FOURIER RESTRICTION FOR AFFINE ARCLENGTH MEASURES IN THE PLANE

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(Communicated by Christopher D. Sogge)

Abstract. We obtain an analog, uniform for a large class of curves in the plane, of the Fefferman-Zygmund theorem on restriction of the Fourier transform.

The purpose of this note is to prove an analog, uniform over a certain class of curves in $\mathbb{R}^2$, of the Fefferman-Zygmund restriction theorem for the circle.

**Theorem.** If $1 < p < \frac{4}{3}$ and $\frac{1}{p} + \frac{1}{3q} = 1$, there is a constant $C = C(p)$ such that

$$
\left( \int_a^b |\hat{f}(t, \phi(t))|^q \phi''(t) \frac{dt}{t} \right)^{\frac{1}{q}} \leq C(p) \|f\|_{L^p(\mathbb{R}^2)}
$$

holds whenever $\phi$ is a real-valued function on an interval $(a, b)$ satisfying $\phi''(t) > 0$, $\phi(3)(t) \geq 0$ on $(a, b)$.

Writing $d\lambda$ for the measure $\phi''(t)\frac{1}{3}dt$ on the curve $\gamma(t) = (t, \phi(t))$, $a < t < b$, we will follow the broad outline of the Fefferman-Zygmund proof and thus establish the dual estimate

$$
\|f d\lambda\|_{L^q(\mathbb{R}^2)} \leq C(p) \|f\|_{L^p(d\lambda)},
$$

where $1 < p < 4$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Interpolation with the case $(p, q) = (1, \infty)$ shows that it is enough to prove a restricted weak type version of (2). Thus if $\frac{1}{p} = \frac{1}{r} - \frac{1}{2}$, so that $\frac{1}{r} + \frac{1}{q} = 1$, we will show that

$$
\|(\chi_{\gamma} d\lambda) \ast (\chi_{\gamma} d\lambda)^\ast\|_{L^r, \infty(\mathbb{R}^2)} \leq C(p) \left( \int_E \phi''(t)^{1/3} dt \right)^{\frac{1}{2}}
$$

whenever $E$ is a Borel subset of $(a, b)$. Then it will follow that

$$
\|\chi_{\gamma} d\lambda\|_{L^q, \infty(\mathbb{R}^2)} = \|\chi_{\gamma} d\lambda\|_{L^r, \infty(\mathbb{R}^2)}^{\frac{1}{2}} \leq C(p) \left( \int_E \phi''(t)^{1/3} dt \right)^{\frac{1}{2}},
$$

Received by the editors March 15, 2000.
1991 Mathematics Subject Classification. Primary 42B10.
Key words and phrases. Fourier transform, restriction.
by Hunt’s generalization of the Hausdorff-Young theorem. Now (3) is true for all \((p, r)\) of interest if it is true for the two extreme cases \((p, r) = (1, 1)\) and \((p, r) = (4, 2)\). The first of these is easy and so it is enough to establish the inequality
\[
(4) \quad \| (\chi_E T d\lambda) * (\chi_E T d\lambda)^* \|_{L^2(\mathbb{R}^2)} \leq C \left( \int_E \phi''(t)^{1/3} dt \right)^{2/3}
\]
for some absolute constant \(C\). Inequality (4) may be regarded as a weak endpoint estimate for (the dual of) Fourier restriction. It is a consequence of
\[
(5) \quad \int_a^b \int_t^b \chi_T (\gamma(t) - \gamma(s)) \chi_E(s) \chi_E(t) \phi''(s)^{1/3} \phi''(t)^{1/3} ds \, dt \leq C \left( \int_E \phi''(t)^{1/3} dt \right)^{2/3} |T|^{1/2},
\]
where \(|T|\) is the two-dimensional Lebesgue measure of an arbitrary Borel subset \(T\) of \(\mathbb{R}^2\). But the LHS of (5) is bounded by
\[
\left( \int_a^b \left( \int_t^b \chi_T (\gamma(t) - \gamma(s)) \phi''(s)^{1/3} ds \right)^2 \phi''(t)^{1/3} dt \right)^{1/2} \left( \int_E \phi''(t)^{1/3} dt \right)^{1/2}.
\]
Thus it is enough to establish the inequality
\[
(6) \quad \int_a^b \left( \int_t^b \chi_T (\gamma(t) - \gamma(s)) \phi''(s)^{1/3} ds \right)^2 \phi''(t)^{1/3} dt \leq 4 |T|.
\]
Inequality (6) is (2) in [O]. We repeat the short proof for the reader’s convenience.

The convexity of the graph of \(\phi\) shows that the change of variables
\[
(s, t) \rightarrow \gamma(s) - \gamma(t) = (s - t, \phi(s) - \phi(t))
\]
is one-to-one. Thus
\[
\int_a^b \int_t^b \chi_T (\gamma(t) - \gamma(s)) \left| \phi'(s) - \phi'(t) \right| ds \, dt \leq |T|
\]
and (6) will follow from the inequality
\[
(7) \quad \phi''(t)^{1/3} \left( \int_t^b \chi_A(s) \phi''(s)^{1/3} ds \right)^2 \leq 4 \int_t^b \chi_A(s) (\phi'(s) - \phi'(t)) ds
\]
if \(A \subseteq (t, b)\). To prove (7) we let \(|A_u|\) stand for the (one-dimensional) Lebesgue measure of \(A \cap (u, b)\) whenever \(t \leq u \leq b\). Then
\[
(8) \quad \int_t^b \chi_A(s) (\phi'(s) - \phi'(t)) ds = \int_t^b \chi_A(s) \int_t^s \phi''(u) \, du \, ds = \int_t^b \phi''(u) |A_u| \, du.
\]
Also,
\[
\int_t^b \chi_A(s) \phi''(s)^{1/3} ds = \int_t^b \chi_A(s) (\phi''(s)^{1/3} |A_s|^{1/3} |A_s|^{-1/3} ds
\]
\[
\leq \left( \int_t^b \chi_A(s) (\phi''(s) |A_s| ds \right)^{1/3} \left( \int_t^b \chi_A(s) |A_s|^{1/3} ds \right)^{2/3}.
\]
Thus it follows from (8) that

\begin{equation}
\left( \int_0^b \chi_A(s) \phi''(s) \frac{1}{3} ds \right)^3 \leq \left( \int_0^b \chi_A(s) \left( \phi'(s) - \phi'(t) \right) ds \right) \left( \int_0^b \chi_A(s)|A_s|^{-1/2} ds \right)^2.
\end{equation}

If 0 \leq \rho \leq |A|, then \( |\{ s \in A : |A_s| \leq \rho \}| = \rho \), and so

\[ \int_0^b \chi_A(s)|A_s|^{-1/2} ds = \int_0^{|A|} y^{-1/2} dy = 2 |A|^{1/2}. \]

With this and the fact that \( \phi'' \) is nondecreasing, (9) yields (7) to complete the proof of the theorem.

The measure \( d\lambda \) is called the affine arclength measure on the curve \( \gamma \). Drury \[D\] was the first to point out its relevance to certain problems in harmonic analysis. In particular, \[D\] contains a restriction theorem using affine arclength measure on certain curves in \( \mathbb{R}^3 \).

**References**


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