of finite-dimensional subspaces \(E_i\) of \(X\) so that for every \(x \in X\) there is a unique sequence \(x_i \in E_i\) so that \(x = \sum x_i\). However, it is an open question of whether a separable weak Hilbert space must have a basis.

References


Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri 65211
E-mail address: pete@math.missouri.edu

Department of Mathematics, Texas A&M University, College Station, Texas 77843–3368
E-mail address: cngarci@math.tamu.edu

Department of Mathematics, Texas A&M University, College Station, Texas 77843–3368
E-mail address: johnson@math.tamu.edu