ON THE BEREZIN-TOEPLITZ CALCULUS

L. A. COBURN

(Communicated by Joseph A. Ball)

Abstract. We consider the problem of composing Berezin-Toeplitz operators on the Hilbert space of Gaussian square-integrable entire functions on complex n-space, \( \mathbb{C}^n \). For several interesting algebras of functions on \( \mathbb{C}^n \), we have \( T_{\varphi}T_{\psi} = T_{\varphi \circ \psi} \) for all \( \varphi, \psi \) in the algebra, where \( T_{\varphi} \) is the Berezin-Toeplitz operator associated with \( \varphi \) and \( \varphi \circ \psi \) is a "twisted" associative product on the algebra of functions. On the other hand, there is a \( C^1 \) function \( \varphi \) for which \( T_{\varphi} \) is bounded but \( T_{\varphi}T_{\psi} \neq T_{\psi} \) for any \( \psi \).

1. Introduction

For \( z = (z_1, ..., z_n) \) in complex n-space, \( \mathbb{C}^n \), with \( z \cdot w = z_1 \overline{w_1} + ... + z_n \overline{w_n} \), consider the space \( L^2(\mathbb{C}^n, d\mu) \) of Gaussian square-integrable complex-valued functions on \( \mathbb{C}^n \), with \( d\mu(z) = \exp\{-|z|^2/2\} dv(z)(2\pi)^{-n} \) with \( dv(z) \) Lebesgue measure. The entire functions in \( L^2(\mathbb{C}^n, d\mu) \) form a closed subspace \( H^2(\mathbb{C}^n, d\mu) \) which arises naturally as a representation space of the Heisenberg group \( [B, F] \), \( [BC1], [C] \). On this (Segal-Bargmann) space, there are natural operators, formally introduced by Berezin \( [B2] \), defined densely for \( \varphi(\cdot) \) with \( \varphi(w)e^{w \cdot a} \) in \( L^2(\mathbb{C}^n, d\mu) \) for all \( a \) in \( \mathbb{C}^n \), by

\[
(T_{\varphi}f)(z) = \int_{\mathbb{C}^n} e^{z \cdot w/2} \varphi(w)f(w)d\mu(w).
\]

The (possibly unbounded) operator \( T_{\varphi} \) is called the Berezin-Toeplitz operator associated to \( \varphi \). Note that \( H^2(\mathbb{C}^n, d\mu) \) is a Bergman space with reproducing kernel function \( e^{z \cdot a/2} \) for the functional of "evaluation at \( a \" [B]. Note also that \( T_{\varphi} = 0 \) if and only if \( \varphi = 0 \) \( [B] \) p. 140).

The operators \( T_{\varphi} \) are closely related to pseudodifferential operators on \( L^2(\mathbb{R}^n, dv) \). For \( \varphi \) bounded, and somewhat more generally, the relation is given by

\[
B^{-1}T_{\varphi}B = \mathcal{W}_{\beta_{\varphi}}
\]

where \( B \) is the Bargmann isometry \( [Gn] \), \( \mathcal{W}_\beta \) is the Weyl operator on \( L^2(\mathbb{R}^n, dv) \) given by

\[
(W_{\beta}g)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \beta(\xi, \frac{x+y}{2})e^{i(x+y) \cdot \xi}g(y)dyd\xi,
\]

Received by the editors December 21, 1999 and, in revised form, March 21, 2000.
2000 Mathematics Subject Classification. Primary 47B35; Secondary 47B32.
The author’s research was supported by a grant of the NSF and a visiting membership in the Erwin Schrödinger Institute.

©2001 American Mathematical Society
and
\[ \beta_\varphi(\xi, x) = \pi^{-n} \int_{\mathbb{C}^n} \varphi(w) e^{-(w-x, \xi)^2} dv(w). \]

The operators $T_\varphi$ might, therefore, be expected to share many of the properties of pseudodifferential operators. It is not easy to demonstrate a complete equivalence, partly because $T_\varphi$ is a “very smoothed” version of $\varphi$. The analytic structure of $H^2(\mathbb{C}^n, d\mu)$ also enters the picture so that, for example,
\[ T_\varphi T_{\varphi_j} = T_{\varphi_{\varphi_j}}. \]

Moreover, the available function-theoretic machinery on $H^2(\mathbb{C}^n, d\mu)$ is relatively rudimentary, limited primarily to the Bergman space structure and the structure inherited as a representation space of the Heisenberg group.

In this note, we deal with the composition problem: is there a function $\varphi \circ \psi$ so that
\[ T_\varphi T_\psi = T_{\varphi \circ \psi}? \]

As a consequence of representation-theoretic results in [C], we do have (*), for a reasonably large class of bounded $\varphi, \psi$ and there is an explicit formula for $\varphi \circ \psi$. The same “Moyal-type” formula also holds for a large class of unbounded $\varphi, \psi$ (with unbounded $T_\varphi, T_\psi, T_{\varphi \circ \psi}$) – precisely, $\varphi, \psi$ can be arbitrary polynomials in $\{z_j, \overline{z}_j : 1 \leq j \leq n\}$.

On the other hand, we will exhibit a $\varphi$ (unbounded, but $C^\infty$), for which $T_\varphi$ is a bounded operator but $T_\varphi T_{\varphi_j}$ cannot be approximated in norm by bounded Berezin-Toeplitz operators. Thus, there is a genuine limitation on our ability to compose Berezin-Toeplitz operators.

I thank Samuel D. Schack for useful comments.

2. COMPOSITION OF BEREZIN-TOEPLITZ OPERATORS

For $C^\infty$ functions $\varphi, \psi$ we consider the (formal) twisted product
\[ \varphi \circ \psi = \sum_k \frac{(-2)^{|k|} k!}{k!} (\partial^k \varphi)(\overline{\partial}^k \psi) \]

where $k = (k_1, \ldots, k_n)$ with $k_j$ non-negative integers, and
\[ \partial_j = \frac{\partial}{\partial z_j}, \quad \overline{\partial}_j = \frac{\partial}{\partial \overline{z}_j}, \]
\[ \partial^k = \partial_1^{k_1} \ldots \partial_n^{k_n}, \quad \overline{\partial}^k = \overline{\partial}_1^{k_1} \ldots \overline{\partial}_n^{k_n}, \]
\[ |k| = k_1 + k_2 + \ldots + k_n, \]
\[ k! = k_1! k_2! \ldots k_n!. \]

In the cases we will consider, the sum in (**) will converge.

The first case we consider arises from representation-theoretic considerations of the Heisenberg group [C]. We consider $\varphi, \psi$ in the “smooth Bochner algebra” $B_a(\mathbb{C}^n)$ which consists of all Fourier-Stieltjes transforms of compactly supported, regular, bounded complex-valued Borel measures on $\mathbb{C}^n$. More precisely, let
\[ \chi_a(z) = \exp\{i \text{Im}(z \cdot a)\}. \]

Then $B_a(\mathbb{C}^n)$ consists of all functions
\[ \hat{\sigma}(z) = \int_{\mathbb{C}^n} \chi_a(z) \, d\sigma(a) \]
where σ is a compactly supported, regular, bounded complex-valued Borel measure. It is well known that such functions are bounded, uniformly continuous, with bounded derivatives of all orders.

As our first positive result, we have

**Theorem 1.** For φ, ψ in $B_1(C^n)$, $φ \circ ψ$ is also in $B_1(C^n)$ and $T_φ T_ψ = T_{φ \circ ψ}$. The series in (**) converges uniformly and absolutely.

**Proof.** In [C], it was shown that for $φ = σ$, $ψ = τ$ in $B_1(C^n)$,

$$T_φ T_ψ = T_{(σ τ)^-}.$$  

Here, we defined $σ \circ τ$ for all $φ \in C_0(C^n)$ by

$$\int_{C^n} φ(c) \, d(σ \circ τ)(c) = \int_{C^n} \int_{C^n} φ(a + b) e^{-\frac{1}{2} dσ(a)} \, dτ(b)$$

so that

$$(* *) \quad (σ \circ τ)^-(z) = \int_{C^n} \int_{C^n} χ_{a+b}(z) e^{-\frac{1}{2} dσ(a)} dτ(b)$$

is in $B_1(C^n)$.

Expanding $e^{-\frac{1}{2} dσ/2}$ in MacLaurin series in (**) gives

$$\sum_{s=0}^{∞} \frac{1}{s!} \frac{1}{2^s} \sum_{1 ≤ j_i ≤ n} \int π_{j_1} \ldots π_{j_s} χ_a(z) \, dσ(a) \int b_{j_1} \ldots b_{j_s} \, χ_b(z) \, dτ(b)$$

$$= \sum_{s=0}^{∞} \frac{1}{s!} \frac{1}{2^s} \sum_{1 ≤ j_i ≤ n} 2^s (\overline{j}_{j_1} \ldots \overline{j}_{j_s}) (φ (-2)^s (\overline{j}_{j_1} \ldots \overline{j}_{j_s}) \psi)$$

$$= \sum_{s=0}^{∞} \frac{(-2)^s}{s!} \sum_{1 ≤ j_i ≤ n} (\overline{j}_{j_1} \ldots \overline{j}_{j_s}) (φ \overline{j}_{j_1} \ldots \overline{j}_{j_s} \psi)$$

$$= \sum_{k=0}^{∞} \frac{(-2)^{|k|}}{k!} (φ \overline{k} \psi)$$

and it is clear that the series converges uniformly and absolutely. Comparison with (**) shows that

$$T_φ T_ψ = T_{φ \circ ψ}$$

and completes the proof.

Our second case consists of $φ, ψ$ arbitrary polynomials in $\{z_j, \overline{z}_j : 1 ≤ j ≤ n\}$. Here, the operators $T_φ, T_ψ$ are unbounded and we need to be a little more careful. Nevertheless, we have for $φ \circ ψ$ given by (**),

**Theorem 2.** For $φ, ψ$ polynomials in $(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n)$, we have $T_φ T_ψ$ defined on a dense domain consisting of linear combinations of functions of the form $\{p(z)e^{z^-a} : a \in C^n \text{ and } p(z) \text{ polynomial in } (z_1, ..., z_n)\}$. On this domain

$$T_φ T_ψ = T_{φ \circ ψ}$$

and $φ \circ ψ$ is polynomial in the $z_j, \overline{z}_j$.  

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. Clearly, $T_{z_j}^2 = 2\partial_j$ and it is now easy to check that $T_{\varphi} p(z)e^{z^*a} = q(z)e^{z^*a}$ where $p, q$ are polynomial in $z_1, \ldots, z_n$. The proof of the composition formula is inductive, in several steps.

We note first that, for $\varphi$ polynomial in $\{z_j, \bar{z}_j : 1 \leq j \leq n\}$, $T_{\varphi}T_{z_j} = T_{\varphi z_j}$ implies $T_{\varphi|z_j|^2} = T_{\varphi(z_j)}$. This is because

$$T_{\varphi}T_{z_j}|^2 = (T_{\varphi}T_{z_j})T_{z_j} = T_{\varphi(z_j)}.$$

Next, we check inductively that $T_{\varphi}T_{z_j} = T_{\varphi z_j}$ for all $\varphi$ polynomial in $\{z_j, \bar{z}_j : 1 \leq j \leq n\}$. It is enough to consider $\varphi$ monomial. Assume the result for $\varphi$ of fixed degree ($\varphi$ constant is trivial). The inductive step is:

$$T_{\varphi z_k}T_{z_j} = T_{\varphi z_k}T_{z_k} = T_{\varphi z_k^*z_j}, \quad k \neq j,$$

$$T_{\varphi z_j}T_{z_j} = T_{\varphi(T_{z_j}T_{z_j})} = T_{\varphi(T_{z_j})} - T_{z_j} - T_{z_j}^2 = T_{\varphi|z_j|^2} - T_{z_j} - T_{\varphi|z_j|^2} = T_{\varphi z_j^*z_j},$$

$$T_{z_k z_j}T_{z_j} = T_{z_k(T_{z_j}T_{z_j})} = T_{z_k(z_j)} = T_{z_k(z_j^*z_j)} = T_{z_k z_j^*z_j}.$$

Thus, $T_{\varphi}T_{z_j} = T_{\varphi z_j}$ for all $\varphi$.

Next, for arbitrary $\varphi$ we consider $T_{\varphi}T_{\psi}$ and do induction on the degree of $\psi$. We can assume $\psi$ is monomial. Assume the result for all $\varphi$ and for $\psi$ of fixed degree ($\psi$ constant is trivial). The inductive step is, first,

$$T_{\varphi}T_{z_j} = (T_{\varphi}T_{\psi})T_{z_j} = T_{\varphi z_j} = T_{z_j}.$$
This is a direct calculation. We note that
\[ \varphi \circ \overline{\zeta}_j = \varphi \overline{\zeta}_j - 2(\partial_j \varphi) \]
so
\[ (\varphi \circ \overline{\zeta}_j) \circ \psi = \varphi \overline{\zeta}_j \circ \psi - 2(\partial_j \varphi) \circ \psi \]
\[ = \sum_k \frac{(-2)^{|k|}}{k!} \overline{\zeta}_j (\partial^k \varphi)(\partial^k \psi) \]
\[ -2 \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \partial_j \varphi)(\partial^k \psi). \]
Using
\[ \overline{\partial}^k (\overline{\zeta}_j \psi) = \overline{\zeta}_j (\overline{\partial}^k \psi) + k_j (\overline{\partial}^{k-\delta_j} \psi) \]
where
\[ k - \delta_j = (k_1, k_2, ..., k_j - 1, k_{j+1}, ..., k_n), \]
we see that
\[ \varphi \circ \overline{\zeta}_j \psi = \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi)(\partial^k \overline{\zeta}_j \psi) \]
\[ = \sum_k \frac{(-2)^{|k|}}{k!} \overline{\zeta}_j (\partial^k \varphi)(\partial^k \psi) \]
\[ + \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi) k_j (\overline{\partial}^{k-\delta_j} \psi). \]
Thus, we need only check that
\[ \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi) k_j (\overline{\partial}^{k-\delta_j} \psi) = -2 \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \partial_j \varphi)(\partial^k \psi). \]
Reindexing the sum on the left by \( \ell = k - \delta_j \) completes the proof.

**Remark.** Since \( \overline{\zeta}_j \circ \psi = \overline{\zeta}_j \psi \), the identity
\[ \varphi \circ \overline{\zeta}_j \psi = (\varphi \circ \overline{\zeta}_j) \circ \psi \]
follows from the reasonably well-known associativity of \( \circ \). Our computational proof has the advantage of giving associativity of \( \circ \) as an immediate corollary of Theorem 2 since
\[ T_{\varphi \circ (\psi \circ \gamma)} = T_{\varphi}(T_{\psi}T_{\gamma}) = (T_{\varphi}T_{\psi})T_{\gamma} = T_{(\varphi \circ \psi) \circ \gamma}. \]

3. \( T_{\varphi} \) with \( T_{\varphi}T_{\psi} \neq T_{\psi} \) for any \( \psi \)

In this section, we produce the promised obstruction to composition of Berezin-Toeplitz operators. We use some calculations from [BC2] and we begin with a needed improvement of [BC2, Theorem 17]. In this section, we work on \( H^2(C, d\mu) \) (\( n = 1 \)). Here, the Bergman reproducing kernel function for evaluation at \( z \) is just
\[ K(w, z) = e^{w\overline{z}/2} \]
and it follows that
\[ k_z(w) = K(w, z)/\sqrt{K(z, z)} = e^{w\overline{z}/2 - |z|^2/4} \]
is a unit vector in $H^2(\mathbb{C}, d\mu)$. We consider the unitary operator

$$(R_a f)(z) = f(az)$$

on $H^2(\mathbb{C}, d\mu)$ for $|a| = 1$.

**Theorem 3.** For $|a| = 1$ and $\text{Re } a < 0$, we have

$$\|R_a - T_\psi\| \geq 1$$

for all $\psi$ such that $\psi K(\cdot, z)$ is in $L^2(\mathbb{C}, d\mu)$ for every $z$ in $\mathbb{C}$.

**Proof.** We consider

$$\|T_\psi - R_a\| \geq |\langle T_\psi k_z, R_a k_z \rangle - \langle R_a k_z, R_a k_z \rangle|$$

$$\geq |\langle T_\psi k_z, R_a k_z \rangle - 1|.$$  

Now,

$$\langle T_\psi k_z, R_a k_z \rangle = \langle \psi \chi_z, K(\cdot, (1 + \overline{a})z) \rangle e^{-|z|^2/2}$$

so we have

$$|\langle T_\psi k_z, R_a k_z \rangle| \leq e^{-|z|^2/2} \| \psi \| \sqrt{K((1 + \overline{a})z, (1 + \overline{a})z)}$$

$$\leq \| \psi \| e^{-|z|^2/2} e^{1 + |a|^2 |z|^2 / 4}$$

$$\leq \| \psi \| e^{jz^2/2 \text{Re } a/2}.$$  

Since $\text{Re } a < 0$, we see that

$$|\langle T_\psi k_z, R_a k_z \rangle| \to 0$$

as $|z| \to \infty$. Thus, $\|T_\psi - R_a\| \geq 1$.

The function $\varphi$ will be chosen to have the form $\varphi(z) = e^{jz^2}$ where $\text{Re } \lambda < \frac{1}{4}$ so that $T_{\varphi}$ makes sense.

**Lemma.** For $\lambda = \frac{1}{4} + \frac{i}{2}$ and $\varphi(z) = e^{jz^2}$, we have $T_{\varphi}$ unitary with

$$T_{\varphi} T_{\varphi} = a R_a$$

for $a = \frac{(1 - 2\lambda)^2}{2} = -\frac{7}{25} + \frac{24}{25}$.

**Proof.** $\text{Re } \lambda < \frac{1}{4}$ and calculations outlined in [BC2, p. 582] show that $T_{\varphi}$ is diagonal in the basis

$$e_k = (2k)!^{-1/2} z^k, \quad k = 0, 1, \ldots,$$

for $H^2(\mathbb{C}, d\mu)$, with

$$T_{\varphi} e_k = (1 - 2\lambda)^{-k-1} e_k.$$  

Now

$$\lambda = \frac{1}{4} + \frac{i}{2}$$

and so

$$T_{\varphi} T_{\varphi} e_k = (1 - 2\lambda)^{-2(k+1)} e_k = a^{k+1} e_k.$$  

But

$$a R_a e_k = a^{k+1} e_k$$

and we are done.
We now have the promised

**Theorem 4.** For \( \lambda = \frac{1}{8} + i \frac{3}{8} \) and \( a = \frac{1}{25} - \frac{7}{25} + \frac{24}{25} i \), with \( \varphi(z) = e^{\lambda|z|^2} \),
\[
\|T_\varphi T_\psi - T_\psi \| \geq 1
\]
for all \( \psi \) such that \( \psi K(\cdot, z) \) is in \( L^2(C, d\mu) \) for every \( z \) in \( C \).

**Proof.** Direct combination of Theorem 3 and the Lemma.

**Remark.** In fact, for \( \varphi(z) = e^{\lambda|z|^2} \), (***) yields
\[
\varphi \circ \varphi = e^{\mu|z|^2},
\]
where \( \mu = 2\lambda(1 - \lambda) \). Thus, for \( \lambda = \frac{1}{8} + i \frac{3}{8} \), we have \( \mu = \frac{16}{25} + i \frac{12}{25} \) and \( e^{\mu|z|^2} f(z) \) cannot
be in \( L^2(C, d\mu) \) for any \( f \neq 0 \) in \( H^2(C, d\mu) \).

4. **Remarks**

There is a considerable space between Theorems 1 and 2 and Theorem 4. It does not seem easy to lift the known much stronger positive results directly over from the setting of pseudodifferential operators. It does seem likely that (**) provides a composition formula for Berezin-Toeplitz operators in a setting substantially larger than those of Theorems 1 and 2. For non-\( C^\infty \) \( \varphi, \psi \) or even for general \( C^\infty \) \( \varphi, \psi \), the problem of determining whether there is a \( \varphi \circ \psi \) with \( T_\varphi T_\psi = T_{\varphi \circ \psi} \), as well as the form of \( \varphi \circ \psi \), remains open.

Theorems 1 and 2 can be extended to the natural family of Gaussian measures on \( C^n \) which provide representation spaces for the Heisenberg group \( \mathbb{Q} \). For \( d\mu_r(z) = (2\pi)^n e^{-|z|^2} dv(z) \) with \( r > 0 \) and \( H^2(C^n, d\mu_r) \) as before, we have Bergman kernels
\[
K_r(w, z) = e^{r w \cdot z}
\]
and Berezin-Toeplitz operators on \( H^2(C^n, d\mu_r) \)
\[
(T_\varphi^{(r)} f)(z) = \int_{C^n} e^{r z \cdot w} \varphi(w) f(w) \, d\mu_r(w).
\]
Then minor modifications yield

**Theorem 1’.** For \( \varphi, \psi \) in \( B_a(C^n) \), \( \varphi \circ_r \psi \) is also in \( B_a(C^n) \) for
\[
(\ddagger) \quad \varphi \circ_r \psi = \sum_k \left( \frac{-1}{r} \right)^{|k|} \frac{1}{k!} (\partial^k \varphi)(\overline{\partial}^k \psi)
\]
and \( T_\varphi^{(r)} T_\psi^{(r)} = T_{\varphi \circ \psi} \). The series in (\ddagger) converges uniformly and absolutely. Moreover, for \( r > 1 \)
\[
\| \varphi \circ_r \psi - \sum_{|k| \leq K} \left( \frac{-1}{r} \right)^{|k|} \frac{1}{k!} (\partial^k \varphi)(\overline{\partial}^k \psi) \| \leq \frac{1}{r^{K+1}} C(\varphi, \psi, K)
\]
for \( C(\varphi, \psi, K) \) a constant independent of \( r \).

**Theorem 2’.** For \( \varphi, \psi \) polynomials in \( (z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n) \), we have \( T_\varphi^{(r)} T_\psi^{(r)} \) defined on a dense domain consisting of linear combinations of functions of the form \( \{ p(z) e^{a \cdot z} : a \in C^n \) and \( p(z) \) polynomial in \( (z_1, \ldots, z_n) \) \}. On this domain
\[
T_\varphi^{(r)} T_\psi^{(r)} = T_{\varphi \circ \psi}
\]
for \( \varphi \circ_r \psi \) given by (\ddagger) and \( \varphi \circ_r \psi \) is polynomial in the \( z_j, \overline{z}_j \).
While Theorems 1 and 2 provide some basis for optimism about the development of a reasonably extensive Berezin-Toeplitz calculus on $C^n$, the situation is considerably less promising on the classical Bergman space of the disc, $H^2(D, \frac{dA}{\pi})$, where $D = \{z \in \mathbb{C} : |z| < 1\}$ and $\frac{dA}{\pi}$ is normalized Lebesgue area measure. In this case, the Bergman kernel function is just $K(z, w) = (1 - z \overline{w})^{-2}$ and the Berezin-Toeplitz operator $T_\varphi$ on $H^2(D, \frac{dA}{\pi})$ is given by

$$(T_\varphi f)(z) = \int_D K(z, w) \varphi(w) f(w) \frac{dA(w)}{\pi}.$$ 

Direct calculation shows, first, that

$$T_z T_z^* = T_{1+\log|z|^2}.$$ 

Moreover,

$$T_z^* T_z^* = T_{1+2\log|z|^2} + P_0$$ 

where $P_0 f = \int_D f(z) \frac{dA(z)}{\pi}$ and $P_0 \neq T_\varphi$ for any $\varphi$. For asymptotic results on composition of Berezin-Toeplitz operators on $H^2(D, \frac{dA}{\pi})$ see [KL].

REFERENCES


DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, BUFFALO, NEW YORK 14260

E-mail address: lacoburn@acsu.buffalo.edu