THE CANONICAL SOLUTION OPERATOR TO ∂
RESTRICTED TO BERGMAN SPACES

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Abstract. We first show that the canonical solution operator to ∂ restricted to (0,1)-forms with holomorphic coefficients can be expressed by an integral operator using the Bergman kernel. This result is used to prove that in the case of the unit disc in \( \mathbb{C} \) the canonical solution operator to ∂ restricted to (0,1)-forms with holomorphic coefficients is a Hilbert-Schmidt operator. In the sequel we give a direct proof of the last statement using orthonormal bases and show that in the case of the polydisc and the unit ball in \( \mathbb{C}^n \), \( n > 1 \), the corresponding operator fails to be a Hilbert-Schmidt operator. We also indicate a connection with the theory of Hankel operators.

1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \) and let \( A^2(\Omega) \) denote the Bergman space of all holomorphic functions \( f : \Omega \to \mathbb{C} \) such that
\[
\int_{\Omega} |f(z)|^2 d\lambda(z) < \infty,
\]
where \( \lambda \) denotes the Lebesgue measure in \( \mathbb{C}^n \).

We solve the \( \partial \)-equation \( \partial u = g \), where \( g = \sum_{j=1}^n g_j \partial z_j \) is a (0,1)-form with coefficients \( g_j \in A^2(\Omega) \), \( j = 1, \ldots, n \).

It is pointed out in [FS1] that in the proof that compactness of the solution operator for ∂ on (0,1)-forms implies that the boundary of \( \Omega \) does not contain any analytic variety of dimension greater than or equal to 1, only the fact that there is a compact solution operator to ∂ on the (0,1)-forms with holomorphic coefficients is used. In this case compactness of the solution operator restricted to (0,1)-forms with holomorphic coefficients already implies compactness of the solution operator on general (0,1)-forms.

The question of compactness is of interest for various reasons; see [FS2] for an excellent survey.

A similar situation appears in [SSU] where the Toeplitz \( C^* \)-algebra \( T(\Omega) \) is considered and the relation between the structure of \( T(\Omega) \) and the ∂-Neumann problem is discussed (see [SSU], Corollary 4.6).

We first show that the canonical solution operator to ∂ restricted to (0,1)-forms with holomorphic coefficients can be expressed by an integral operator using the

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Bergman kernel. This result is used to prove that in the case of the unit disc in \( \mathbb{C} \), the canonical solution operator to \( \bar{\partial} \) restricted to \((0,1)\)-forms with holomorphic coefficients is a Hilbert-Schmidt operator.

In the sequel we give a direct proof of the last statement using orthonormal bases and show that in the case of the polydisc and the unit ball in \( \mathbb{C}^n, \ n \geq 2 \), the corresponding operator fails to be a Hilbert-Schmidt operator.

The canonical solution operator to \( \bar{\partial} \) restricted to \((0,1)\)-forms with holomorphic coefficients can also be interpreted as the Hankel operator

\[
H_z(g) = (I - P)(\overline{g}),
\]

where \( P : L^2(\Omega) \to A^2(\Omega) \) denotes the Bergman projection. See [A], [AFP], [B], [J], [R], [W] and [Z] for details.

2. The Integral Representation

The canonical solution operator

\[
S_1 : A^2_{(0,1)}(\Omega) \to L^2(\Omega)
\]

has the properties \( \overline{\partial}S_1(g) = g \) and \( S_1(g) \perp A^2(\Omega) \).

**Proposition 1.** The canonical solution operator

\[
S_1 : A^2_{(0,1)}(\Omega) \to L^2(\Omega)
\]

has the form

\[
S_1(g)(z) = \int_\Omega B(z, w)(g(w), z - w) \, d\lambda(w),
\]

where \( B \) denotes the Bergman kernel of \( \Omega \) and

\[
\langle g(w), z - w \rangle = \sum_{j=1}^n g_j(w)(\overline{z}_j - \overline{w}_j)
\]

for \( z = (z_1, \ldots, z_n) \) and \( w = (w_1, \ldots, w_n) \).

Integral operators of similar type have been used to settle questions on compactness of the solution operator to \( \bar{\partial} \); see [CD] and [L].

**Proof.** Let \( v(z) = \sum_{j=1}^n \overline{z}_j g_j(z) \). Then it follows that

\[
\overline{\partial}v = \sum_{j=1}^n \frac{\partial v}{\partial \overline{z}_j} d\overline{z}_j = \sum_{j=1}^n g_j d\overline{z}_j = g.
\]

Hence the canonical solution operator \( S_1 \) can be written in the form \( S_1(g) = v - P(v) \), where \( P : L^2(\Omega) \to A^2(\Omega) \) is the Bergman projection. If \( \tilde{v} \) is another solution to \( \bar{\partial}u = g \), then \( v - \tilde{v} \in A^2(\Omega) \); hence \( v = \tilde{v} + h \), where \( h \in A^2(\Omega) \). Therefore

\[
v - P(v) = \tilde{v} + h - P(\tilde{v}) - P(h) = \tilde{v} - P(\tilde{v}).
\]

Since \( g_j \in A^2(\Omega), \ j = 1, \ldots, n \), we have

\[
g_j(z) = \int_\Omega B(z, w) g_j(w) \, d\lambda(w).
\]
Now we get
\[
S_1(g)(z) = \sum_{j=1}^{n} \bar{z}_j g_j(z) - \int_{\Omega} B(z, w) \left( \sum_{j=1}^{n} \bar{w}_j g_j(w) \right) d\lambda(w)
\]
\[
= \int_{\Omega} \left[ \left( \sum_{j=1}^{n} \bar{z}_j g_j(w) \right) B(z, w) - \left( \sum_{j=1}^{n} \bar{w}_j g_j(w) \right) B(z, w) \right] d\lambda(w)
\]
\[
= \int_{\Omega} B(z, w) \langle g(w), z - w \rangle d\lambda(w).
\]

\[\square\]

Remark. It is pointed out that a \((0,1)\)-form \(g = \sum_{j=1}^{n} g_j \partial \bar{z}_j\) with holomorphic coefficients is not invariant under the pull back by a holomorphic map \(F = (F_1, \ldots, F_n): \Omega_1 \to \Omega\). It can be shown that
\[
F^* g = \sum_{j=1}^{n} \left( \sum_{l=1}^{n} g_l \frac{\partial F_l}{\partial \bar{z}_j} \right) \partial \bar{z}_j
\]
and the expressions \(\frac{\partial F_l}{\partial \bar{z}_j}\) are not holomorphic.

Nevertheless it is true that \(\overline{\partial} u = g\) implies \(\overline{\partial} (u \circ F) = F^* g\).

Now let \(\omega\) be a holomorphic \((n,n-1)\)-form, i.e.
\[
\omega = \bar{\omega} \, dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n,
\]
where \(\bar{\omega} \in A^2(\Omega)\). In this case we can express the canonical solution to \(\overline{\partial} u = \omega\) in the following form:

**Proposition 2.** Let \(u\) be the \((n,n-1)\)-form
\[
u = \sum_{j=1}^{n} u_j \, dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge [d\bar{z}_j] \wedge \cdots \wedge d\bar{z}_n,
\]
where
\[
u_j(z) = \frac{(-1)^{n+j-1}}{n} \int_{\Omega} (\bar{z}_j - \bar{w}_j) B(z, w) \bar{\omega}(w) d\lambda(w).
\]
Then \(u_j \perp A^2(\Omega), \ j = 1, \ldots, n, \) and \(\overline{\partial} u = \omega\).

**Proof.** It follows that
\[
u_j(z) = \frac{(-1)^{n+j-1}}{n} (\bar{z}_j \bar{\omega}(z) - \nu(\bar{w}_j \bar{\omega})(z)).
\]
From this we obtain
\[
\frac{\partial \nu_j}{\partial \bar{z}_k} = \frac{(-1)^{n+j-1}}{n} \left( \frac{\partial \bar{z}_j}{\partial \bar{z}_k} \bar{\omega} + \bar{z}_j \frac{\partial \bar{\omega}}{\partial \bar{z}_k} \right) = \frac{(-1)^{n+j-1}}{n} \delta_{jk} \bar{\omega},
\]
where $\delta_{jk}$ is the Kronecker delta symbol. Hence

\[
\bar{\partial} u = \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial u_j}{\partial \bar{z}_k} \, d\bar{z}_k \wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge [d\bar{z}_j] \wedge \cdots \wedge d\bar{z}_n
\]

\[
= \sum_{k=1}^{n} \sum_{j=1}^{n} (-1)^{n+j-1} / n \, \delta_{jk} \, \bar{\omega} \, d\bar{z}_k \\
\wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge [d\bar{z}_j] \wedge \cdots \wedge d\bar{z}_n
\]

\[
= \bar{\omega} \, dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.
\]

**Remark.** The pull back by a holomorphic map $F$ has in this case the form

\[
F^* \omega = \left| \det \frac{\partial F}{\partial z_k} \right|^2 \bar{\omega} \, dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.
\]

**Proposition 3.** Suppose that $\Omega$ is a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^n$. Let $T : L^2_{(0,1)}(\Omega) \to L^2(\Omega)$ be the operator defined by

\[
T(f)(z) = \int_{\Omega} B(z,w) \langle f(w), z-w \rangle \, d\lambda(w), \quad f \in L^2_{(0,1)}(\Omega).
\]

Then $T$ is a compact operator.

This follows from Theorem 1 in [CD].

The last result implies that the restriction of $T$ to $A^2_{(0,1)}(\Omega)$, which is the canonical solution operator to $\bar{\partial}$, is also a compact operator. This fact also follows from [C], where it is shown that the $\bar{\partial}$-Neumann operator is compact.

Next we consider the integral kernel of the canonical solution operator $S_1$ for the unit disc $D$ in $\mathbb{C}$ and prove that this kernel is square integrable over $D \times D$.

**Proposition 4.**

\[
\int_D \int_D \frac{|z-w|^2}{|1-z\bar{w}|^4} \, d\lambda(z) \, d\lambda(w) < \infty.
\]

**Proof.** It is easily seen that $|z-w| \leq |1-z\bar{w}|$ for $z, w \in D$. Hence we get

\[
\int_D \int_D \frac{|z-w|^2}{|1-z\bar{w}|^4} \, d\lambda(z) \, d\lambda(w) \leq \int_D \int_D \frac{1}{|1-z\bar{w}|^4} \, d\lambda(z) \, d\lambda(w).
\]

Using polar coordinates $z = re^{i\theta}$ and $w = se^{i\phi}$ we can write the last integral in the following form:

\[
\int_D \int_D \frac{1}{|1-z\bar{w}|^4} \, d\lambda(z) \, d\lambda(w) = \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{r \, s \, d\theta \, d\phi \, dr \, ds}{1 - 2r \, s \, \cos(\theta - \phi) + r^2 \, s^2}
\]

\[
= \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2 \, s^2}{1 - 2r \, s \, \cos(\theta - \phi) + r^2 \, s^2} \, r \, s \, d\theta \, d\phi \, dr \, ds.
\]

Integration of the Poisson kernel with respect to $\theta$ yields

\[
\int_0^{2\pi} \frac{1 - r^2 \, s^2}{1 - 2r \, s \, \cos(\theta - \phi) + r^2 \, s^2} \, d\theta = 2\pi, \quad 0 < r < 1.
\]
Hence
\[ \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2 s^2}{1 - 2r s \cos(\theta - \phi) + r^2 s^2} \frac{r s}{1 - r^2 s^2} d\theta \ d\phi \ dr \ ds = (2\pi)^2 \int_0^1 \int_0^1 \frac{r s}{1 - r^2 s^2} dr \ ds = -(2\pi)^2 \int_0^1 \log(1 - s^2) \frac{ds}{2s} < \infty. \]

**Remark.** The last proposition implies that the operator \( T : L^2(\mathbb{D}) \to L^2(\mathbb{D}) \) defined by
\[ T(f)(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{\overline{w} - z}{(1 - z \overline{w})^2} f(w) \, d\lambda(w), \]
for \( f \in L^2(\mathbb{D}) \), is a Hilbert-Schmidt operator; see [MV], 16.12.

If we restrict this operator to the closed subspace \( A^2(\mathbb{D}) \) we obtain

**Proposition 5.** The canonical solution operator to \( \overline{\partial} \),
\[ S_1 : A^2(\mathbb{D}) \to L^2(\mathbb{D}), \]
is a Hilbert-Schmidt operator.

**Proof.** By [MV], 16.8, we have to show that there exists a complete orthonormal system \( (\phi_k)_{k=0}^\infty \) of \( A^2(\mathbb{D}) \) such that
\[ \sum_{k=0}^\infty \| S_1(\phi_k) \|^2 < \infty. \]

For this purpose we take a complete orthonormal system \( (\phi_k)_{k=0}^\infty \) of \( A^2(\mathbb{D}) \) and extend it to a complete orthonormal system \( (\psi_j)_{j=0}^\infty \) of \( L^2(\mathbb{D}) \). Again by [MV], 16.8, and Proposition 3, it follows that
\[ \sum_{j=0}^\infty \| T(\psi_j) \|^2 < \infty, \]
which implies that
\[ \sum_{k=0}^\infty \| S_1(\phi_k) \|^2 < \infty. \]

\[ \square \]

### 3. Hilbert-Schmidt Operators

Now we show directly that the canonical solution operator to \( \overline{\partial} \),
\[ S_1 : A^2_{(0,1)}(\mathbb{D}) \to L^2(\mathbb{D}), \]
is a Hilbert-Schmidt operator if \( \mathbb{D} \) is the open unit disc in \( \mathbb{C} \), and is not Hilbert-Schmidt if \( \mathbb{B} \) is the open unit ball in \( \mathbb{C}^n \) for \( n > 1 \).

Let \( \mathbb{D} \subset \mathbb{C} \) and let \( \| . \| \) denote the norm in \( A^2(\mathbb{D}) \) and consider the orthonormal basis
\[ \{ u_n(z) = [(n + 1)/\pi]^{1/2} z^n : n \in \mathbb{N}_0 \} \]
of \( A^2(\mathbb{D}) \).
Proposition 6. The canonical solution operator \( S_1 \) for the unit disc \( \mathbb{D} \) in \( \mathbb{C} \) has the following property:

\[
\sum_{n=0}^{\infty} \| S_1(u_n \, d\overline{z}) \| < \infty,
\]

which implies that \( S_1 : A^2_{0,(1)}(\mathbb{D}) \rightarrow L^2(\mathbb{D}) \) is a Hilbert-Schmidt operator (see [MV]).

Proof. Using calculations in [J] we can show that

\[
S_1(u_n \, d\overline{z})(z) = \left[(n+1)/\pi\right]^{1/2} \ z^n \ \overline{z} - \left[n^2/((n+1)\pi)\right]^{1/2} \ z^{n-1}, \ n \in \mathbb{N}_0.
\]

The Bergman kernel \( B \) of \( \mathbb{D} \) has the form

\[
B(z, \zeta) = \frac{1}{\pi} \frac{1}{(1-z\overline{\zeta})^2},
\]

hence by Proposition 1 we can express \( \| S_1(u_n d\overline{z}) \|^2 \) in the form

\[
\int_{\mathbb{D}} \overline{z} \ u_n(z) - \frac{1}{\pi} \int_{\mathbb{D}} \frac{\overline{z} \ u_n(\zeta)}{(1-z\overline{\zeta})^2} \ d\lambda(\zeta) \ d\lambda(z).
\]

Therefore we get

\[
\| S_1(u_n d\overline{z}) \|^2 = \int_{\mathbb{D}} \left( \frac{n+1}{\pi} \right)^{1/2} \ z^n \ \overline{z} - \left(\frac{n \ z^{n-1}}{((n+1)\pi)^{1/2}}\right)^2 \ d\lambda(z)
\]

\[
= \int_{\mathbb{D}} \left( \frac{n+1}{\pi} \right)^{1/2} z^n \ \overline{z} - \frac{2n \ |z|^{2n}}{(n+1)^{1/2}} + \left( \frac{n^2 \ |z|^{2n-2}}{(n+1)^{1/2}} \right) \ d\lambda(z)
\]

\[
= 2\pi \int_0^1 \left( \frac{(n+1)^{1/2} \ r^{2n+3}}{\pi} - \frac{2n \ r^{2n+1}}{(n+1)^{1/2}} + \frac{n^2 \ r^{2n-1}}{(n+1)^{1/2}} \right) \ dr
\]

\[
= \frac{1}{(n+1)(n+2)}.
\]

Hence

\[
\sum_{n=0}^{\infty} \| S_1(u_n d\overline{z}) \|^2 < \infty.
\]

\[\square\]

Remark. It can be shown that the set \( \{ S_1(u_n d\overline{z}) : n \in \mathbb{N}_0 \} \) consists of pairwise orthogonal elements of \( L^2(\mathbb{D}) \).

In the following part we consider the case of the polydisc. For the sake of simplicity we concentrate on \( \mathbb{C}^2 \); let

\[
\mathbb{D}^2 = \{ z = (z_1, z_2) : |z_1| < 1, \ |z_2| < 1 \}.
\]

Now \( \{ z_1^{n_1} z_2^{n_2} : n_1, n_2 \in \mathbb{N}_0 \} \) is an orthogonal basis in \( A^2(\mathbb{D}^2) \). It is easily seen that the norms of the functions \( z_1^{n_1} z_2^{n_2} \) are \( \pi^{1/2}/((n_1+1)(n_2+1))^{1/2} \). The functions

\[
u_{n_1, n_2}(z_1, z_2) = \frac{((n_1+1)(n_2+1))^{1/2}}{\pi} \ z_1^{n_1} z_2^{n_2}, \ n_1, n_2 \in \mathbb{N}_0,
\]
form an orthonormal basis of $A^2(D^2)$, and the system
\[ \{ u_{n_1, n_2} \, d\zbar_1, u_{n_1, n_2} \, d\zbar_2 : n_1, n_2 \in \mathbb{N}_0 \} \]
constitutes an orthonormal basis for $A^2_{(0,1)}(D^2)$.

Next we compute the Bergman projections of the functions
\[ (z_1, z_2) \mapsto \zbar_1 \, u_{n_1, n_2}(z_1, z_2) \quad \text{and} \quad (z_1, z_2) \mapsto \zbar_2 \, u_{n_1, n_2}(z_1, z_2) \]
and obtain
\[ P(\zbar_1 \, u_{n_1, n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{(n_1 + 1)(n_2 + 1)}{\pi} \frac{n_1}{n_1 + 1} z_1^{n_1 - 1} z_2^{n_2}, \]
where we used similar computations as in Proposition 6.

The Bergman projection of the second function is
\[ P(\zbar_2 \, u_{n_1, n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{(n_1 + 1)(n_2 + 1)}{\pi} \frac{n_2}{n_2 + 1} z_1^{n_1} z_2^{n_2 - 1}. \]
Now we can compute the norms of the images under the canonical solution operator of the elements of our orthonormal basis of $A^2_{(0,1)}(D^2)$:
\[ \frac{(n_1 + 1)(n_2 + 1)}{\pi^2} \int_{D^2} \left| \zbar_1 \, z_1^{n_1} z_2^{n_2} - \frac{n_1}{n_1 + 1} z_1^{n_1 - 1} z_2^{n_2} \right|^2 d\lambda(z) = \frac{1}{(n_1 + 2)(n_1 + 1)}. \]
where we used the corresponding computation of Proposition 6 for the integral with respect to $z_1$.

In a similar way we obtain
\[ \frac{(n_1 + 1)(n_2 + 1)}{\pi^2} \int_{D^2} \left| \zbar_2 \, z_1^{n_1} z_2^{n_2} - \frac{n_2}{n_2 + 1} z_1^{n_1} z_2^{n_2 - 1} \right|^2 d\lambda(z) = \frac{1}{(n_2 + 2)(n_2 + 1)}. \]
Since
\[ \sum_{n_1, n_2 = 1}^{\infty} \left( \frac{1}{(n_1 + 2)(n_1 + 1)} + \frac{1}{(n_2 + 2)(n_2 + 1)} \right) = \infty, \]
the canonical solution operator
\[ S_1 : A^2_{(0,1)}(D^2) \longrightarrow L^2(D^2) \]
is not Hilbert-Schmidt.

Remark. With results from [K2] it can be shown that the canonical solution operator
\[ S_1 : A^2_{(0,1)}(D^2) \longrightarrow L^2(D^2) \]
is not even compact.

We now consider the case of the unit ball $B^2$ in $\mathbb{C}^2$. Here we can use calculations from the proof of Proposition 1 in [W].

The norms of the functions $z_1^{n_1} z_2^{n_2}$ are now $\pi [n_1! n_2! / (n_1 + n_2 + 2)!]^{1/2}$ (see [K1]). The functions
\[ U_{n_1, n_2}(z_1, z_2) = \frac{[n_1 + n_2 + 2]!^{1/2}}{\pi [n_1! n_2!]^{1/2}} z_1^{n_1} z_2^{n_2}, \quad n_1, n_2 \in \mathbb{N}_0, \]
form an orthonormal basis of $A^2(\mathbb{B}^2)$, and the system
\[\{U_{n_1,n_2} \varphi_1, U_{n_1,n_2} \varphi_2 : n_1, n_2 \in \mathbb{N}_0\}\]
constitutes an orthonormal basis for $A^2_{(0,1)}(\mathbb{B}^2)$.

We compute the Bergman projections of the functions
\[(z_1, z_2) \mapsto \varphi_1 U_{n_1,n_2}(z_1, z_2) \quad \text{and} \quad (z_1, z_2) \mapsto \varphi_2 U_{n_1,n_2}(z_1, z_2)\]
and obtain
\[P(\varphi_1 U_{n_1,n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{[(n_1 + n_2 + 2)]^{1/2}}{\pi (n_1!n_2!)^{1/2}} \frac{n_1}{n_1 + n_2 + 2} z_1^{n_1-1} z_2^{n_2},\]
\[P(\varphi_2 U_{n_1,n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{[(n_1 + n_2 + 2)]^{1/2}}{\pi (n_1!n_2!)^{1/2}} \frac{n_2}{n_1 + n_2 + 2} z_1 z_2^{n_2-1}.
\]
Finally we compute the norms of the images of the basis elements under the canonical solution operator $S_1$, and obtain
\[\frac{(n_1 + n_2 + 2)!}{\pi^2 n_1!n_2!} \int_{\mathbb{B}^2} \left| z_1 z_1^{n_1-2} - \frac{n_1}{n_1 + n_2 + 2} z_1 z_2^{n_2-1} \right|^2 d\lambda(z) = \frac{n_2 + 2}{(n_1 + n_2 + 2)(n_1 + n_2 + 3)},\]
and
\[\frac{(n_1 + n_2 + 2)!}{\pi^2 n_1!n_2!} \int_{\mathbb{B}^2} \left| z_2 z_1^{n_1-2} - \frac{n_2}{n_1 + n_2 + 2} z_1 z_2^{n_2-1} \right|^2 d\lambda(z) = \frac{n_1 + 2}{(n_1 + n_2 + 2)(n_1 + n_2 + 3)}.\]
Since
\[\sum_{n_1, n_2 = 1}^{\infty} \left( \frac{n_2 + 2}{(n_1 + n_2 + 2)(n_1 + n_2 + 3)} + \frac{n_1 + 2}{(n_1 + n_2 + 2)(n_1 + n_2 + 3)} \right) = \infty,
\]
the canonical solution operator
\[S_1 : A^2_{(0,1)}(\mathbb{B}^2) \rightarrow L^2(\mathbb{B}^2)\]
is also not Hilbert-Schmidt.

Remark. In [2] it is shown that there are no nonzero Hilbert-Schmidt Hankel operators on the Bergman space of the unit ball in $\mathbb{C}^n$ with antiholomorphic symbol when $n \geq 2$.

**Proposition 7.** The integral kernel
\[\frac{|z_1 - w_1|^2 + |z_2 - w_2|^2}{|1 - z_1 \overline{w}_1|^4 |1 - z_2 \overline{w}_2|^4}\]
does not belong to $L^2(\mathbb{D}^2 \times \mathbb{D}^2)$ and the integral kernel
\[\frac{|z_1 - w_1|^2 + |z_2 - w_2|^2}{|1 - z_1 \overline{w}_1 - z_2 \overline{w}_2|^6}\]
does not belong to $L^2(\mathbb{B}^2 \times \mathbb{B}^2)$. 

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The canonical solution operator to $\mathcal{J}$

Proof. Suppose the first kernel belongs to $L^2(D^2 \times D^2)$. Then the corresponding integral operator from $L^2_{(0,1)}(D^2)$ to $L^2(D^2)$ is a Hilbert-Schmidt operator, which would imply that the restriction to $A^2_{(0,1)}(D^2)$ is also Hilbert-Schmidt. But this restriction coincides with the canonical solution operator $S_1$, from which we already know that it is not Hilbert-Schmidt. The proof for the second integral is analogous to the first.

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