ON THE HOMOLOGY OF SPLIT EXTENSIONS
WITH $p$–ELEMENTARY KERNEL

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Abstract. We study a Hochschild–Serre spectral sequence associated to a
split group extension with kernel $(\mathbb{Z}/p)^n$. It is shown that a large part of $E_2^{0*}$
must survive to infinity. We also sketch the general procedure of computing
this surviving group.

1. INTRODUCTION

It is often useful to decompose a spectral sequence into eigenspaces of an au-
tomorphism of the sequence. In the case of a Hochschild–Serre spectral sequence
associated to a split extension with abelian kernel the Liebermann trick (see [Sa], p.
262) provides an important example of this. One takes the automorphism induced
by multiplication by a scalar in the kernel of the extension. For example, it is easy
using this method to show that in a split extension with abelian kernel

$$0 \to A \to H \to G \to 1$$

we have $H_i(H, Q) = \bigoplus_{0 \leq j \leq i} (H_{i-j}(G, \Lambda^j(A \otimes Q)))$. If one considers the homology
with $\mathbb{F}_p$–coefficients, the situation becomes more involved. The first problem is that
scalars have only finite multiplicative order and the second is that the homology of
an abelian group also contains a part generated by elements of degree 2 (for $p$ odd).
For these reasons a Hochschild–Serre spectral sequence can have many nontrivial
differentials and is generally hard to understand.

In the present paper we use the Liebermann trick together with an analysis
of a scalar extension to show triviality of some differentials when one takes $\mathbb{F}_p$–
coefficients. We apply this technique in Section 2 to show that a part of the 0–th
column survives which is close to $\Lambda^*(A)_G$. In Section 3 we discuss some examples,
in particular we show that $\Lambda^*(A)_G$ does not always embed into $H_*(H, \mathbb{F}_p)$.

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strong encouragement.

2. THE THEOREM

Let

$$0 \to A \to H \to G \to 1$$

(1)
be a split extension with $A = (\mathbb{Z}/p)^n$ (we specialize to this case only to simplify notation, for general (abelian) $A$ our Theorem 1 remains true if we replace $A$ by $A \otimes \mathbb{Z} F_p$). We will consider a homological Hochschild–Serre spectral sequence with $F_p$-coefficients corresponding to this extension. Denoting by $D^j$ a $j$-th divided power we have the natural identification of the $E^2$-term:

\[(2) \quad E^2_{ij} = \bigoplus_{k+2l=j} H_i(G, \Lambda^k(A) \otimes D^l(A)) \quad \text{for } p \text{ odd},\]

\[(3) \quad E^2_{ij} = H_i(G, D^j(A)) \quad \text{for } p = 2\]

(unless otherwise stated $\otimes$ means $\otimes_{F_p}$).

Let us define $H_i(G, \Lambda^*(A))_{\text{reg}}$ to be

\[H_i(G, \Lambda^*(A))/\ker(H_i(G, \Lambda^*(A)) \to H_i(G, \Lambda^*(\mathbb{Z})))\]

where the arrow is induced by the natural embedding $\Delta : \Lambda^*(A) \hookrightarrow A^\otimes$. Similarly, we put

\[H_i(G, D^*(A))_{\text{reg}} = H_i(G, D^*(A))/\ker(H_i(G, D^*(A)) \to H_i(G, A^\otimes))\]

for $\Gamma : D^*(A) \hookrightarrow A^\otimes$.

In our spectral sequence only some pieces of the groups we are interested in survive, hence we should carefully differ between $E^2$ and the higher $E^r$. Thus, in order to make the formulation of Theorem 1 clear and to avoid a confusion in its proof we introduce some notation. For $i \leq 1$ we have natural epimorphisms $\alpha_{ij}^r : E^2_{ij} \to E^r_{ij}$. Then we put $B_{ij} = \bigcup_{r \geq 2} \ker(\alpha_{ij}^r) \cap H_i(G, \Lambda^j(A))$ for $p$ odd and $B_{ij} = \bigcup_{r \geq 2} \ker(\alpha_{ij}^r) \cap H_i(G, D^j(A))$ for $p = 2$.

**Theorem 1.** In the sequence (2) we have $B_{ij} \subset \ker(\Delta_*)$ for $i \leq 1$. In other words, the spaces $H_i(G, \Lambda^j(A))_{\text{reg}}$ for $i \leq 1$ survive to infinity. The same holds for the sequence (3) when we replace $\Lambda^j(A)$ by $D^j(A)$.

**Proof.** We begin with some remarks concerning functoriality of semidirect products. It is well known that the class of split extensions of a fixed group $G$ by abelian kernels is in bijection with the class of $\mathbb{Z}[G]$-modules via construction of semidirect product. Moreover, this assignment yields an isomorphism of the category of $\mathbb{Z}[G]$-modules and the category of split extensions of $G$ with abelian kernels where morphisms are morphisms of extensions being identity on $G$. The practical consequence is that any $G$-homomorphism between kernels of two extensions induces a morphism of spectral sequences.

The idea of the proof (for $p$ odd) is as follows. We look at the automorphism of the spectral sequence (2) induced by the $G$-automorphism of $A$ defined by the formula $x \mapsto cx$ for a given scalar $c \in F_p^*$ (we will frequently use the structure of $F_p$-linear space on $A$). Then it is easy to see that the space

\[H_s(G, \Lambda^k(A) \otimes D^l(A))\]

belongs to the eigenspace of the induced automorphism for the eigenvalue $c^{k+l}$ and that the whole spectral sequence is a direct sum of eigensequences for eigenvalues $1, c, c^2, \ldots, c^{p-1}$. At this point it is clear for example that there are no differentials coming to $H_s(G, \Lambda^k(A))$ for $k < p$ because all $E^r_{ij}$ for $j < k$ belong to eigenspaces of $c^s$ for $s < k$ and differentials must preserve the decomposition. Thus $H_0(G, \Lambda^k(A))$ and $H_1(G, \Lambda^k(A))$ for $k < p$ survives. Unfortunately, this argument fails for $k \geq p$
since $c^p = c$ for any $c \in \mathbb{F}_p^*$. We partially overcome this difficulty comparing the sequence (2) with a sequence associated to a split extension with kernel equipped with $G$–automorphism of higher order.

Therefore let us consider a split extension

$$0 \to A \otimes L \to H(L) \to G \to 1$$

where $L$ is a one–dimensional space over a field $\mathbb{F}_q$ with $q = p^d$ elements regarded as a trivial $G$–module. We shall describe $\cdots E^2$—the second page of a Hochschild–Serre spectral sequence (with coefficients in $\mathbb{F}_q$) associated to it. We focus here on the case when $p$ is odd. We should take into account the $\mathbb{F}_q$–structure appearing in this new sequence. More precisely, we describe $\cdots E^2_{ij}$ as evaluations on $L$ of functors from the category of finite $\mathbb{F}_q$–spaces to itself assigning to a $\mathbb{F}_q$–space $V$ the entries in the spectral sequence associated to the extension

$$0 \to A \otimes V \to H(V) \to G \to 1.$$
Proof. Let \( j = f(p - 1) + g \) where \( g < p - 1 \). We put
\[
k_i = \begin{cases} 
0 & \text{for } i < d - f - 1, \\
g & \text{for } i = d - f - 1, \\
p - 1 & \text{for } d - f - 1 < i \leq d - 1.
\end{cases}
\]
We are going to show that for any \( k' \) satisfying \( |k'| = |k| \) we have \( r(k') > r(k) \).
Let us take such \( k' \). If there exists \( k'' \) with \( k'' \geq p \), we may replace \( k' \) by \( k'' \) having the same \( | | \) but smaller \( r \) defining
\[
k''_i = \begin{cases} 
k''_i = k_i' - p & \text{for } i = i_0, \\
k''_i = k''_i + 1 & \text{for } i = i_0 + 1, \\
k''_i & \text{for } i \neq i_0, i_0 + 1
\end{cases}
\]
(we use here the convention \( k_0 = k_0 \)). Thus, iterating this procedure, we may assume that all \( k_i' \) are smaller than \( p \). But in this case the only possibility for \( |k'\| = |k| \) is \( k' = k \).
Let \( \Lambda^k(A \otimes L) \) denote \( \Lambda^{k_0}(A \otimes L) \otimes \ldots \otimes \Lambda^{k_{d-1}}(A \otimes L^{d-1}) \). If for \( j < d(p - 1) \) we take \( k \) as in Lemma 1, then there cannot be any differentials coming to \( H_*(G, \Lambda^k(A \otimes L)) \) (and its subspaces in the higher \( 'E'' \)). Thus the spaces \( H_i(G, \Lambda^k(A \otimes L)) \) for \( i \leq 1 \) survive in sequence (4).

We now want to construct a morphism from the spectral sequence (2) to (4). First we should replace (2) by (5)—its counterpart with \( F_q \)-coefficients. In this new sequence we have
\[
''_i E^2_{ij} = \bigoplus_{k+2l=j} H_i(G, \Lambda^k(A \otimes F_q) \otimes_{F_q} D^l(A \otimes F_q))
\]
There is a morphism \( \Phi \) from (2) to (5) which is, by the Kunneth formula, on \( E^2 \) just induced by scalar extension in all tensors appearing as the coefficients of the homology of \( G \). Now to obtain a morphism from (5) to (4) it suffices to choose a \( G \)-homomorphism from \( A \) to \( A \otimes L \) which is possible since \( L \) is a trivial \( G \)-module. In order to make formulas explicit let us identify \( L \) with \( F_q \). Then we take the homomorphism from \( A \) to \( A \otimes F_q \) sending \( a \) to \( a \otimes 1 \) and we will consider the morphism of spectral sequences \( \Psi \) induced by this \( G \)-homomorphism. Under the isomorphism \( (A \otimes F_q) \otimes F_q = \bigoplus_{i=0}^{d-1} (A \otimes F_q^{(i)}) \) the morphism \( \Psi \) from (5) to (4) is induced on \( E^2 \) by the morphism of coefficients \( \psi : A \otimes F_q \rightarrow \bigoplus_{i=0}^{d-1} (A \otimes F_q^{(i)}) \) sending \( a \otimes x \) to \( \bigoplus_{i=0}^{d-1} (a \otimes x^{p^i}) \). We focus on the groups \( H_*(G, \Lambda^j(A \otimes F_q)) \). We would like to describe the map \( \pi_k \circ \psi : H_* (G, \Lambda^j(A \otimes F_q)) \rightarrow H_* (G, \Lambda_k^k(A \otimes F_q)) \) for given a sequence \( k \) with \( r(k) = j \) the map \( \pi_k \) is the projection from \( 'E''_{ij} \) onto \( H_* (G, \Lambda^k(A \otimes F_q)) \). According to the above formulas, \( \pi_k \circ \psi \) may be factorized as \( f_* \circ \text{com}_* \) where
\[
\text{com}_* : H_* (G, \Lambda^j(A \otimes F_q)) \rightarrow H_* (G, \Lambda^{k_0}(A \otimes F_q) \otimes \ldots \otimes \Lambda^{k_{d-1}}(A \otimes F_q))
\]
is induced by the iterated comultiplication map in the \( F_q \)-Hopf algebra \( \Lambda^*_{F_q} (A \otimes F_q) \) while
\[
f_* : H_* (G, \Lambda^{k_0}(A \otimes F_q) \otimes \ldots \otimes \Lambda^{k_{d-1}}(A \otimes F_q)) \rightarrow H_* (G, \Lambda^{k_0}(A \otimes F_q) \otimes \ldots \otimes \Lambda^{k_{d-1}}(A \otimes F_q))
\]
is determined by the \( G \)-isomorphism \( f_{k_0} \otimes \ldots \otimes f_{k_{d-1}} \) defined on a factor \( \Lambda^{k_i} \) by the formula \( f_{k_i} (a \otimes x) = a \otimes x^{p^i} \).
Now we are in a position to prove the theorem. Given \( x \in H_i(G, \Lambda^i(A)) \) \((i \leq 1)\) belonging to \( \ker(\alpha^r_{ij}) \), choose \( k \) as in Lemma 1 and consider the commutative diagram

\[
\begin{array}{ccc}
E^2_{ij} & \xrightarrow{\psi_* \circ \Phi^2} & \pi^2_{|k|} E^2_{ij} \\
\downarrow {\alpha^r_{ij}} & & \downarrow {\alpha^r_{ij}} \\
E^r_{ij} & \xrightarrow{\psi_* \circ \Phi^r} & \pi^r_{|k|} E^r_{ij} \\
\end{array}
\]

where \( |k|E \) denotes the subsequence corresponding to the eigenvalue \( |k| \), \( \pi^r_{|k|} \) is a natural projection (it is a morphism of spectral sequences in contrast to \( \pi_k \)).

Now suppose that
\[
'\alpha^r_{ij} \circ \pi^r_{|k|} \circ \psi_* \circ \Phi^r(x) = 0.
\]

Since \( \pi^r_{|k|} \circ \psi_* \circ \Phi^r(x) = \pi_k \circ \psi_* \circ \Phi^r(x) \in H_i(G, \Lambda^k(A \otimes F_q)) \), which by the paragraph after Lemma 1 survives to infinity, we thus obtain
\[
\pi_k \circ \psi_* \circ \Phi^r(x) = 0.
\]

Now by identifications we have worked out earlier we get
\[
0 = \pi_k \circ \psi_* \circ \Phi^r(x) = f_* \circ com_* \circ \Phi^r(x).
\]

But since \( f_* \) is an isomorphism, we have
\[
com_* \circ \Phi^r(x) = 0.
\]

At last, by the Kunneth formula, \( ker(com_* \circ \Phi^r) = ker(com'_*) \) where \( com' \) is iterated comultiplication in \( F_p \)-Hopf algebra \( \Lambda^p_{F_p}(A) \). Thus we get that \( com'_*(x) = 0 \). Since \( \Delta \) is also iterated comultiplication (corresponding to the partition \((1, \ldots, 1)\)), then \( \Delta \) factors through \( com' \) and we obtain \( ker(com'_*) \subset ker(\Delta_*) \) concluding the proof.

We note that in fact \( ker(com'_*) = ker(\Delta_*) \), and we have introduced \( \Delta \) only in order to simplify the statement of the theorem, since \( com \) depends on \( j \) in a more complicated way.

For the case \( p = 2 \) we proceed analogously. The only difference is the different description of the homology of an abelian group which does not affect our arguments.

\[\square\]

3. Remarks and examples

This paper was motivated by the following example. We consider a split extension of finite \( F_p \)-algebras
\[
(6) \quad F_p \to F_p[x]/x^2 \to F_p.
\]

This extension induces a split group extension
\[
(7) \quad 0 \to M(J) \to GL(R) \to GL(S) \to 1
\]
where \( GL \) is the colimit of general linear groups, \( M \) is the colimit of additive groups of matrices and \( GL(F_p) \) acts on \( M(F_p) \) by conjugation (of course, a group extension exists already at the level of \( GL_n \) and \( M_n \)). It was shown by Goodwillie that for any split extension of rings \( J \rightarrow R \rightarrow S \), where \( J \) is a free \( S \)-bimodule regarded as an ideal with trivial multiplication, that \( \Lambda^*(M(J) \otimes Q)_{GL(S)} = H_*(F,Q) \) where \( F \) is the homotopy fiber of the induced map \( BGL^+(R) \rightarrow BGL^+(S) \) (\cite{gp}, p. 395). This result awakened my interest to the space \( \Lambda^*(A)_G \). For example, if Goodwillie’s theorem was also true with coefficients in \( F_p \), then thanks to \( H_*(GL(F_p),F_p) = 0 \) (\cite{qs}) we would obtain
\[
H_*(GL(F_p[x]/x^2),F_p) = \Lambda^*(M(F_p),F_p)_{GL(F_p)}.
\]

It has been known since the early eighties (see e.g. \cite{ef}) that this equality cannot hold because \( H_*(GL(F_p[x]/x^2),F_p) \) is too big, but initially I conjectured that \( \Lambda^*(M(F_p),F_p)_{GL(F_p)} \) embeds into \( H_*(GL(F_p[x]/x^2),F_p) \) through the edge homomorphism in the sequence (2) associated to the extension (7). This hope was destroyed by results of \cite{hm}. Hesselholt and Madsen have computed \( K_*(F_p[x]/x^2) \), but since \( BGL^+(F_p[x]/x^2)_p \) is a generalized Eilenberg–Mac Lane spectrum, it also determines \( H_*(GL(F_p[x]/x^2),F_p) \). In particular, their formulas give
\[
H_2(GL(F_2[x]/x^2),F_2) = F_2^2,
\]
but it is easy to see that \( H_0(GL(F_2),D^2(M(F_2))) = F_2^2 \). This shows that the spectral sequence (3) corresponding to the extension \( F_2 \rightarrow F_2[x]/x^2 \rightarrow F_2 \) must have a nontrivial differential arriving at \( H_0(GL(F_2),D^2(M(F_2))) \). A similar example may be also constructed for \( p = 3 \). Thus the restriction to \( (H_i)_{reg} \) in Theorem 1 is necessary.

We look more closely at the groups in Theorem 1, and focus on \( H_0(G,\Lambda^*(A))_{reg} \) (for \( p \) odd) which, as we have seen, appears in another context but is also more computable than \( H_1(G,\Lambda^*(A))_{reg} \). In general, the process of computing \( H_0(G,\Lambda^*(A))_{reg} \) divides into two steps. The first requires knowledge not only about \( H_0(G,A) \) but also about the whole representation \( G \rightarrow Aut(A) \) to determine \( H_0(G,A^{\otimes j}) \). The second is to describe the action of the group \( \Sigma_j \) on \( H_0(G,A^{\otimes j}) \) induced by permutation of factors in \( A^{\otimes j} \). If one completes this program, in order to obtain a formula for \( H_0(G,\Lambda^*(A))_{reg} \) it suffices to observe that it may be identified with the image of the endomorphism \( Alt_* \) of \( H_0(G,A^{\otimes j}) \) given by the antisymmetrization formula
\[
Alt_*(x) = \sum_{\sigma \in \Sigma_j} sgn(\sigma) \sigma x
\]
(8)

(8) is analogous fact is not true for \( p = 2 \) because \( D^j(A) \) is not an image of \( A^{\otimes j} \).

To illustrate this algorithm we apply it to extension (7). By the First Fundamental Theorem of (co)Invariant Theory \cite{tc} we have
\[
H_0(GL(F_p),M(F_p)^{\otimes j}) = F_p[\Sigma_j]
\]
and the action of the symmetric group on the group algebra is given by the formula \( \sigma \cdot e_\tau = e_{\sigma \tau}^{-1} \). Now we should describe the image of the antisymmetrization map (8). Let us take \( e_\tau \in F_p[\Sigma_j] \) and consider two different cases: if the centralizer of \( \tau \) contains an odd permutation, and if it does not. In the first case we have \( Alt_*(e_\tau) = 0 \) so we focus on the case when the centralizer consists of only even
permutations. Then choosing representatives for \( \Sigma_j/\text{Centr}(\tau) \) we may write

\[
\text{Alt}_*(e_\tau) = |\text{Centr}(\tau)| \cdot \sum_{\tau' \in \Sigma_j/\text{Centr}(\tau)} \text{sgn}(\tau') \cdot e_{\tau' \tau}.
\]

From this formula the following consequences may be immediately derived: \( \text{Alt}_*(e_\tau) \) depends only on the conjugacy class of \( \tau \), it is nontrivial if \( \text{Centr}(\tau) \) contains no odd permutation and is of order prime to \( p \), elements in different conjugacy classes have images linearly independent. Using elementary combinatorics of the symmetric group we may translate these conditions into the language of partitions of \( j \). The result is

\[
\dim(H_0(G, \Lambda^j(A)))_{\text{reg}} = \{ \text{the number of partitions of } j \text{ into different odd numbers prime to } p \}.
\]

We point out that the last requirement is nothing but the condition for regularity of a conjugacy class in the sense of representation theory. This explains our notation for \( (H_*)_{\text{reg}} \).

We finish by making one disappointing remark concerning the group \( H_1(\text{GL}(F_p), \Lambda^j(M(F_p)))_{\text{reg}} \). Namely, in contrast to \( H_0(\text{GL}(F_p), \Lambda^j(M(F_p)))_{\text{reg}} \), it quite easily follows from \[\text{Be1}\] and \[\text{Be2}\] that \( H_1(\text{GL}(F_p), \Lambda^j(M(J))) = 0 \) for \( p > 2 \) and \( j < p \).

REFERENCES


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