ON THE BEREZIN-TOEPLITZ CALCULUS

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Abstract. We consider the problem of composing Berezin-Toeplitz operators on the Hilbert space of Gaussian square-integrable entire functions on complex n-space, \( \mathbb{C}^n \). For several interesting algebras of functions on \( \mathbb{C}^n \), we have \( T_{\psi} T_{\phi} = T_{\phi \circ \psi} \) for all \( \phi, \psi \) in the algebra, where \( T_{\phi} \) is the Berezin-Toeplitz operator associated with \( \phi \) and \( \phi \circ \psi \) is a "twisted" associative product on the algebra of functions. On the other hand, there is a \( C^1 \) function \( \phi \) for which \( T_{\phi} \) is bounded but \( T_{\phi} T_{\psi} \neq T_{\phi} \) for any \( \psi \).

1. Introduction

For \( z = (z_1, ..., z_n) \) in complex n-space, \( \mathbb{C}^n \), with \( z_j \) in \( \mathbb{C} \), and \( z \cdot w = z_1 \overline{w_1} + ... + z_n \overline{w_n} \), consider the space \( L^2(\mathbb{C}^n, d\mu) \) of Gaussian square-integrable complex-valued functions on \( \mathbb{C}^n \), with \( d\mu(z) = \exp\{-|z|^2/2\} dv(z) (2\pi)^{-n} \) with \( dv(z) \) Lebesgue measure. The entire functions in \( L^2(\mathbb{C}^n, d\mu) \) form a closed subspace \( H^2(\mathbb{C}^n, d\mu) \) which arises naturally as a representation space of the Heisenberg group [B], [F], [BC1], [C]. On this (Segal-Bargmann) space, there are natural operators, formally introduced by Berezin [B2], defined densely for \( \phi(\cdot) \) with \( \phi(w)e^{w \cdot a} \) in \( L^2(\mathbb{C}^n, d\mu) \) for all \( a \) in \( \mathbb{C}^n \), by

\[
(T_{\phi} f)(z) = \int_{\mathbb{C}^n} e^{z \cdot w/2} \phi(w) f(w) d\mu(w).
\]

The (possibly unbounded) operator \( T_{\phi} \) is called the Berezin-Toeplitz operator associated to \( \phi \). Note that \( H^2(\mathbb{C}^n, d\mu) \) is a Bergman space with reproducing kernel function \( e^{z \cdot a/2} \) for the functional of "evaluation at \( a \" [B]. Note also that \( T_{\phi} = 0 \) if and only if \( \phi = 0 \) [B, p. 140].

The operators \( T_{\phi} \) are closely related to pseudodifferential operators on \( L^2(\mathbb{R}^n, dv) \). For \( \phi \) bounded, and somewhat more generally, the relation is given by

\[
B^{-1} T_{\phi} B = W_{\phi},
\]

where \( B \) is the Bargmann isometry [Gn], \( W_{\beta} \) is the Weyl operator on \( L^2(\mathbb{R}^n, dv) \) given by

\[
(W_{\beta} g)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \beta(\xi, \frac{x-y}{2}) e^{i(x-y) \cdot \xi} g(y) dy d\xi,
\]

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and
\[ \beta_p(\xi, x) = \pi^{-n} \int_{C^n} \varphi(w) e^{-|w-(x-i\xi)|^2} dv(w). \]

The operators \( T_p \) might, therefore, be expected to share many of the properties of pseudodifferential operators. It is not easy to demonstrate a complete equivalence, partly because \( \beta_p \) is a “very smoothed” version of \( \varphi \). The analytic structure of \( H^2(C^n, d\mu) \) also enters the picture so that, for example,
\[ T_p T_{z_j} = T_{p z_j}. \]

Moreover, the available function-theoretic machinery on \( H^2(C^n, d\mu) \) is relatively rudimentary, limited primarily to the Bergman space structure and the structure inherited as a representation space of the Heisenberg group.

In this note, we deal with the composition problem: is there a function \( \varphi \circ \psi \) so that
\[ (\ast) \quad T_p T_\psi = T_{p \circ \psi}? \]

As a consequence of representation-theoretic results in [C], we do have (\ast) for a reasonably large class of bounded \( \varphi, \psi \) and there is an explicit formula for \( \varphi \circ \psi \). The same “Moyal-type” formula also holds for a large class of unbounded \( \varphi, \psi \) (with unbounded \( T_p, T_\psi, T_{p \circ \psi} \)) – precisely, \( \varphi, \psi \) can be arbitrary polynomials in \( \{z_j, \overline{z}_j : 1 \leq j \leq n\} \).

On the other hand, we will exhibit a \( \varphi \) (unbounded, but \( C^\infty \)), for which \( T_p \) is a bounded operator but \( T_p T_\psi \) cannot be approximated in norm by bounded Berezin-Toeplitz operators. Thus, there is a genuine limitation on our ability to compose Berezin-Toeplitz operators.

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2. Composition of Berezin-Toeplitz operators

For \( C^\infty \) functions \( \varphi, \psi \) we consider the (formal) twisted product
\[ \varphi \circ \psi = \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi)(\overline{\partial}^k \psi) \]

where \( k = (k_1, \ldots, k_n) \) with \( k_j \) non-negative integers, and
\[ \partial_j = \frac{\partial}{\partial z_j}, \quad \overline{\partial}_j = \frac{\partial}{\partial \overline{z}_j}, \]
\[ \partial^k = \partial_1^{k_1} \cdots \partial_n^{k_n}, \quad \overline{\partial}^k = \overline{\partial}_1^{k_1} \cdots \overline{\partial}_n^{k_n}, \]
\[ |k| = k_1 + k_2 + \ldots + k_n, \]
\[ k! = k_1! k_2! \ldots k_n!. \]

In the cases we will consider, the sum in (\ast\ast) will converge.

The first case we consider arises from representation-theoretic considerations of the Heisenberg group [C]. We consider \( \varphi, \psi \) in the “smooth Bochner algebra” \( B_0(C^n) \) which consists of all Fourier-Stieltjes transforms of compactly supported, regular, bounded complex-valued Borel measures on \( C^n \). More precisely, let
\[ \chi_z = \exp\{i \operatorname{Im}(z \cdot a)\}. \]

Then \( B_0(C^n) \) consists of all functions
\[ \hat{\sigma}(z) = \int_{C^n} \chi_z \, d\sigma(a) \]
where $\sigma$ is a compactly supported, regular, bounded complex-valued Borel measure. It is well known that such functions are bounded, uniformly continuous, with bounded derivatives of all orders.

As our first positive result, we have

**Theorem 1.** For $\varphi, \psi$ in $B_0(C^n)$, $\varphi \circ \psi$ is also in $B_0(C^n)$ and $T_\varphi T_\psi = T_{\varphi \circ \psi}$. The series in (***) converges uniformly and absolutely.

**Proof.** In [C], it was shown that for $\varphi = \sigma$, $\psi = \tau$ in $B_0(C^n)$,

$$T_\varphi T_\psi = T_{(\sigma \circ \tau)}.$$

Here, we defined $\sigma \circ \tau$ for all $\phi$ in $C_0(C^n)$ by

$$\int_{C^n} \phi(c) \ d(\sigma \circ \tau)(c) = \int_{C^n} \int_{C^n} \phi(a + b) e^{b/a} \ d\sigma(a) \ d\tau(b)$$

so that

$$(***) \quad (\sigma \circ \tau)^\wedge(z) = \int_{C^n} \int_{C^n} \chi_{a+b}(z) e^{b/a} \ d\sigma(a) \ d\tau(b)$$

is in $B_0(C^n)$.

Expanding $e^{b/a}$ in MacLaurin series in (***) gives

$$= \sum_{s=0}^{\infty} \sum_{1 \leq j_s \leq n} \frac{1}{s! \cdot 2^s} \sum_{1 \leq j_s \leq n} 2^s (\partial_{j_1} \ldots \partial_{j_s} \varphi)(-2)^s (\partial_{j_1} \ldots \partial_{j_s} \psi)$$

and it is clear that the series converges uniformly and absolutely. Comparison with (***) shows that

$$T_\varphi T_\psi = T_{\varphi \circ \psi}$$

and completes the proof.

Our second case consists of $\varphi, \psi$ arbitrary polynomials in $\{z_j, \overline{z}_j : 1 \leq j \leq n\}$. Here, the operators $T_\varphi, T_\psi$ are unbounded and we need to be a little more careful. Nevertheless, we have for $\varphi \circ \psi$ given by (**),

**Theorem 2.** For $\varphi, \psi$ polynomials in $(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n)$, we have $T_\varphi T_\psi$ defined on a dense domain consisting of linear combinations of functions of the form $\{p(z) e^{z \cdot a} : a \in C^n \text{ and } p(z) \text{ polynomial in } (z_1, \ldots, z_n)\}$. On this domain

$$T_\varphi T_\psi = T_{\varphi \circ \psi}$$

and $\varphi \circ \psi$ is polynomial in the $z_j, \overline{z}_j$.
Proof. Clearly, $T_{z_j} = 2\partial_j$ and it is now easy to check that $T_\varphi p(z)e^{z^a} = q(z)e^{z^a}$ where $p, q$ are polynomial in $z_1, ..., z_n$. The proof of the composition formula is inductive, in several steps.

We note first that, for $\varphi$ polynomial in $\{z_j, \pi_j : 1 \leq j \leq n\}$, $T_\varphi T_{z_j} = T_{\varphi \circ \pi_j}$ implies $T_{\varphi \circ |z_j|^2} = T_\varphi T_{|z_j|^2}$. This is because

$$T_\varphi T_{|z_j|^2} = (T_\varphi T_{z_j})T_{z_j} = T_{(\varphi \circ \pi_j)z_j} = T_{\varphi \circ |z_j|^2}.$$

Next, we check inductively that $T_\varphi T_{z_j} = T_{\varphi \circ \pi_j}$ for all $\varphi$ polynomial in $\{z_j, \pi_j : 1 \leq j \leq n\}$. It is enough to consider $\varphi$ monomial. Assume the result for $\varphi$ of fixed degree ($\varphi$ constant is trivial). The inductive step is:

$$T_{\varphi z_k} T_{z_j} = T_{\varphi} T_{z_k} T_{z_j} = T_{(\varphi \circ \pi_j)z_k} = T_{\varphi z_k \circ \pi_j}, \quad k \neq j,$$

$$T_{\varphi z_j} T_{z_j} = T_{\varphi} (T_{z_j} T_{z_j}) = T_{\varphi}(T_{|z_j|^2} - 2I) = T_{\varphi} T_{|z_j|^2} - T_{2\varphi} = T_{\varphi \circ |z_j|^2 - 2\varphi} = T_{\varphi z_j \circ \pi_j},$$

$$T_{z_k \circ \varphi} T_{z_j} = T_{z_k} (T_{\varphi} T_{z_j}) = T_{z_k \circ (\varphi \circ \pi_j)} = T_{z_k \circ \varphi \circ \pi_j}.$$

Thus, $T_\varphi T_{z_j} = T_{\varphi \circ \pi_j}$ for all $\varphi$.

Next, for arbitrary $\varphi$ we consider $T_\varphi T_\psi$ and do induction on the degree of $\psi$. We can assume $\psi$ is monomial. Assume the result for all $\varphi$ and for $\psi$ of fixed degree ($\psi$ constant is trivial). The inductive step is, first,

$$T_\varphi T_{\psi z_j} = (T_\varphi T_\psi)T_{z_j} = T_{(\varphi \circ \psi)z_j} = T_{\varphi \circ \psi z_j}.$$
This is a direct calculation. We note that
\[ \varphi \circ \overline{\varphi} = \varphi \overline{\varphi} - 2(\partial_j \varphi) \]
so
\[ (\varphi \circ \overline{\varphi}) \circ \psi = \varphi \overline{\varphi} \circ \psi - 2(\partial_j \varphi) \circ \psi \]
\[ = \sum_k \frac{(-2)^{|k|}}{k!} \overline{\varphi}(\overline{\partial^k \varphi})(\overline{\partial^k \psi}) \]
\[ - 2 \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi)(\partial_j \varphi)(\partial_j \overline{\partial^k \psi}). \]

Using
\[ \overline{\partial^k (\overline{\varphi} \psi)} = \overline{\varphi}(\overline{\partial^k \psi}) + k_j \overline{\partial^{k-\delta_j} \psi} \]
where
\[ k - \delta_j = (k_1, k_2, ..., k_j - 1, k_j + 1, ..., k_n), \]
we see that
\[ \varphi \circ \overline{\varphi} \psi = \sum_k \frac{(-2)^{|k|}}{k!} (\overline{\partial^k \varphi})(\overline{\partial^k \overline{\varphi} \psi}) \]
\[ = \sum_k \frac{(-2)^{|k|}}{k!} \overline{\varphi}(\overline{\partial^k \varphi})(\overline{\partial^k \psi}) \]
\[ + \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi)k_j (\overline{\partial^{k-\delta_j} \psi}). \]
Thus, we need only check that
\[ \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi)k_j (\overline{\partial^{k-\delta_j} \psi}) = -2 \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi)(\partial_j \varphi)(\partial_j \overline{\partial^k \psi}). \]
Reindexing the sum on the left by \( \ell = k - \delta_j \) completes the proof.

**Remark.** Since \( \overline{\varphi} \circ \psi = \overline{\varphi} \psi \), the identity
\[ \varphi \circ \overline{\varphi} \psi = (\varphi \circ \overline{\varphi}) \circ \psi \]
follows from the reasonably well-known associativity of \( \circ \) in \( \mathbb{C} \). Our computational proof has the advantage of giving associativity of \( \circ \) as an immediate corollary of Theorem 2 since
\[ T_{\varphi \circ (\psi \circ \gamma)} = T_{\varphi}(T_{\psi}T_{\gamma}) = (T_{\varphi}T_{\psi})T_{\gamma} = T_{(\varphi \circ \psi) \circ \gamma}. \]
3. \( T_{\varphi} \) with \( T_{\varphi}T_{\psi} \neq T_{\psi} \) for any \( \psi \)

In this section, we produce the promised obstruction to composition of Berezin-Toeplitz operators. We use some calculations from [BC2] and we begin with a needed improvement of [BC2, Theorem 17]. In this section, we work on \( H^2(\mathbb{C}, d\mu) \) \((n = 1)\). Here, the Bergman reproducing kernel function for evaluation at \( z \) is just
\[ K(w, z) = e^{w\overline{z}/2} \]
and it follows that
\[ k_z(w) = K(w, z)/\sqrt{K(z, z)} = e^{w\overline{z}/2 - |z|^2/4} \]
is a unit vector in $H^2(\mathbb{C}, d\mu)$. We consider the unitary operator

$$(R_a f)(z) = f(az)$$

on $H^2(\mathbb{C}, d\mu)$ for $|a| = 1$.

**Theorem 3.** For $|a| = 1$ and $\Re a < 0$, we have

$$\|R_a - T_\psi\| \geq 1$$

for all $\psi$ such that $\psi K(\cdot, z)$ is in $L^2(\mathbb{C}, d\mu)$ for every $z$ in $\mathbb{C}$.

**Proof.** We consider

$$\|T_\psi - R_a\| \geq \|\langle T_\psi k_z, R_a k_z \rangle - \langle R_a k_z, R_a k_z \rangle\|$$

$$\geq |\langle T_\psi k_z, R_a k_z \rangle - 1|.$$ 

Now,

$$\langle T_\psi k_z, R_a k_z \rangle = \langle \psi \chi_z, K(\cdot, (1 + \bar{a})z) \rangle e^{-|z|^2/2}$$

so we have

$$|\langle T_\psi k_z, R_a k_z \rangle| \leq e^{-|z|^2/2} \|\psi\| \sqrt{K((1 + \bar{a})z, (1 + \bar{a})z)}$$

$$\leq \|\psi\| e^{-|z|^2/2} e^{1 + |a|^2|z|^2/4}$$

$$\leq \|\psi\| e^{|z|^2} e^{2} |\Re a/2|.$$ 

Since $\Re a < 0$, we see that

$$|\langle T_\psi k_z, R_a k_z \rangle| \to 0$$

as $|z| \to \infty$. Thus, $\|T_\psi - R_a\| \geq 1$.

The function $\varphi$ will be chosen to have the form $\varphi(z) = e^{\lambda|z|^2}$ where $\Re \lambda < \frac{1}{4}$ so that $T_\varphi$ makes sense.

**Lemma.** For $\lambda = \frac{1}{5} + i\frac{2}{5}$ and $\varphi(z) = e^{\lambda|z|^2}$, we have $T_\varphi$ unitary with

$$T_\varphi T_\varphi = aR_a$$

for $a = (1 - 2\lambda)^{-1/2} = -\frac{1}{25} + i\frac{24}{25}$.

**Proof.** $\Re \lambda < \frac{1}{4}$ and calculations outlined in [BC2, p. 582] show that $T_\varphi$ is diagonal in the basis

$$e_k = (2^k k!)^{-1/2} z^k, \quad k = 0, 1, \ldots,$$

for $H^2(\mathbb{C}, d\mu)$, with

$$T_\varphi e_k = (1 - 2\lambda)^{-(k+1)} e_k.$$ 

Now

$$\lambda = \frac{1}{5} + i\frac{2}{5}$$

and so

$$T_\varphi T_\varphi e_k = (1 - 2\lambda)^{2(k+1)} e_k$$

$$= a^{k+1} e_k.$$ 

But

$$aR_a e_k = a^{k+1} e_k$$

and we are done.
The series in \( \varphi \circ \psi \) converges uniformly and absolutely. Moreover, for \( r > 1 \)
\[
\| \varphi \circ_r \psi - \sum_{|k| \leq K} \left( \frac{1}{r} \right)^{|k|} \frac{1}{k!} (\overline{\partial}^k \varphi)(\overline{\partial}^k \psi) \|_\infty \leq \frac{1}{r^{K+1}} C(\varphi, \psi, K)
\]
for \( C(\varphi, \psi, K) \) a constant independent of \( r \).

**Theorem 2'.** For \( \varphi, \psi \) polynomials in \( (z_1, ..., z_n, \overline{z}_1, ..., \overline{z}_n) \), we have \( T^{(r)}_\varphi T^{(r)}_\psi \) defined on a dense domain consisting of linear combinations of functions of the form \( \{ p(z) e^{az} : a \in \mathbb{C}^n \text{ and } p(z) \text{ polynomial in } (z_1, ..., z_n) \} \). On this domain
\[
T^{(r)}_\varphi T^{(r)}_\psi = T^{(r)}_{\varphi \circ_r \psi}
\]
for \( \varphi \circ_r \psi \) given by (\dag) and \( \varphi \circ_r \psi \) is polynomial in the \( z_j, \overline{z}_j \).
While Theorems 1 and 2 provide some basis for optimism about the development of a reasonably extensive Berezin-Toeplitz calculus on \( C^n \), the situation is considerably less promising on the classical Bergman space of the disc, \( H^2(D, dA) \), where 
\( D = \{ z \in C : |z| < 1 \} \) and \( dA \) is normalized Lebesgue area measure. In this case, the Bergman kernel function is just \( K(z, w) = (1 - z \overline{w})^{-2} \) and the Berezin-Toeplitz operator \( T_\phi \) on \( H^2(D, dA) \) is given by
\[
(T_\phi f)(z) = \int_D K(z, w) \varphi(w) f(w) \frac{dA(w)}{\pi}.
\]
Direct calculation shows, first, that
\[
T_z T_\varphi = T_{1 + \log |z|^2}.
\]
Moreover,
\[
T^* z T_\varphi = T_{1 + 2 \log |z|^2} + P_0
\]
where \( P_0 f = \int_D f(z) \frac{dA(z)}{\pi} \) and \( P_0 \neq T_\varphi \) for any \( \varphi \). For asymptotic results on composition of Berezin-Toeplitz operators on \( H^2(D, dA) \) see [KL].

References


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