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DEFENDING THE NEGATED KAPLANSKY CONJECTURE

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Dedicated to Professor Yukio Tsushima on his sixtieth birthday

ABSTRACT. To answer in the negative a conjecture of Kaplansky, four recent papers independently constructed four families of Hopf algebras of fixed finite dimension, each of which consisted of infinitely many isomorphism classes. We defend nevertheless the negated conjecture by proving that the Hopf algebras in each family are cocycle deformations of each other.

1. INTRODUCTION

Throughout we work over a fixed field k.

In 1975, I. Kaplansky proposed:

Conjecture. *The Hopf algebras of a given finite dimension are finitely many up to isomorphism, if k is algebraically closed and the characteristic* $\operatorname{ch} k$ *does not divide the given dimension.*

Recently this was answered in the negative independently by Gelaki [G, Cor. 3.3], Andruskiewitsch and Schneider [AS, Thm. 0.3], Beattie, Dăscălescu and Grünenfelder [BDG, Thm. 1], and Müller [M, Thm. 5.13]; each of them constructed a family of Hopf algebras, listed below, of a fixed finite dimension which consists of infinitely many isomorphism classes:

[G] $\mathcal{U}_{\alpha,\beta,\gamma}$, where α, β, $\gamma \in k$;

[AS] $\mathcal{B}(M, N, q, \lambda)$, where $\lambda \in k$ (and M, N, q are fixed);

[BDG] $H(a)$, where $a \in k$;

[M] A_α, where $\alpha : \Gamma \to k^\times$ is a group map defined on any cyclic subgroup $\Gamma \subset SL_2(k)$ of some fixed order.

The Hopf algebras in the first three families look like each other, and are indeed special examples of the pointed Hopf algebra $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$ of finite dimension defined also in [AS, Sect. 5]; see Section 3 below. Thus those families are included in

[AS] $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$, where $\mathcal{D}$ is a *compatible datum* [AS, p. 679] for fixed Γ, $\mathcal{R}$,

if Γ and $\mathcal{R}$ are suitably chosen. On the other hand, Müller's A_α looks quite different from the others; it can be regarded as the coordinate algebra of some finite quantum group; see Section 4.

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In this paper we prove:

Theorem 1. *In each family of those five listed above, the Hopf algebras contained are cocycle deformations of each other.*

Let H be a Hopf algebra with antipode S. By a 2-cocycle for H, we mean a convolution-invertible linear form $\sigma : H \otimes H \to k$ satisfying the normalized 2-cocycle condition given in [P, p. 3805], [T2, p. 24]. The *cocycle deformation* H^σ of H by a 2-cocycle σ is the Hopf algebra defined to be the coalgebra H equipped with the twisted product $\cdot$ and antipode S^σ given by

$$x \cdot y = \sum \sigma(x_1, y_1) x_2 y_2 \sigma^{-1}(x_3, y_3),$$
$$S^\sigma(x) = \sum \sigma(x_1, S(x_2)) S(x_3) \sigma^{-1}(S(x_4), x_5),$$

where $x, y \in H$; see [D, Thm. 1.6]. The convolution-inverse σ^{-1} of σ is a 2-cocycle for H^σ, and $(H^\sigma)^{\sigma^{-1}} = H$; see [T2, p. 25]. This allows us to say that H and H^σ are cocycle deformations of each other.

The category $\operatorname{Comod} H$ of right (or left) H-comodules forms a k-linear, abelian monoidal category with the obvious monoidal structure. If two Hopf algebras H, L are cocycle deformations of each other, then they are *monoidally Morita-Takeuchi equivalent* (or *monoidally co-Morita equivalent* [P, Def. 5.6]) in the sense that there exists a k-linear monoidal equivalence $\operatorname{Comod} H \approx \operatorname{Comod} L$ between the categories of the right (or equivalently left) comodules. The converse holds true if H, L are finite-dimensional; see [P, Cor. 5.9].

The theorem allows us to expect that the Kaplansky conjecture might be true in a weaker form. Thus we would propose:

Conjecture. *The Hopf algebras of a given finite dimension are finitely many up to cocycle deformation, at least if k is as in the original conjecture.*

2. The pushout construction

Let K be a Hopf algebra with antipode S. The set $\operatorname{Alg}(K, k)$ of all algebra maps $g : K \to k$ forms a group under the convolution-product, where the inverse g^{-1} is the composite $g \circ S$. The group acts on K from the left and the right by

$$gx = \sum x_1 g(x_2), \quad xg = \sum g(x_1) x_2 \qquad (x \in K),$$

so that $(gx)g' = g(xg')$. The automorphisms of K defined by

$$x \mapsto gx, \quad x \mapsto xg, \quad x \mapsto gxg^{-1}$$

are those of left K-comodule algebra, right K-comodule algebra and Hopf algebra, respectively.

Definition. Two Hopf ideals I, J in K are said to be *conjugate* if there is an algebra map $g : K \to k$ such that $J = gIg^{-1}$.

The following is a key for the proof of Theorem 1.

Theorem 2. *Suppose that K is a Hopf subalgebra of a Hopf algebra H. If Hopf ideals I, J in K are conjugate, then the quotient Hopf algebras $H/(I)$, $H/(J)$ by the Hopf ideals (I), (J) in H generated by I, J are monoidally Morita-Takeuchi equivalent.*

In general, two Hopf algebras H, L are monoidally Morita-Takeuchi equivalent if and only if there exists an (L, H)-biGalois object; see [P, Cor. 5.7]. An (L, H)-bicomodule algebra $A \neq 0$ with structure maps $\lambda : A \to L \otimes A$, $\rho : A \to A \otimes H$ is called an (L, H)-*biGalois object* (or an (L, H)-*biGalois extension over* k [P, Def. 3.4]) if A is a left L- and right H-Galois object in the sense that the canonical maps

$$\overline{\lambda} : A \otimes A \to L \otimes A, \quad \overline{\lambda}(x \otimes y) = \lambda(x)y$$

and

$$\overline{\rho} : A \otimes A \to A \otimes H, \quad \overline{\rho}(x \otimes y) = x\rho(y)$$

are isomorphisms.

Proof of Theorem 2 (improved by referee's suggestion). The comultiplication $\Delta : K \to K \otimes K$ of K gives rise to the canonical isomorphism

$$\overline{\Delta} : K \otimes K \xrightarrow{\simeq} K \otimes K, \ \overline{\Delta}(x \otimes y) = \sum x_1 \otimes x_2 y,$$

so that K is a left K-Galois object; the inverse of $\overline{\Delta}$ is given by $\overline{\Delta}^{-1}(x \otimes y) = \sum x_1 \otimes S(x_2)y$. Let $I \subset K$ be a Hopf ideal. Then, $\Delta(I) \subset K \otimes I + I \otimes K$ and $\overline{\Delta}(K \otimes I + I \otimes K) = K \otimes I + I \otimes K$. Let $g \in \mathrm{Alg}(K, k)$, and set $M = gI$. Since we have $\Delta(gx) = \sum x_1 \otimes gx_2$, $\overline{\Delta}(gx \otimes gy) = \sum x_1 \otimes g(x_2 y)$ for $x, y \in K$, it follows that

$$\begin{aligned} &\Delta(M) \subset K \otimes M + I \otimes K, \\ &\overline{\Delta}(K \otimes M + M \otimes K) = K \otimes M + I \otimes K. \end{aligned}$$

Hence the quotient algebra K/M is naturally a left K/I-comodule algebra, which is further a left K/I-Galois object. (This follows alternatively, since we see that the automorphism $x \longmapsto gx$ of K induces an isomorphism $K/I \xrightarrow{\simeq} K/M$ of left K/I-comodule algebras, which is necessarily convolution-invertible, and so that K/M is a cleft K/I-comodule algebra with coinvariants k.)

We easily see that the last two formulae hold with K, I, M replaced by H, (I), (M), respectively; this implies that $H/(M)$ is naturally a left $H/(I)$-Galois object. Suppose $J = gIg^{-1}$. Since then $M = Jg$, the mirror argument shows that $H/(M)$ is right $H/(J)$-Galois and so $(H/(I), H/(J))$-biGalois, which implies the desired result. □

Remark. Let $K \subset H$ be as above. For a Hopf ideal $I \subset K$, the natural diagram

$$\begin{array}{ccc} K & \longrightarrow & H \\ \downarrow & & \downarrow \\ K/I & \longrightarrow & H/(I) \end{array}$$

is a pushout in the category of Hopf algebras.

3. The Hopf algebras $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$ due to Andruskiewitsch and Schneider

Let us recall the definition of $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$; it is defined over an arbitrary field k and is determined by three data Γ, $\mathcal{R}$, $\mathcal{D}$.

First, let Γ be a finite abelian group with minimal generators $y_1, \cdots, y_m$. Suppose y_i has order $M_i(>1)$. Then the group Γ is defined by the relations

$$y_l^{M_l} = 1 \qquad (1 \leq l \leq m), \tag{1}$$

$$y_l y_t = y_t y_l \quad (1 \leq t < l \leq m). \tag{2}$$

Next, let $\mathcal{R}$ denote elements $x_1, \cdots, x_n$ in Γ together with the same number of group maps $\chi_1, \cdots, \chi_n : \Gamma \to k^\times (= k \backslash \{0\})$ such that $\chi_i(x_j)\chi_j(x_i) = 1$, where $1 \leq i < j \leq n$ (though [AS] means by $\mathcal{R}$ the quantum linear space determined by the $2n$ elements). Let N_i be the order of $\chi_i(x_i)$, and set

$$Z_\mathcal{R} = \{i \mid 1 \leq i \leq n, \chi_i^{N_i} = 1\},$$
$$W_\mathcal{R} = \{(i,j) \mid 1 \leq i < j \leq n,\ \chi_i\chi_j = 1\}.$$

Finally, let $\mathcal{D}$ be a *compatible datum* [AS, p. 679] for Γ, $\mathcal{R}$; thus $\mathcal{D}$ consists of scalars μ_i $(1 \leq i \leq n)$ and λ_{ij} $(1 \leq i < j \leq n)$ such that $\mu_i = 0$ if $i \notin Z_\mathcal{R}$ and $\lambda_{ij} = 0$ if $(i,j) \notin W_\mathcal{R}$. (We have dropped the requirement $\mu_i \in \{0,1\}$ given in [AS, (5.1)].)

The Hopf algebra $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$ is generated by $m+n$ elements $y_1, \cdots, y_m, a_1, \cdots, a_n$ with the relations (1), (2) and

$$y_l a_i = \chi_i(y_l) a_i y_l \qquad (1 \leq l \leq m,\ 1 \leq i \leq n), \tag{3}$$

$$a_i^{N_i} = \mu_i(1 - x_i^{N_i}) \qquad (1 \leq i \leq n), \tag{4}$$

$$a_j a_i = \chi_i(x_j) a_i a_j + \lambda_{ij}(1 - x_i x_j) \quad (1 \leq i < j \leq n), \tag{5}$$

where each y_l is grouplike and a_i is $(1, x_i)$-primitive, so $\Delta(a_i) = a_i \otimes 1 + x_i \otimes a_i$, $\varepsilon(a_i) = 0$ and $S(a_i) = -x_i^{-1} a_i$.

In a bialgebra, an ideal generated by nearly primitives is a bi-ideal. It follows that if one divides a bialgebra by such relations in each of which both sides are grouplikes or nearly primitives of the same type, then a quotient bialgebra is obtained. As seen from the proof of [AS, Lemma 5.1], this fact proves that $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$ is a quotient bialgebra of the free bialgebra $k\langle y_1, \cdots, y_m, a_1, \cdots, a_n\rangle$ in which each y_l is grouplike and a_i is $(1, x_i)$-primitive; the x_i here denotes a monomial in $y_1, \cdots, y_m$ reduced to the x_i in Γ. We remark in particular that by the q-binomial formula, the left-hand side of (4) as well as the right-hand side is $(1, x_i^{N_i})$-primitive modulo the relation (3). By [R, Lemma 1 c)], $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$ is pointed and its grouplikes, say G, are generated by $y_1, \cdots, y_m$, so $G = \Gamma$. Since this is a group, $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$ is indeed a Hopf algebra.

Proposition 3. *Fix data* Γ, $\mathcal{R}$ *such as above. The Hopf algebras* $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$, *where* $\mathcal{D}$ *runs through all compatible data for* Γ, $\mathcal{R}$, *are cocycle deformations of each other.*

Proof. Let $\mathcal{D}$ be an arbitrary compatible datum given above. Let $\mathcal{D}_0$ be the special one consisting of zeros. By applying Theorem 2, we are going to prove that $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$ is a cocycle deformation of $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D}_0)$.

Let B be the quotient bialgebra of the free bialgebra $k\langle y_1, \cdots, y_m, a_1, \cdots, a_n\rangle$ defined by the relations (2), (3) and

$$a_s^{N_s} = 0,\ a_j a_i = \chi_i(x_j) a_i a_j,$$

where $s \notin Z_\mathcal{R}$, $(i,j) \notin W_\mathcal{R}$. Let H be the quotient Hopf algebra of B obtained by adding the relation (1). Let C (resp., K) be the sub-bialgebra of B (resp., the Hopf subalgebra of H) generated by

$$y_l,\ a_s^{N_s},\ u_{ij} := a_j a_i - \chi_i(x_j) a_i a_j,$$

where $1 \le l \le m$, $s \in Z_{\mathcal{R}}$ and $(i,j) \in W_{\mathcal{R}}$. Let $R \subset C$ be the subalgebra generated by all $a_s^{N_s}$ and u_{ij}. One sees that each y_l is central in C. In addition by applying the diamond lemma to B, we see that $C = R \otimes P$ as an algebra, where $P := k[y_1, \cdots, y_m]$ denotes the polynomial algebra in the indeterminates $y_1, \cdots, y_m$. Hence, R is mapped isomorphically into K, and $K = R \otimes k\Gamma$, where $k\Gamma$ denotes the group algebra of Γ.

We claim that there exists a (unique) algebra map $g : K \to k$ such that

$$g(y_l) = 1,\ g(a_s^{N_s}) = \mu_s,\ g(u_{ij}) = \lambda_{ij},$$

where $1 \le l \le m$, $s \in Z_{\mathcal{R}}$ and $(i,j) \in W_{\mathcal{R}}$. To see this, divide B further by the relations

$$a_s^{N_s} = \mu_s(1 - x_s^{N_s}),\ u_{ij} = \lambda_{ij}(1 - x_i x_j),$$

where $s \in Z_{\mathcal{R}}$, $(i,j) \in W_{\mathcal{R}}$, and let $\overline{B}$ denote the quotient bialgebra. Let $\overline{C}$ denote the natural image of C in $\overline{B}$. Since the diamond lemma gives a canonical basis for $\overline{B}$ in the same way as in the proof of [AS, Prop. 5.2], we see that $\overline{C}$ includes P as a central subalgebra and that $\overline{B}$ is free as a left (and right) P-module. It follows that the canonical map

$$k = P/(y_1, \cdots, y_m) \to \overline{C}/(y_1, \cdots, y_m)$$

is a monomorphism, and so an isomorphism. If one extends

$$R \hookrightarrow C \twoheadrightarrow \overline{C}/(y_1, \cdots, y_m) = k$$

to $g : K = R \otimes k\Gamma \to k$ by setting $g(x) = 1$ for all $x \in \Gamma$, this is the desired map.

Let $I \subset K$ be the Hopf ideal generated by all $a_s^{N_s}$ and u_{ij}. Then, $H/(I) = \mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D}_0)$. Let g be as above. One computes then that $g^{-1}(a_s^{N_s}) = -\mu_s$, $g^{-1}(u_{ij}) = -\lambda_{ij}$, and so that $g a_s^{N_s} g^{-1} = a_s^{N_s} - \mu_s(1 - x_x^{N_s})$, $g u_{ij} g^{-1} = u_{ij} - \lambda_{ij}(1 - x_i x_j)$. Hence, $H/(gIg^{-1}) = \mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$, which is a cocycle deformation of $H/(I) = \mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D}_0)$ by Theorem 2. □

To see that the Hopf algebras $\mathcal{U}_{\alpha,\beta,\gamma}$, $\mathcal{B}(M, N, q, \lambda)$, $H(a)$ given in the list in the Introduction are of the form $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$, let M, N be positive integers with $N > 1$, and suppose that k contains a primitive N-th root q of 1, and so that $\mathrm{ch}\, k \not| N$. Let $\Gamma = \langle y \rangle$ be the cyclic group of order MN generated by y, and define group maps $\chi_1, \chi_2 : \Gamma \to k^\times$ by $\chi_1(y) = q$, $\chi_2(y) = q^{-1}$.

Given $\lambda \in k$, define data by

$$\mathcal{R} = (x_1 = x_2 = y; \chi_1, \chi_2),\ \mathcal{D} = (\mu_1 = \mu_2 = 1, \lambda_{12} = \lambda).$$

Then by definition [AS, Sect. 6], $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D}) = \mathcal{B}(M, N, q, \lambda)$.

Suppose $M = N = p$, an odd prime. Given $a \in k$, let $\mathcal{R}$ be as above, and define

$$\mathcal{D} = (\mu_1 = \mu_2 = -1, \lambda_{12} = -a).$$

Then, $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D}) = H(a)$.

Let ν be a positive integer prime to N. Given $\alpha, \beta, \gamma \in k$, define

$$\mathcal{R} = (x_1 = x_2 = y^\nu; \chi_1, \chi_2),\ \mathcal{D} = (\mu_1 = -\alpha, \mu_2 = -\beta, \lambda_{12} = q^\nu \gamma).$$

Assign the generators y, a_1, a_2 of $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$ to those a, x, y [G, p. 648, line -3] of $\mathcal{U}_{\alpha,\beta,\gamma} = \mathcal{U}_{(N, MN, \nu, q, \alpha, \beta, \gamma)}$, respectively. Then one obtains an isomorphism

$\mathcal{A}(\Gamma,\mathcal{R},\mathcal{D}) \simeq (\mathcal{U}_{\alpha,\beta,\gamma})^{\mathrm{op\ cop}}$, and also $\mathcal{A}(\Gamma,\mathcal{R},\mathcal{D}) \simeq \mathcal{U}_{\alpha,\beta,\gamma}$, since $\mathcal{U}_{\alpha,\beta,\gamma} \simeq (\mathcal{U}_{\alpha,\beta,\gamma})^{\mathrm{op\ cop}}$ through the antipode.

Thus the first three families in the list are included respectively in some family of $\mathcal{A}(\Gamma,\mathcal{R},\mathcal{D})$. Hence, Proposition 3 implies the assertion of Theorem 1 for those families.

4. The Hopf algebras A_α due to Müller

The Hopf algebras seem suitably described in terms of quantum groups, following M. Takeuchi's idea stated in his comments on an earlier version of [M].

As is often the case, we regard the category of quantum groups (over k) as the dual of the category of Hopf algebras (over k). Thus each quantum group G corresponds to a Hopf algebra, $O(G)$. We say G is finite of order m if $\dim O(G) = m$. A quantum subgroup G' of G corresponds to a quotient Hopf algebra $O(G')$ of $O(G)$. A short exact sequence of quantum groups corresponds to such a short sequence of Hopf algebras that is *strictly exact* in the sense of Schneider [S].

In the following, let q be a root of 1 in k of odd order $N > 2$. For each integer $n > 1$, we have the short exact sequence

$$1 \to SL_q^{[[N]]}(n) \to SL_q(n) \xrightarrow{F} SL(n) \to 1 \tag{6}$$

of quantum groups, where F denotes the q-analogue of the Frobenius map; see [T1, Sect. 5]. This is *co-central* in the sense that $O(F) : O(SL(n)) \to O(SL_q(n))$ is central, that is, its image is included in the center of $O(SL_q(n))$.

We have a natural epimorphism from $O(SL(n))$ onto the coordinate algebra $O(SL_n(k))$ of the affine algebraic group $SL_n(k)$, which is an isomorphism if k is infinite. Let $\Gamma \subset SL_n(k)$ be a finite subgroup. Then it is an affine algebraic group with $O(\Gamma) = (k\Gamma)^*$, the dual of the group algebra $k\Gamma$. The composite $O(SL(n)) \to O(SL_n(k)) \to O(\Gamma)$ of the natural epimorphisms makes Γ into a finite quantum subgroup of $SL(n)$.

Let $\mu_N(k)$ denote the group of N-th roots of 1 in k, which is generated by q. For a finite subgroup $\Gamma \subset SL_n(k)$ and a group map $\alpha : \Gamma \to \mu_N(k)$, Müller [M] constructs some finite quantum subgroups $G_{\Gamma,\alpha}$ of $GL_q(n)$ such that a (co-central) short exact sequence

$$1 \to SL_q^{[[N]]}(n) \to G_{\Gamma,\alpha} \to \Gamma \to 1 \tag{7}$$

is induced from the GL-version of (6). If Γ is of order m, then $G_{\Gamma,\alpha}$ is of order mN^3. If $\alpha : \Gamma \to \mu_N(k)$ is trivial, we simply write $G_\Gamma = G_{\Gamma,\alpha}$. This is defined to be the quantum subgroup of $SL_q(n)$ which makes

$$\begin{array}{ccc} SL_q(n) & \xrightarrow{F} & SL(n) \\ \uparrow & & \uparrow \\ G_\Gamma & \longrightarrow & \Gamma \end{array}$$

a pullback in the obvious sense.

Remark. We have restricted Müller's construction [M, Thm. 4.11] to the very special case where $s = 1$, $T = \mathcal{T}$, $Q = \mathbb{Z}$ in the notation of [M, Sect. 4]; $O(G_{\Gamma,\alpha})$ here coincides with $A_{\mathcal{T}}/J_\alpha$ in [M].

Proposition 4. *If two finite subgroups* Γ, Γ' *in* $SL_n(k)$ *are conjugate, then* $O(G_\Gamma)$ *and* $O(G_{\Gamma'})$ *are cocycle deformations of each other.*

Proof. In general, let K be a finitely generated Hopf algebra and let I, $J \subset K$ be Hopf ideals. Then, $G := \mathrm{Alg}(K, k)$ is an affine algebraic group over k with closed subgroups $G_1 := \mathrm{Alg}(K/I, k)$ and $G_2 := \mathrm{Alg}(K/J, k)$. Suppose that $K/I = O(G_1)$, $K/J = O(G_2)$, the coordinate algebras. This means that an element $x \in K$ is in I (or in J) if and only if $g(x) = 0$ for all g in G_1 (or in G_2). We see easily that, if G_1 and G_2 are conjugate in G, then I and J are conjugate.

Apply the result to $K = O(SL(n))$, $K/I = O(\Gamma)$, $K/J = O(\Gamma')$, and combine with Theorem 2. Then the proposition follows immediately. □

Lemma 5. *Let* $\Gamma \subset SL_n(k)$ *be a finite subgroup of order* m*. If the g.c.d.* (m, N) *is prime to* n*, then the short exact sequences* (7)*, where* $\alpha : \Gamma \to \mu_N(k)$ *are group maps, are equivalent with each other, and especially the quantum groups* $G_{\Gamma,\alpha}$*, with* Γ *fixed, are isomorphic with each other.*

Proof. In the group of group maps $\Gamma \to \mu_N(k)$, each α generates a cyclic group, $\langle\alpha\rangle$, whose order $|\alpha|$ divides $d := (m, N)$. If n is prime to d and hence to $|\alpha|$, there is β in $\langle\alpha\rangle$ such that $\alpha = \beta^n$. Therefore the lemma follows from the 'if' part of [M, Prop. 5.9]. □

Suppose $n = 2$. Theorem 5.13 in [M] proves that if k is infinite, then for all cyclic subgroups $\Gamma \subset SL_2(k)$ of order N and group maps $\alpha : \Gamma \to \mu_N(k)$, the Hopf algebras $A_\alpha = O(G_{\Gamma,\alpha})$ of dimension N^4 consist of infinitely many isomorphism classes.

Proposition 6. *For any* k *(containing* q*), the Hopf algebras* $A_\alpha = O(G_{\Gamma,\alpha})$*, where* Γ *and* α *are as above, are cocycle deformations of each other.*

Proof. By Lemma 5, $O(G_{\Gamma,\alpha}) \simeq O(G_\Gamma)$ for any α. Proposition 4 implies the result, since any Γ is conjugate to the subgroup of $SL_2(k)$ generated by the diagonal matrix $\mathrm{diag}(q, q^{-1})$. □

This completes the proof of Theorem 1.

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