A CHARACTERIZATION OF BILATERAL OPERATOR WEIGHTED SHIFTS BEING COWEN-DOUGLAS OPERATORS

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Abstract. It is characterized when a bilateral operator weighted shift is a Cowen-Douglas operator.

1. Introduction

Let \( \mathbb{C} \) denote the complex plane, \( \mathbb{C}^n = \bigoplus_{n=1}^{\infty} \mathbb{C} \) \((n < \infty)\) and \( \{W_k\}_{k=-\infty}^{\infty} \) a sequence of uniformly bounded invertible linear operators on \( \mathbb{C}^n \). A bounded linear operator \( S \) on \( K := \bigoplus_{k=-\infty}^{\infty} \mathbb{C}^n \) is called a (forward) bilateral operator weighted shift with the weight sequence \( \{W_k\}_{k=-\infty}^{\infty} \), denoted by \( S_{\{W_k\}_{k=-\infty}^{\infty}} \), if

\[
S(\cdots, x_{-1}, x_0, x_1, \cdots) = (\cdots, W_{-1}x_{-2}, W_0x_{-1}, W_1x_0, \cdots), \quad \forall x = (x_k) \in K.
\]

It can be easily shown that \( K_+ := \bigoplus_{k=0}^{\infty} \mathbb{C}^n \) is an invariant subspace of \( S \). Also, \( S_+ := S|_{K_+} \) is called a (forward) unilateral operator weighted shift with the weight sequence \( \{W_k\}_{k=1}^{\infty} \), denoted by \( S \sim \{W_k\}_{k=1}^{\infty} \). In general, \( S \) and \( S_+ \) are called by a joint name operator weighted shift. Their adjoint operators \( S^* \) and \( S_+^* \) are referred to as backward operator weighted shifts, and \( n \) is said to be their multiplicity.

Operator weighted shifts were first defined by A. Lambert [4]. When \( n = 1 \), they are exactly scalar weighted shifts which have been widely studied (see [6]).

First, we recall some notations and terminologies (see, for example, [2]). Let \( \mathcal{H} \) be a complex separable Hilbert space and let \( \mathcal{L}(\mathcal{H}) \) denote the set of all bounded linear operators acting on \( \mathcal{H} \). For \( T \in \mathcal{L}(\mathcal{H}) \), let \( \sigma(T) \) and \( \sigma_p(T) \) denote the spectrum and the point spectrum of \( T \), respectively. Set \( \text{null} T = \dim \ker T \). We write \( r(T) \) for the spectral radius of \( T \) and let \( r_1(T) = \lim_{k \to \infty} (m(T^k))^{1/k} \), where \( m(T) := \inf \{ \|Tx\| : \|x\| = 1 \} \). Recall that \( T \) is called a Fredholm operator if \( \text{ran} T \) is closed and both \( \text{null} T \) and \( \text{null} T^* \) are finite. In this case, the index of \( T \) is defined by \( \text{ind} T = \text{null} T - \text{null} T^* \). Moreover,

\[
\rho_F(T) := \{ \lambda \in \mathbb{C} : T - \lambda \text{ is Fredholm} \}
\]

and \( \sigma_F(T) := \mathbb{C} \setminus \rho_F(T) \) will denote the Fredholm domain and the essential spectrum of \( T \), respectively.

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Let $\Omega$ be a connected open subset of $\mathbb{C}$ and $m$ a natural number. $B_m(\Omega)$ denotes the set of operators $T$ in $L(H)$ satisfying:

(a) $\Omega \subset \sigma(T)$,
(b) $\text{ran} \ (T - \lambda) = H$, $\forall \lambda \in \Omega$,
(c) $\bigvee \{\ker (T - \lambda) : \lambda \in \Omega\} = H$ and
(d) $\text{mul} (T - \lambda) = m$, $\forall \lambda \in \Omega$.

Call an operator in $B_m(\Omega)$ a Cowen-Douglas operator.

Clearly, if $T \in B_m(\Omega)$, then $\rho_F(T)$ and $\text{ind} (T - \lambda) = m$ for every $\lambda \in \Omega$.

Originally, Cowen-Douglas operators were introduced as using the method of complex geometry to developing operator theory (see [1]). However, it has been presented recently that they are closely related to the structure of bounded linear operators (see [3]).

The backward unilateral shift, i.e., all its weights $W_k$’s are the identity operator on $\mathbb{C}$, is the simplest Cowen-Douglas operator. Thus, it is a proper problem when an operator weighted shift is a Cowen-Douglas operator. It is not hard to show $\sigma_p(S_+) = \emptyset$. Hence, $S_+$ cannot be a Cowen-Douglas operator. When $S_+^*$ is a Cowen-Douglas operator has been characterized in [5]. In this note, we will prove the following theorem.

**Theorem.** Let $S \sim \{W_k\}_{k=-\infty}^{+\infty}$ be a bilateral operator weighted shift with multiplicity $n$. Then $S$ is a Cowen-Douglas operator if and only if there exists a $\lambda_0 \in \rho_F(S)$ such that $\text{ind} (S - \lambda_0) = n$.

**Remark 1.** For $S \sim \{W_k\}_{k=-\infty}^{+\infty}$, it can be shown that $S^*$ is unitarily equivalent to the bilateral operator weighted shift with the weight sequence $\{W_{-k}\}_{k=1}^{+\infty}$. Thus, the theorem is also applicable for backward bilateral operator weighted shifts.

### 2. Some Lemmas

Note that a bilateral operator weighted shift $S \sim \{W_k\}_{k=-\infty}^{+\infty}$ can be represented as the following operator matrix:

$$
S = \left(\begin{array}{cc}
\begin{array}{c}
\ddots \\
\vdots \\
W_{-1} \\
W_0
\end{array}
& 0 \\
0 & 0 \\
0 & W_1 \\
\vdots & \ddots
\end{array} \right)
= \left(\begin{array}{cc}
S_- & 0 \\
F & S_+
\end{array} \right),
$$

where $S_+$ (resp. $S_-$) is a forward (resp. backward) unilateral operator weighted shift with the weight sequence $\{W_{k}\}_{k=1}^{+\infty}$ (resp. $\{W_{-k}\}_{k=1}^{+\infty}$), and $F$ is an $n$ rank operator. Therefore, we can immediately obtain the following lemma.

**Lemma 1.** Let $S$ be given as in (2.1). Then $\sigma_e(S) = \sigma_e(S_-) \cup \sigma_e(S_+)$ and

$$
\text{ind} (S - \lambda) = \text{ind} (S_- - \lambda) + \text{ind} (S_+ - \lambda), \ \forall \lambda \in \rho_F(S).
$$
In [5], \( \sigma_e(S_+) \) and the indices associated with all holes in \( \sigma_e(S_+) \) have been described. For convenience, we draw the following conclusions.

**Lemma 2 ([5]).** Each \( S_n \) is unitarily equivalent to an upper triangular operator matrix \((S_{ij})_{1\leq i,j\leq n} \) on \((l^2)_{(n)} := \bigoplus_{i=1}^{n} l^2_i \), where \( l^2_i = \bigoplus_{k=0}^{\infty} C \), each \( S_{ij} \) is a unilateral scalar weighted shift, and each \( S_{ii} \) is injective. Moreover, \( r(S_{ii}) \geq r(S_{(i+1)(i+1)}) \), \( \sigma_e(S_+) = \bigcup_{i=1}^{n} \sigma_e(S_{ii}) \), and \( \sigma(S_+) = \bigcup_{i=1}^{n} \sigma(S_{ii}) \).

By Theorem 4 and Theorem 6 in [6], we have that
\[
\sigma(S_{ii}) = \{ \lambda \in \mathbb{C} : |\lambda| \leq r(S_{ii}) \}
\]
and
\[
\sigma_e(S_{ii}) = \{ \lambda \in \mathbb{C} : r_1(S_{ii}) \leq |\lambda| \leq r(S_{ii}) \}.
\]
Let \( \Omega \) be a hole of \( \sigma_e(S_+) \). It follows from Lemma 2 and the above formulas that one and only one of the following two situations must occur:

(a) \( \Omega = \{ \lambda : |\lambda| < \delta \} \), where \( \delta = \min\{r_1(S_{ii}) : 1 \leq i \leq n \} > 0 \);
(b) there exist \( i_0 \) and \( j_0 \), \( 1 \leq i_0 < j_0 \leq n \), such that \( \Omega = \{ \lambda : r(S_{j_0j_0}) < |\lambda| < r_1(S_{i_0i_0}) \} \).

**Lemma 3 ([5]).** (1) If the situation (a) occurs, then \( \text{ind}(S_+ - \lambda) = -n, \forall \lambda \in \Omega \);
(2) If the situation (b) occurs, then \( \text{ind}(S_+ - \lambda) = -i_0, \forall \lambda \in \Omega \).

### 3. Proof of the Theorem and its corollaries

Now, we are ready to prove our theorem.

Proof of the Theorem. Suppose that there exists a \( \lambda_0 \in \rho_F(S) \) such that \( \text{ind}(S - \lambda_0) = n \). Taking a basis \( \{e_j\}_{j=1}^{n} \) of \( \mathbb{C}^n \), there exists a basis \( \{d_j\}_{j=1}^{n} \) of \( \mathbb{C}^n \) such that \( W_0 = \sum_{j=1}^{n} e_j \otimes d_j \), identifying \( d_j \) and \( e_j \) with \((\cdots, 0, d_j) \) in \( K_- \) and \((e_j, 0, \cdots) \) in \( K_+ \), respectively. Then \( F \), as in (2.1), is identical with \( \sum_{j=1}^{n} e_j \otimes d_j \).

Since \( 0 \leq \text{nul}(S_- - \lambda_0) \leq n \) and \( \text{nul}(S_- - \lambda_0)^* = \text{nul}(S_+ - \lambda_0) = 0 \) it follows from (2.1) that
\[
\text{nul}(S_+ - \lambda_0)^* = \text{ind}(S_+ - \lambda_0) = 0
\]
and
\[
\text{ind}(S_- - \lambda_0) = \text{nul}(S_- - \lambda_0) = \text{ind}(S - \lambda_0) = n.
\]
Thus, we have that \( \lambda_0 \notin \sigma_e(S_+) = \{ \lambda : |\lambda| \leq r(S_+) \} \) (see [3]). Let \( \Omega_0 \) be the component of \( \rho_F(S_-) \) containing \( \lambda_0 \). Then it follows from Lemma 3 that \( \Omega_0 \) is an open disk with the center at 0. It implies that \( \sigma(S_-) \subset \Omega_0 \). By Proposition 2.3 in [3], moreover, we have that \( S_- \in B_n(\Omega_0) \). Set \( \Omega = \Omega_0 \setminus \sigma(S_+) \). Then \( \Omega \) is a connected open subset of \( \mathbb{C} \). We will prove that \( S \in B_n(\Omega) \). For this purpose, it suffices to show the following statements:

(i) \( \Omega \subset \rho_F(S) \);
(ii) \( \text{ind}(S - \lambda) = n \) and \( \text{ker}(S - \lambda)^* = \{0\}, \forall \lambda \in \Omega \);
(iii) \( \mathcal{K} = \mathcal{K}_- \oplus \mathcal{K}_+ = \bigvee \{\text{ker}(S - \lambda) : \lambda \in \Omega\} \).

Obviously, \( \Omega \subset \rho_F(S_-) \cap \rho_F(S_+) \). From Lemma 1, (i) is immediate. Also, by the continuity of index, (ii) holds. Finally, it follows from Proposition 1.51 in [3] that there exist holomorphic \( \mathcal{K}_- \)-valued functions \( \{f_i(\lambda)\}_{i=1}^{n} \) on \( \Omega \) such that \( \{f_i(\lambda)\}_{i=1}^{n} \) forms a basis of \( \ker(S_- - \lambda) \) for each \( \lambda \in \Omega \). Now, set
\[
g_i(\lambda) = \sum_{j=1}^{n} (f_i(\lambda), d_j)(S_+ - \lambda)^{-1} e_j, \quad \forall \lambda \in \Omega,
\]
and

\[ h_i(\lambda) = f_i(\lambda) + g_i(\lambda), \quad \forall \lambda \in \Omega. \]

Then \( h_i(\lambda) \in \ker (S - \lambda) \). To prove (iii), it suffices to show that \( \mathcal{M} := \bigvee \{ h_i(\lambda) : \lambda \in \Omega, 1 \leq i \leq n \} = \mathcal{K} \). Suppose that there exists some \( x \in \mathcal{K} \cap \mathcal{M} \). Write \( x = y + z \) with \( y \) in \( \mathcal{K}_- \) and \( z \) in \( \mathcal{K}_+ \). It follows that

\[ 0 = (h_i(\lambda), x) = (f_i(\lambda), y) + (g_i(\lambda), z), \quad \forall \lambda \in \Omega. \]

Namely,

\[ (f_i(\lambda), y) = -(g_i(\lambda), z) = \sum_{j=1}^{n} (f_i(\lambda), d_j)((S_+ - \lambda)^{-1} e_j, z), \quad \forall \lambda \in \Omega. \]

Thus, we have that

\[
\begin{pmatrix}
(f_1(\lambda), y) \\
\vdots \\
(f_n(\lambda), y)
\end{pmatrix} = G(\lambda) \begin{pmatrix}
((S_+ - \lambda)^{-1} e_1, z) \\
\vdots \\
((S_+ - \lambda)^{-1} e_n, z)
\end{pmatrix}, \quad \forall \lambda \in \Omega,
\]

where \( G(\lambda) = ((f_i(\lambda), d_j))_{1 \leq i, j \leq n} \) is a Gram matrix. So, it is invertible because both \( \{f_i(\lambda)\}_{i=1}^{n} \) and \( \{d_j\}_{j=1}^{n} \) are linear independent. It implies that

\[
(G(\lambda))^{-1} \begin{pmatrix}
(f_1(\lambda), y) \\
\vdots \\
(f_n(\lambda), y)
\end{pmatrix} = \begin{pmatrix}
((S_+ - \lambda)^{-1} e_1, z) \\
\vdots \\
((S_+ - \lambda)^{-1} e_n, z)
\end{pmatrix}, \quad \forall \lambda \in \Omega.
\]

The left side of the equality is holomorphic on \( \Omega_0 \). Meanwhile, the right side is holomorphic on \( \{ \lambda : |\lambda| > r(S_+) \} \) and \( ((S_+ - \lambda)^{-1} e_i, z) \longrightarrow 0 \) \( (\lambda \longrightarrow \infty) \) for \( i = 1, 2, \cdots, n \). Hence, it follows from the Liouville Theorem that

\[ (f_i(\lambda), y) = 0, \quad \forall \lambda \in \Omega_0, \quad i = 1, 2, \cdots, n, \]

and

\[ ((\lambda - S_+)^{-1} e_i, z) = 0, \quad \forall \lambda \in \mathbb{C} \setminus \sigma(S_+), \quad i = 1, 2, \cdots, n. \]

However, \( \bigvee \{ f_i(\lambda) : \lambda \in \Omega_0, 1 \leq i \leq n \} = \bigvee \{ \ker(S_- - \lambda) : \lambda \in \Omega_0 \} = \mathcal{K}_- \). So, \( y = 0 \). Also, note that

\[ 0 = ((\lambda - S_+)^{-1} e_i, z) = \sum_{k=0}^{+\infty} (S_+^k e_i, z) \lambda^{-(k+1)}, \quad \forall \lambda > ||S_+||. \]

It follows that \( (S_+^k e_i, z) = 0 \) for all \( k \geq 0 \) and \( 1 \leq i \leq n \). Moreover, it can be verified that \( \bigvee \{ S_+^k e_i : k \geq 0, 1 \leq i \leq n \} = \mathcal{K}_+ \). Thus, \( z = 0 \). This proves that \( \mathcal{M} = \mathcal{K} \).

Conversely, if \( S \) is a Cowen-Douglas operator, i.e., \( S \in \mathcal{B}_m(\Omega) \), then it is clear that \( m \leq n \). Imitating the case \( n = 1 \), we can show that \( \sigma(S), \sigma_e(S) \) and \( \sigma_p(S) \) are circular symmetric about the origin. Thus, without loss of generality, we can assume that \( \Omega \) is an annular domain that contains 1. For each natural number \( l \), set \( \lambda_l = e^{-\frac{i\pi}{l}} \), where \( i \) denotes the imaginary unit. Then \( \lambda_l \in \Omega \) for all \( l \) and \( \lim_{l \to \infty} \lambda_l = 1 \). Now, it follows from Proposition 1.41 in \[3\] that

\[ \bigvee \{ \ker(S - \lambda_l) : l \geq 1 \} = \bigvee \{ \ker(S - \lambda) : \lambda \in \Omega \} = \mathcal{K}. \]
Taking a basis \( \{ f^{(0)}_j \}_{j=1}^m \) of \( \ker(S-1) \), let \( f^{(0)}_j = (x^{(j)}_k)_{-\infty < k < +\infty} \), where \( x^{(j)}_k \in \mathbb{C}^n \) for all \( k \). Then \( S f^{(0)}_j = f^{(0)}_j \) implies that \( W_{k+1} x^{(j)}_k = x^{(j)}_{k+1} \) for all \( k \) and \( 1 \leq j \leq m \).

Since \( \sum_{k=-\infty}^{+\infty} \| e^{i\frac{k\pi}{j}} x^{(j)}_k \|^2 = \sum_{k=-\infty}^{+\infty} \| x^{(j)}_k \|^2 < \infty \), \( f^{(i)}_j := (e^{i\frac{k\pi}{j}} x^{(j)}_k)_{-\infty < k < +\infty} \) is in \( K \) and

\[
W_{k+1} e^{i\frac{k\pi}{j}} x^{(j)}_k = e^{i\frac{k\pi}{j}} W_{k+1} x^{(j)}_k = e^{i\frac{k\pi}{j}} x^{(j)}_{k+1} = \lambda_k e^{i\frac{(k+1)\pi}{j}} x^{(j)}_{k+1}.
\]

Thus, \( S f^{(i)}_j = \lambda_k f^{(i)}_j \). Clearly, \( \{ f^{(i)}_1, \ldots, f^{(i)}_m \} \) is linear independent. So, it is a basis of \( \ker(S-\lambda_k) \). Hence,

\[
\bigvee \{ f^{(i)}_j : l \geq 1, 1 \leq j \leq m \} = K.
\]

Assume that \( m < n \). Then there exists a non-zero vector \( x \in \mathbb{C}^n \) such that \( x \perp x^{(j)}_0 \) for \( j = 1, 2, \ldots, m \). So, \( x \perp \bigvee \{ f^{(i)}_j : l \geq 1, 1 \leq j \leq m \} \). This contradiction completes the proof.

By the Theorem and its proof, we immediately get the following corollaries.

**Corollary 1.** Let \( S \) be a bilateral scalar weighted shift. Then either \( S \) or \( S^* \) is a Cowen-Douglas operator if and only if \( \sigma_e(S) \neq \sigma(S) \).

**Corollary 2.** Let \( S \) be a bilateral operator weighted shift. If either \( S \) or \( S^* \) is a Cowen-Douglas operator, then \( \sigma_e(S) \) is not connected.

**Remark 2.** An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be strongly irreducible if it does not commute with any nontrivial idempotents. A lot of work has been done studying strongly irreducible operators (cf. [3]). By the Riesz decomposition theorem, if \( \sigma(T) \) is not connected, then \( T \) is not strongly irreducible. For a unilateral operator weighted shift \( S_+ \), it follows from Theorem 3.1 in [5] that if \( \sigma_e(S_+) \) is not connected, then \( S_+ \) is not strongly irreducible. For a bilateral operator weighted shift \( S \), however, the following example shows that a similar claim is false even though \( S \) is a Cowen-Douglas operator.

**Example.** Let \( w_k = 2 \) for \( k < 0 \), \( w_k = 1 \) for \( k \geq 0 \) and let \( S \sim \{ w_k \}_{k=\infty}^{+\infty} \) be a bilateral scalar weighted shift. By Lemma 1, \( \sigma_e(S) = \{ \lambda : |\lambda| = 1 \text{ or } 2 \} \), which is not connected. Also, by Theorem 9 in [6], \( \text{ind}(S - \lambda) = 1 \) for all \( \lambda \in \Omega := \{ \lambda : 1 < |\lambda| < 2 \} \). Thus, it follows from our theorem that \( S \in B_1(\Omega) \). Hence, \( S \) is strongly irreducible (see [2]).

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