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# NON-TANGENTIAL LIMITS, FINE LIMITS AND THE DIRICHLET INTEGRAL

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ABSTRACT. Let $B$ denote the unit ball in $\mathbb{R}^n$. This paper characterizes the subsets $E$ of $B$ with the property that $\sup_E h = \sup_B h$ for all harmonic functions $h$ on $B$ which have finite Dirichlet integral. It also examines, in the spirit of a celebrated paper of Brelot and Doob, the associated question of the connection between non-tangential and fine cluster sets of functions on $B$ at points of the boundary.

## 1. Introduction

Let $B(x,r)$ denote the open ball of centre $x$ and radius $r$ in Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, and let $B = B(0,1)$. If $\mathcal{A}$ is a collection of harmonic functions on $B$, then it is natural to ask which non-empty subsets $E$ of $B$ have the property that

$$\sup_E h = \sup_B h \quad \text{for all } h \in \mathcal{A}. \tag{1}$$

In the case where $\mathcal{A} = h^\infty$, the collection of all bounded harmonic functions on $B$, it is known (cf. [2]) that (1) holds if and only if $\sigma(E_{NT}) = \sigma(\partial B)$, where $\sigma$ denotes surface area measure on $\partial B$ and $E_{NT}$ is the (Borel) set of points of $\partial B$ which can be approached non-tangentially by a sequence in $E$. In the case where $\mathcal{A} = h^1$, the collection of differences of positive harmonic functions on $B$, it has been shown (see [10] and [8]) that (1) holds if and only if

$$\int_{E(1/2)} |x-y|^{-n}\, dx = +\infty \quad \text{for all } y \in \partial B,$$

where $E(1/2) = \bigcup_{x\in E} B(x,(1-|x|)/2)$. Below we present the corresponding result when $\mathcal{A} = \mathcal{D}$, the collection of all harmonic functions $h$ on $B$ which have finite Dirichlet integral; that is, $\int_B |\nabla h(x)|^2\, dx < +\infty$. We will use $\mathcal{C}(\cdot)$ to denote Newtonian (if $n \geq 3$) or logarithmic (if $n = 2$) capacity on $\mathbb{R}^n$.

**Theorem 1.** *Let $\emptyset \neq E \subseteq B$ and $\mathcal{A} = \mathcal{D}$. Then* (1) *holds if and only if $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$.*

When $n = 2$, Theorem 1 is closely related to a recent result of Stray [13] concerning holomorphic functions in the Dirichlet space. However, our methods are completely different. If $\sigma(E_{NT}) = \sigma(\partial B)$, then it follows easily, by considering

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Poisson integrals in $B$ of suitable potentials, that $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$. In fact, the capacitary condition is much weaker, as the following example shows.

**Example 1.** Let $n = 2$, let $\mathbb{R}^2$ be identified with $\mathbb{C}$ in the usual manner, and let

$$E = \left\{ \left(1 - 2^{-2j}\right) \exp\left(ik2^{1-j}\pi\right) : j \in \mathbb{N} \text{ and } k = 0, 1, ..., 2^j - 1 \right\}.$$

Then $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$, but $\sigma(E_{NT}) = 0$. (See §3.4 for details.)

Brelot and Doob, in their landmark paper [3], were able to relate classical and potential theoretic boundary limit theorems by establishing the relationship between non-tangential and minimal fine cluster sets of functions. Inspired by their work and Theorem 1 we will now provide corresponding results which describe the relationship between non-tangential and fine cluster sets of functions.

Recall that the *fine topology* on $\mathbb{R}^n$ is the coarsest topology for which all superharmonic functions are continuous. A set $A$ is said to be *thin* at a point $x$ if $x$ is not a fine limit point of $A$. (For an account of these concepts see Chapter 1.XI of the book by Doob [7].) By Wiener's criterion, this is equivalent to the condition

$$\sum_k 2^{k(n-2)} \mathcal{C}^*(\{y \in A : 2^{-k-1} \le |x - y| \le 2^{-k}\}) < +\infty \quad (n \ge 3)$$

or

$$\sum_k \frac{k}{\log 1/\mathcal{C}^*(\{y \in A : 2^{-k-1} \le |x - y| \le 2^{-k}\})} < +\infty \quad (n = 2),$$

where $\mathcal{C}^*(\cdot)$ denotes outer (Newtonian or logarithmic) capacity. If, instead, the weaker condition

$$2^{k(n-2)} \mathcal{C}^*(\{y \in A : 2^{-k-1} \le |x - y| \le 2^{-k}\}) \to 0 \text{ as } k \to \infty \quad (n \ge 3)$$

or

$$\frac{k}{\log 1/\mathcal{C}^*(\{y \in A : 2^{-k-1} \le |x - y| \le 2^{-k}\})} \to 0 \text{ as } k \to \infty \quad (n = 2)$$

holds, then $A$ is said to be *semi-thin* at $x$. Now let $f : S \to [-\infty, +\infty]$, where $S \subseteq \mathbb{R}^n$, let $x \in \overline{S}$ and $l \in [-\infty, +\infty]$. We say that $l$ is a *fine* (respectively, *semi-fine*) *cluster value of* $f$ *at* $x$ if, for every neighbourhood $N$ of $l$ in $[-\infty, +\infty]$, the set $f^{-1}(N)$ is not thin (respectively, not semi-thin) at $x$. Finally, we define the non-tangential approach region

$$K(z, \delta, \varepsilon) = \{x : \varepsilon > 1 - |x| > \delta\, |x - z|\} \; (z \in \partial B; 0 < \delta < 1; 0 < \varepsilon < 1).$$

The following result is straightforward to prove, using ideas from [14].

**Proposition 1.** *Let $h$ be a harmonic function on $K(z, \delta, \varepsilon)$ such that*

$$\int_{K(z,\delta,\varepsilon)} (1 - |x|)^{2-n} \, |\nabla h(x)|^2 \, dx < +\infty,$$

*and let $\delta < \delta_1 < 1$. If there is a sequence $(x_k)$ of points in $K(z, \delta_1, \varepsilon)$ such that $x_k \to z$ and $h(x_k) \to l$, then $l$ is a semi-fine (and hence a fine) cluster value of $h$ at $z$.*

Less obvious is the next result, which goes in the opposite direction. If $f : B \to [-\infty, +\infty]$, then the *non-tangential* and *fine cluster sets* of $f$ at a point $z \in \partial B$ are defined respectively by

$$\begin{aligned} C_{NT}(f,z) \quad = \quad & \{l \in [-\infty,+\infty] : f(x_k) \to l \text{ for some sequence } (x_k) \text{ of} \\ & \text{points in } B \text{ which approaches } z \text{ non-tangentially}\} \end{aligned}$$

and

$$C_F(f,z) = \{l \in [-\infty,+\infty] : l \text{ is a fine cluster value of } f \text{ at } z\}.$$

**Theorem 2.** *Let $f : B \to [-\infty, +\infty]$. Then there is a Euclidean-$G_\delta$ set $A \subseteq \partial B$ such that $\mathcal{C}(A) = \mathcal{C}(\partial B)$ and $C_F(f,z) \subseteq C_{NT}(f,z)$ whenever $z \in A$.*

Theorem 2 can be viewed as a fine topology analogue of a maximality theorem of Collingwood and Lohwater (see Theorem 4.10 in [5]). We note that Mizuta [12] has also considered the relationship between non-tangential, normal and fine cluster sets. His results, which are specific to harmonic functions satisfying a Dirichlet-type integral condition, are of a completely different nature. The sharpness of Theorem 2, even for harmonic functions, is demonstrated by the next result.

**Theorem 3.** *Let $A \subseteq \partial B$ be a Euclidean-$G_\delta$ set such that $\mathcal{C}(A) = \mathcal{C}(\partial B)$. Then there is a harmonic function $h$ on $B$ such that $C_F(h,z) = [-\infty, +\infty]$ and $C_{NT}(h,z) = \{0\}$ whenever $z \in \partial B \backslash A$.*

Proposition 1 and Theorems 1 and 2 will be proved in §§2 - 4 respectively. Theorem 3 will be proved in §5 using recent results concerning approximation by harmonic functions.

## 2. Proof of Proposition 1

Let $h$ be a harmonic function on $K(z,\delta,\varepsilon)$ such that

$$\int_{K(z,\delta,\varepsilon)} (1-|x|)^{2-n} |\nabla h(x)|^2 \, dx < +\infty,$$

let $\delta < \delta_0 < \delta_1 < 1$ and $0 < \varepsilon_1 < \varepsilon_0 < \varepsilon$. Then there is a decreasing function $\phi : (0,+\infty) \to (0,+\infty)$ such that $\phi(t) \le 2\phi(2t)$ for all $t$ and $\phi(t) \to +\infty$ as $t \to 0$, and such that $L < +\infty$, where

$$L = \int_{K(z,\delta,\varepsilon)} (1-|x|)^{2-n} \phi(1-|x|) |\nabla h(x)|^2 \, dx.$$

Let $\eta_0 > 0$ be small enough so that $B(x, \eta_0(1-|x|)) \subseteq K(z,\delta,\varepsilon)$ whenever $x \in K(z,\delta_0,\varepsilon_0)$. By the volume mean value inequality, applied to the subharmonic function $|\nabla h|^2$ and the ball $B(x,\eta_0(1-|x|))$, we see that there is a positive constant $M$, depending only on $\eta_0$ and $n$, such that

$$(1-|x|)^{2-n} \phi(1-|x|) |\nabla h(x)|^2 \le ML(1-|x|)^{-n} \quad (x \in K(z,\delta_0,\varepsilon_0)),$$

and hence

$$(1-|x|)|\nabla h(x)| \le \sqrt{\frac{ML}{\phi(1-|x|)}} \quad (x \in K(z,\delta_0,\varepsilon_0)). \tag{2}$$

Let $\eta_1 > 0$ be small enough so that $B(x,\eta_1(1-|x|)) \subseteq K(z,\delta_0,\varepsilon_0)$ whenever $x \in K(z,\delta_1,\varepsilon_1)$. Also, let $(x_k)$ be a sequence of points in $K(z,\delta_1,\varepsilon_1)$ such that

$x_k \to z$ and $h(x_k) \to l$, and let $D = \bigcup_k B(x_k, \eta_1(1-|x_k|))$. It follows from (2) and the mean value theorem of differential calculus that

$$h(x) \to l \quad (x \to z; x \in D),$$

in view of the fact that $\phi(t) \to +\infty$ as $t \to 0$. Since $\mathcal{C}(B(x,r))$ is $r^{n-2}$ $(n \geq 3)$ or $r$ $(n = 2)$, it is clear that $D$ is not semi-thin at $z$. Thus $l$ is a semi-fine (and hence a fine) cluster value of $h$ at $z$.

## 3. Proofs of Theorem 1 and Example 1

3.1. For the proofs of Theorems 1 - 3 we will assume that $n \geq 3$ and omit the minor modifications required to adapt our arguments to the plane. Before proving Theorem 1 we will assemble some preliminary observations. A function $f : S \to [-\infty, +\infty]$ is said to have a *fine limit* at a fine limit point $x$ of $S$ if $C_F(f,x)$ consists of only one point. If, further, $C_F(f,x) = \{f(x)\}$, then $f$ is said to be *finely continuous* at $x$.

**Lemma A.** *Let $f : B \to [0, +\infty)$ be integrable on $B$ and let $0 < \varepsilon < 1$. Then there is a set $Y \subset \partial B$, of zero $(n-2)$-dimensional Hausdorff measure, such that*

$$\int_{K(z,\delta,\varepsilon)} (1-|x|)^{2-n} f(x)dx < +\infty \quad (z \in \partial B \backslash Y; 0 < \delta < 1).$$

**Theorem A.** *If $h \in \mathcal{D}$, then there is a finite-valued extension $\overline{h}$ of $h$ to $\overline{B}$, and a polar set $Z \subset \partial B$, such that*

$$h(rz) \to \overline{h}(z) \quad (r \to 1-;\ z \in \partial B \backslash Z) \tag{3}$$

*and $\overline{h}$ is finely continuous at each point of $\partial B \backslash Z$.*

In proving Lemma A it is clearly enough to show that, for a fixed choice of $\delta$, there is a set $Y_\delta \subset \partial B$, of zero $(n-2)$-dimensional Hausdorff measure, such that

$$\int_{K(z,\delta,\varepsilon)} (1-|x|)^{2-n} f(x)dx < +\infty \quad (z \in \partial B \backslash Y_\delta).$$

This can be done by imitating the proof of the analogous result for functions on half-spaces, which may be found in Lemma 5 of [14].

Theorem A is taken from Deny ([6], Chap. IV, Théorème 3), who proved the result for the more general class of Beppo Levi functions $h$ on $B$. Actually, Deny's result asserts only that $\overline{h}$ has a fine limit at each point of $\partial B \backslash Z$, but the proof makes it clear that $\overline{h}$ is actually finely continuous at each point of a set which has this form.

**Lemma 1.** *The unit sphere $\partial B$, with the topology induced on it by the fine topology on $\mathbb{R}^n$, is a Baire space.*

To prove Lemma 1, we note that the fine topology on $\mathbb{R}^n$ has a base which consists of (Euclidean) compact sets. The same is therefore also true of the topology it induces on $\partial B$. We can now adopt the argument given on pp. 167, 168 of [7] to see that this space is Baire.

If $A$ is a bounded set in $\mathbb{R}^n$, then we use $\widehat{R}_1^A$ to denote the capacitary potential of $A$. Thus the Riesz measure $\nu_A$ associated with $\widehat{R}_1^A$ satisfies $\nu_A(\mathbb{R}^n) = \mathcal{C}^*(A)$.

**Lemma 2.** *Let $A \subseteq \partial B$ and $z \in \partial B$. If $A$ is thin at $z$, then $\widehat{R}_1^A(z) < 1$.*

To see this we note that, if $z \in \overline{A}$, then there is a positive superharmonic function $u$ on $\mathbb{R}^n$ such that

$$\liminf_{x \to z, x \in A} u(x) > u(z).$$

If we define $v(x) = u(x) + a\,|x+z|^{2-n}$, where $a$ is a suitably chosen positive number, then $v$ is a positive superharmonic function on $\mathbb{R}^n$ such that

$$\inf_{A \setminus \{z\}} v > v(z).$$

It follows easily that $\widehat{R}_1^A(z) < 1$, and this inequality is obviously also true when $z \notin \overline{A}$.

3.2. We now turn to the "if" part of Theorem 1. Suppose that $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$ and suppose further, for the sake of contradiction, that $E_{NT}$ is thin at some point $z$ of $\partial B$. Then $\widehat{R}_1^{E_{NT}}(z) < 1$, by Lemma 2, and this leads to the contradictory conclusion that $\mathcal{C}(E_{NT}) = \widehat{R}_1^{E_{NT}}(0) < 1 = \mathcal{C}(\partial B)$. Hence the fine closure of $E_{NT}$ is all of $\partial B$.

Now let $h \in \mathcal{D}$. By Theorem A, there exist a function $\bar{h} : \bar{B} \to \mathbb{R}$ and a polar set $Z \subset \partial B$ such that $\bar{h}\,|_B = h$ and (3) holds, and such that $\bar{h}$ is finely continuous at all points of $\partial B \setminus Z$. Let $M = \sup_E h$ and suppose, to avoid triviality, that $M < +\infty$. By Proposition 1 and Lemma A (with $f = |\nabla h|^2$), there is a polar subset $Y$ of $\partial B$ such that $\overline{h} \le M$ on $E_{NT} \setminus (Y \cup Z)$. (Here we are using the fact that a bounded set of finite $(n-2)$-dimensional Hausdorff measure is polar; see [4], Chap. IV, Theorem 1). The fine closure of $E_{NT} \setminus (Y \cup Z)$ is also $\partial B$, so $\overline{h} \le M$ on $\partial B \setminus Z$. We note that the subharmonic function $h^2$ has a harmonic majorant on $B$ because

$$\int_B (1 - |x|)\,(\Delta\left(h^2\right))(x)dx \le 2 \int_B |\nabla h(x)|^2\,dx < +\infty.$$

Hence $h$ is equal to the Poisson integral of a function $g$ on $\partial B$ (see [7], 1.II.14). Further, from (3) and Fatou's boundary limit theorem, $g = \overline{h}\,|_{\partial B}$ almost everywhere $(\sigma)$ on $\partial B$, and so $h \le M$ on $B$. Thus (1) holds and we have now established the "if" part of Theorem 1.

3.3. Conversely, suppose that (1) holds and let

$$F(r) = \partial B \cap \left( \bigcup_{x \in E \setminus B(0,r)} B(x, 2(1-|x|)) \right) \quad (0 < r < 1).$$

Clearly (1) implies that $\bar{E} \cap \partial B \neq \emptyset$, so $F(r) \neq \emptyset$ for all $r$. Suppose further, for the sake of contradiction, that there exists $r$ such that the fine closure of $F(r)$ is a proper subset of $\partial B$. Then the function $h = 1 - \widehat{R}_1^{F(r)}$, which is harmonic on $B$, is strictly positive there, in view of Lemma 2. We note (see [7], 1.IV.5) that there is an increasing sequence $(u_k)$ of $C^\infty$ Newtonian potentials, with associated measures $(\mu_k)$, such that $u_k \uparrow \widehat{R}_1^{F(r)}$ and $u_k$ is harmonic outside $\overline{B(0, 1+1/k)} \setminus B(0, 1-1/k)$ and that

$$\int_{\mathbb{R}^n} |\nabla u_k(x)|^2\,dx = a_n \int u_k d\mu_k$$

by Green's first identity, where $a_n$ is a positive constant depending only on $n$. Since $|\nabla u_k| \to |\nabla h|$ locally uniformly on $B$, it follows from the reciprocity law that

$$\int_B |\nabla h(x)|^2 \, dx \leq a_n \int \widehat{R}_1^{F(r)} d\nu_{F(r)} \leq a_n \mathcal{C}(F(r)) < +\infty;$$

that is, $h\,|_B \in \mathcal{D}$.

Let $M = \sup_B h$ and let $v_r$ denote the harmonic measure of $\partial B \backslash F(r)$ in $B$. The definition of $F(r)$ ensures that there is a constant $c \in (0,1)$, independent of $E$, such that $v_r \leq c$ on $E \backslash B(0,r)$. Since $h = 1 - \widehat{R}_1^{F(r)} = 0$ on $F(r)$, it follows that $h \leq cM$ on $E \backslash B(0,r)$, and so

$$\sup_E h \leq \max \left\{ cM, \sup_{B(0,r)} h \right\} < M = \sup_B h.$$

This contradicts (1). Hence, for every $r \in (0,1)$, the fine closure of $F(r)$ is all of $\partial B$.

Let

$$F = \bigcap_{k=1}^{\infty} F\left(\frac{k}{k+1}\right).$$

Then, by Lemma 1, the fine closure of $F$ is also all of $\partial B$. Finally, it is clear that $F \subseteq E_{NT}$, so the fine closure of $E_{NT}$ is also $\partial B$. Thus $\hat{R}_1^{E_{NT}} = \hat{R}_1^{\partial B}$, and hence $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$, as required.

3.4. We now present the details of Example 1. Let $E$ be the set defined there. Then

$$E_{NT} = \bigcup_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{j=m}^{\infty} \bigcup_{k=0}^{2^j-1} F_{j,k,l},$$

where

$$F_{j,k,l} = \left\{ e^{i\theta} : \left|\theta - k2^{1-j}\pi\right| < l2^{-2j} \right\}.$$

The set $F_{j,k,l}$ is an open arc of the unit circle and $\mathcal{C}(F_{j,k,l}) \geq l2^{-2j-2}$ (see [11], p. 173, (2.4.4)). It follows easily from Wiener's criterion that the set

$$A_{l,m} = \bigcup_{j=m}^{\infty} \bigcup_{k=0}^{2^j-1} F_{j,k,l}$$

is finely dense in the circle. Hence, for any $l \in \mathbb{N}$, the set $\bigcap_{m=1}^{\infty} A_{l,m}$ is finely dense in the circle, by Lemma 1. It follows that $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$. However,

$$\sigma\left(A_{l,m}\right) \leq \sum_{j=m}^{\infty} \sum_{k=0}^{2^j-1} l2^{1-2j} = l2^{2-m},$$

so $\sigma\left(\bigcap_{m=1}^{\infty} A_{l,m}\right) = 0$ and hence $\sigma(E_{NT}) = 0$, as claimed.

## 4. Proof of Theorem 2

Let $f : B \to [-\infty, +\infty]$ and

$$D = \{z \in \partial B : C_F(f,z) \backslash C_K(f,z) \neq \emptyset\},$$

where, for each $z \in \partial B$,

$$C_K(f,z) = \{l \in [-\infty,+\infty] : f(x_k) \to l \text{ for some sequence } (x_k) \text{ of points in } K(z,1/2,1/2) \text{ such that } x_k \to z\}.$$

Further, let $\mathcal{I}$ denote the collection of closed intervals of $[-\infty, +\infty]$ with endpoints in $\mathbb{Q} \cup \{-\infty, +\infty\}$. Suppose that $z \in D$. Since $C_K(f,z)$ is a compact subset of $[-\infty, +\infty]$, we can find $I \in \mathcal{I}$, a finite union $J$ of intervals from $\mathcal{I}$, and $\varepsilon \in \mathbb{Q} \cap (0,1)$ such that

$$I \cap C_F(f,z) \neq \emptyset, \quad I \cap J = \emptyset \tag{4}$$

and

$$f\left(K\left(z, \frac{1}{2}, \varepsilon\right)\right) \subseteq J. \tag{5}$$

If $I$, $J$ and $\varepsilon$ are as above, then we say that $z \in D(I,J,\varepsilon)$. Thus

$$D \subseteq \bigcup_{I,J,\varepsilon} D(I,J,\varepsilon), \tag{6}$$

where the union is over all possible choices of $I$, $J$, $\varepsilon$.

Now suppose that one of the sets in this union, $D_0 = D(I_0, J_0, \varepsilon_0)$ say, has the property that its (Euclidean) closure has non-empty interior $U$ with respect to the fine-induced topology on $\partial B$, let

$$E = B \setminus \left( \bigcup_{z \in D_0} K\left(z, \frac{1}{2}, \varepsilon_0\right)\right) \tag{7}$$

and $h = 1 - \widehat{R}_1^{\partial B \backslash U}$. Then $h = 0$ on $\partial B \backslash U$ since the latter set is finely perfect. Also, $\overline{E} \cap \partial B \neq \emptyset$ by (4) and (5), and

$$B(x, 2(1-|x|)) \cap U = \emptyset \qquad (x \in E \backslash \overline{B(0, 1-\varepsilon_0)})$$

by the definition of $E$. Reasoning as in §3.3, we now see that $h > 0$ on $B$, and on $U$ by Lemma 2, but $\sup_E h < \sup_B h$. Let $\sup_E h < M < \sup_B h$ and $V = \{x \in \mathbb{R}^n : h(x) > M\}$, and choose $z_1 \in U$ such that $h(z_1) > M$. Then $V$ is a fine neighbourhood of $z_1$ and $V \cap E = \emptyset$. It now follows from (5) and (7) that $C_F(f, z_1) \subseteq J_0$, but this contradicts (4). Thus $U = \emptyset$.

Let

$$A = \bigcap_{I,J,\varepsilon} \left(\partial B \backslash \overline{D(I,J,\varepsilon)}\right). \tag{8}$$

Thus $A$ is a (Euclidean) $G_\delta$-set and each of the sets $\partial B \backslash \overline{D(I,J,\varepsilon)}$ is open and dense in the fine-induced topology on $\partial B$. By Lemma 1, the fine closure of $A$ is $\partial B$ and hence $\mathcal{C}(A) = \mathcal{C}(\partial B)$. Since $C_K(f,z) \subseteq C_{NT}(f,z)$, it follows that

$$C_F(f,z) \subseteq C_{NT}(f,z) \quad (z \in \partial B \backslash D).$$

From (6) and (8) we see that $A \subseteq \partial B \backslash D$, and so Theorem 2 is established.

## 5. Proof of Theorem 3

Let $A \subseteq \partial B$ be a Euclidean-$G_\delta$ set such that $\mathcal{C}(A) = \mathcal{C}(\partial B)$. As we argued at the beginning of §3.2, it follows that $A$ is finely dense in $\partial B$. Thus there is an increasing sequence $(F_k)$ of compact sets such that $\partial B \backslash A = \bigcup_k F_k$ and such that $\partial B \backslash F_k$ is finely dense in $\partial B$ for each $k$. To avoid trivialities we may assume that the sets $\partial B \backslash A$ and $F_1$ are non-empty.

Let $g_k : \partial B \to [0,1)$ be defined by

$$g_k(z) = \frac{\{\mathrm{dist}(z, F_k)\}^2}{5} \quad (z \in \partial B; k \in \mathbb{N}).$$

Then the sets

$$E_k = \left\{ rz : z \in \partial B \backslash F_k \text{ and } 1 - r = \frac{g_k(z)}{3k-1} \right\} \quad (k \in \mathbb{N}),$$

$$D_1 = B \cap \{ rz : z \in \partial B \text{ and } 1 - r \geq g_1(z) \},$$

and

$$D_{k+1} = B \cap \left\{ rz : z \in \partial B \text{ and } \frac{g_k(z)}{3k} \geq 1 - r \geq \frac{g_{k+1}(z)}{3k+1} \right\} \quad (k \in \mathbb{N})$$

are all closed relative to $B$ and are pairwise disjoint. Let

$$E = \left( \bigcup_k E_k \right) \cup \left( \bigcup_k D_k \right) \tag{9}$$

and let $B^*$ denote the Alexandroff (one-point) compactification of $B$. Since, for a given value of $r$ in $(0,1)$, only finitely many sets in the union in (9) meet $B(0,r)$, the set $E$ is also closed relative to $B$.

If we define $u : E \to \mathbb{R}$ by

$$u(x) = \begin{cases} 0 & \text{if} \quad x \in \bigcup_k D_k, \\ q_k & \text{if} \quad x \in E_k; k \geq 1, \end{cases} \tag{10}$$

where $(q_k)$ is an enumeration of $\mathbb{Q}$, then $u$ extends to a locally constant (and hence harmonic) function on an open set which contains $E$. It is easy to check that $B^* \backslash E$ is connected and locally connected (see §3.2 of [9] for a discussion of local connectedness in this context). We note that, if $z \in \partial B \backslash A$ and $0 < \delta < 1$, then there exists a (smallest) number $k_0$ such that $z \in F_{k_0}$ and a number $\varepsilon_{z,\delta}$ in $(0,1)$ such that

$$K(z, \delta, \varepsilon_{z,\delta}) \subseteq D_{k_0} \subseteq \bigcup_k D_k \quad (0 < \delta < 1). \tag{11}$$

We also claim that, if $z \in \partial B \backslash A$, then $E_k$ is non-thin at $z$ for all sufficiently large $k$. To see this, we choose $k_0$ such that $z \in F_{k_0}$ and suppose that $E_k$ is thin at $z$ for some $k \geq k_0$. Since the radial projection map from $E_k$ to $\partial B \backslash F_k$ is a Lipschitz map with Lipschitz constant 2, we have

$$\begin{aligned} & \mathcal{C}\left( \left\{ y \in \partial B \backslash F_k : 2^{-j-1} \leq |y - z| \leq 2^{-j} \right\} \right) \\ \leq \quad & 2^{n-2} \mathcal{C}\left( \left\{ y \in E_k : 2^{-j-2} \leq |y - z| \leq 2^{-j+1} \right\} \right) \end{aligned}$$

for each $j \in \mathbb{N}$, by standard contraction and dilation properties of Newtonian capacity (see [11], Chap. 2, §3). Hence, by Wiener's criterion, we obtain the contradictory conclusion that $\partial B \backslash F_k$ is also thin at $z$. Thus our claim is verified.

We now apply a recent harmonic approximation result (see [1], or Corollary 3.10 in [9]) to observe that there is a harmonic function $h$ on $B$ such that

$$|h(x) - u(x)| < 1 - |x| \quad (x \in E).$$

It follows from (10) and (11) that $C_{NT}(h, z) = \{0\}$ whenever $z \in \partial B \backslash A$. For such $z$, it also follows from (10) and the claim verified in the previous paragraph that $C_F(h, z)$ contains all but a finite number of the rationals, and so $C_F(h, z) = [-\infty, +\infty]$. The proof of Theorem 3 is now complete.

## References


1. D. H. Armitage and M. Goldstein, *Tangential harmonic approximation on relatively closed sets*, Proc. London Math. Soc. (3), **68** (1994), 112-126. MR **94i:**31005
2. F. F. Bonsall, *Domination of the supremum of a bounded harmonic function by its supremum over a countable subset*, Proc. Edinburgh Math. Soc., **30** (1987), 471-477. MR **88k:**31001
3. M. Brelot and J. L. Doob, *Limites angulaires et limites fines*, Ann. Inst. Fourier Grenoble, **13** (1963), 395-415. MR **33:**4299
4. L. Carleson, *Selected problems on exceptional sets*, Van Nostrand, Princeton, 1967. MR **37:**1576
5. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge Univ. Press, Cambridge, 1966. MR **38:**325
6. J. Deny, *Les potentiels d'énergie finie*, Acta Math. **82** (1950), 107-183. MR **12:**98e
7. J. L. Doob, *Classical potential theory and its probabilistic counterpart*, Springer, New York, 1983. MR **85k:**31001
8. S. J. Gardiner, *Sets of determination for harmonic functions*, Trans. Amer. Math. Soc. **338** (1993), 233-243. MR **93j:**31005
9. S. J. Gardiner, *Harmonic approximation*, London Math. Soc. Lecture Note Series 221, Cambridge Univ. Press, Cambridge, 1995. MR **96j:**31001
10. W. K. Hayman and T. J. Lyons, *Bases for positive continuous functions*, J. London Math. Soc. (2) **42** (1990), 292-308. MR **92a:**31002
11. N. S. Landkof, *Foundations of modern potential theory*, Springer, Berlin, 1972. MR **50:**2520
12. Y. Mizuta, *On the behaviour of harmonic functions near a hyperplane*, Analysis **2** (1982), 203-218.
13. A. Stray, *Simultaneous approximation in the Dirichlet space*, in: Advances in Multivariate Approximation, ed. W. Haussmann et al., Wiley, Berlin, 1999, pp. 307-319.
14. H. Wallin, *On the existence of boundary values of a class of Beppo Levi functions*, Trans. Amer. Math. Soc. **120** (1965), 510-525. MR **32:**5911


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