THE STRUCTURE OF QUANTUM SPHERES

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Abstract. We show that the C*-algebra $C(S^2_{q^n+1})$ of a quantum sphere $S^2_{q^n+1}$, $q > 1$, consists of continuous fields \( \{f_t\}_{t \in \mathbb{T}} \) of operators $f_t$ in a C*-algebra $A$, which contains the algebra $\mathcal{K}$ of compact operators with $A = \mathcal{K}$, such that $\rho_*(f_t)$ is a constant function of $t \in \mathbb{T}$, where $\rho: A \to \mathbb{T}$ is the quotient map and $\mathbb{T}$ is the unit circle.

1. Quantum sphere and groupoid

In this section, we identify the C*-algebra $C(S^2_{q^n+1})$ of a quantum sphere $S^2_{q^n+1}$, $q > 1$, with a concrete groupoid C*-algebra $C^*(\mathfrak{G}_n)$ of a concrete groupoid $\mathfrak{G}_n$, independent of $q$, found in [Sh3]. For the background material of groupoid and group C*-algebras, we refer readers to the books of Renault [R] and Pedersen [P].

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spheres $S^{2n+1}_q = SU(n)_q/SU(n+1)_q$ defined as homogeneous quantum spaces can be identified with

$$C(S^{2n+1}_q) = C^*(\{u_{n+1,m} | 1 \leq m \leq n+1\}).$$

Let $\mathbb{Z}_\geq = \mathbb{N} \cup \{0\}$, and regard $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{+\infty\}$ and $\overline{\mathbb{Z}}_\geq := \mathbb{Z}_\geq \cup \{+\infty\}$ as topological spaces with their canonical topologies. We use $\mathcal{H}^n := \mathbb{Z}^n \times \mathbb{Z}^n |_{\overline{\mathbb{Z}}_\geq}$ to denote the transformation group groupoid $\mathbb{Z}^n \times \mathbb{Z}^n$ restricted to the positive “cone” $\overline{\mathbb{Z}}_\geq^n$ of its unit space $\overline{\mathbb{Z}}^n$, and use $\mathcal{F}^n = \mathbb{Z} \times (\mathbb{Z}^n \times \mathbb{Z}^n |_{\overline{\mathbb{Z}}_\geq})$ to denote the direct product of the group $\mathbb{Z}$ and the groupoid $\mathcal{H}^n, \mathcal{R}, \mathcal{MR}, \mathcal{CM}$.

Let $\approx$ be the equivalence relation on $\overline{\mathbb{Z}}_\geq^n := (\overline{\mathbb{Z}}_\geq)^n$ that is generated by $w \approx w'$ for $w, w' \in \overline{\mathbb{Z}}_\geq^n$ such that for some $1 \leq i \leq n$, $w_j = w'_j$ for all $j \leq i$ and $w'_j = \infty$ for all $j \geq i$. This equivalence relation can be canonically extended to equivalence relations $\sim$ on spaces like $\mathcal{H}^n$ or $\mathcal{F}^n$ by defining $(x, w) \sim (x', w')$ if and only if $x = x'$ and $w \approx w'$ for $(x, w), (x', w') \in \mathcal{H}^n$, and $(z, x, w) \sim (z', x', w')$ if and only if $(z, x) = (z', x')$, and $w \approx w'$ for $(z, x, w), (z', x', w') \in \mathcal{F}^n$.

It is proved in [N3] that $C(S_q^{2n+1}) \simeq C^*(\widehat{\mathcal{S}}_n)$ with $\widehat{\mathcal{S}}_n := \widehat{\mathcal{S}}/\sim$ a subquotient groupoid of $\mathcal{F}^n$ where

$$\widehat{\mathcal{S}}_n := \{(z, x, w) \in \mathcal{F}^n | \text{ for any } 1 \leq i \leq n, \text{ if } w_i = \infty, \text{ then } x_i = -z - x_1 - x_2 - ... - x_{i-1} \text{ and } x_{i+1} = ... = x_n = 0\}$$

is a subgroupoid of $\mathcal{F}^n$.

We first note that by a “change of variables” $k := z + x_1 + x_2 + ... + x_n$, the conditions

$$x_i = -z - x_1 - x_2 - ... - x_{i-1} \text{ and } x_{i+1} = ... = x_n = 0$$

in defining $\widehat{\mathcal{S}}_n$, can be replaced by

$$k = 0 \text{ and } x_{i+1} = ... = x_n = 0.$$ More precisely, the bijection

$$(z, x, w) \mapsto (z + x_1 + x_2 + ... + x_n, x, w)$$

defines a homeomorphic groupoid isomorphism from $\mathcal{S}_n$ to the subgroupoid

$$\mathcal{S}_n := \{(k, x, w) \in \mathcal{F}^n | \text{ for any } 1 \leq i \leq n, \text{ if } w_i = \infty, \text{ then } k = 0 \text{ and } x_{i+1} = ... = x_n = 0\}$$

of $\mathcal{F}^n$. Defining $\mathcal{S}_n := \mathcal{S}_n/\sim$, we get a groupoid $\mathcal{S}_n$ isomorphic to $\mathcal{S}_n$ since the above groupoid isomorphism preserves the equivalence relation $\sim$.

**Proposition 1.** For $q > 1$,

$$C(S_q^{2n+1}) \simeq C^*(\mathcal{S}_n).$$
2. Structure theorem

In this section, we recursively characterize $C(S^2_{q^n})$ as an algebra of fields of operators and hence determine $C(S^2_{q^n})$ up to isomorphism.

We first note that $\mathfrak{S}_n \subset \mathbb{Z} \times \mathfrak{S}_n \subset \mathcal{F}^n$ and

$$\mathfrak{S}_n \subset \mathbb{Z} \times \mathfrak{S}_n$$

where $\mathfrak{S}_n$ is the subgroupoid

$$\mathfrak{S}_n := \{(x, w) \in \mathcal{H}^n \mid \text{for any } 1 \leq i \leq n, \text{ if } w_i = \infty,$$

then $x_{i+1} = \ldots = x_n = 0 \}$

of $\mathcal{H}^n$ and $\mathfrak{S}_n := \mathfrak{S}_n/\sim$. The unit space of $\mathfrak{S}_n$ (or $\mathbb{Z} \times \mathfrak{S}_n$, or $\mathfrak{S}_n$) is $\widetilde{W} := \mathbb{Z}^n_{\geq 0}$ while the unit space of $\mathfrak{S}_n$ (or $\mathbb{Z} \times \mathfrak{S}_n$, or $\mathfrak{S}_n$) is the quotient space $W := \mathbb{Z}^n_{\geq 0}$. The closed subset $\widetilde{W}_n := \mathbb{Z}^n_{\geq 0} \setminus \mathbb{Z}^n_{\geq 2}$ of $\widetilde{W}$ and its complement $\widetilde{W} \setminus \widetilde{W}_n = \mathbb{Z}^n_{\geq 2}$ are closed under the equivalence relation $\approx$ and are invariant (under the $\mathfrak{S}_n$-action) subsets of $\widetilde{W}$. Correspondingly, we have the closed subset $W_n := \mathbb{Z}^n_{\geq 0} \approx W$ and its complement $W \setminus W_n$ as invariant subsets of the unit space $W$ of $\mathfrak{S}_n$. By the general theory of groupoid $C^*$-algebras [R], we have the short exact sequence

$$0 \to C^* (\mathfrak{S}_n |_{W \setminus W_n}) \overset{\rho_*}{\to} C^* (\mathfrak{S}_n) \overset{\iota_*}{\to} C^* (\mathfrak{S}_n |_{W_n}) \to 0$$

where $\rho_*$ is induced by the restriction map $\rho$ on $C_c (\mathfrak{S}_n)$ and $\iota_*$ is induced by the inclusion map $\iota$ on $C_c (\mathfrak{S}_n |_{W \setminus W_n})$, and similarly the short exact sequence

$$0 \to C^* ((\mathbb{Z} \times \mathfrak{S}_n) |_{W \setminus W_n}) \to C^* (\mathbb{Z} \times \mathfrak{S}_n) \to C^* ((\mathbb{Z} \times \mathfrak{S}_n) |_{W_n}) \to 0.$$

Since clearly $(\mathbb{Z} \times \mathfrak{S}_n) |_{W \setminus W_n} \cong \mathbb{Z} \times (\mathfrak{S}_n |_{W \setminus W_n})$ and $(\mathbb{Z} \times \mathfrak{S}_n) |_{W_n} \cong \mathbb{Z} \times (\mathfrak{S}_n |_{W_n})$, we get the commutative diagram

$$\begin{array}{cccc}
0 & \to & C^* (\mathbb{Z} \times (\mathfrak{S}_n |_{W \setminus W_n})) & \overset{\sim}{\to} \ C^* (\mathbb{Z} \times \mathfrak{S}_n) & \overset{\sim}{\to} \ C^* (\mathfrak{S}_n |_{W_n}) & \to & 0 \\
0 & \to & C^* (\mathbb{Z} \times (\mathfrak{S}_n |_{W \setminus W_n})) & \overset{\sim}{\to} \ C^* (\mathbb{Z} \times \mathfrak{S}_n) & \overset{\sim}{\to} \ C^* (\mathfrak{S}_n |_{W_n}) & \to & 0 \\
0 & \to & C^* (\mathbb{Z} \times (\mathfrak{S}_n |_{W \setminus W_n})) & \overset{\sim}{\to} \ C^* (\mathbb{Z} \times \mathfrak{S}_n) & \overset{\sim}{\to} \ C^* (\mathfrak{S}_n |_{W_n}) & \to & 0 \\
0 & \to & C^* (\mathbb{Z} \times (\mathfrak{S}_n |_{W \setminus W_n})) & \overset{\sim}{\to} \ C^* (\mathbb{Z} \times \mathfrak{S}_n) & \overset{\sim}{\to} \ C^* (\mathfrak{S}_n |_{W_n}) & \to & 0
\end{array}$$

of exact rows.

Clearly the equivalence relation $\approx$ on $\mathbb{Z} \times \mathfrak{S}_n = \mathbb{Z}^n_{\geq 0}$ is trivial, and hence $\mathfrak{S}_n |_{W \setminus W_n}$ $\cong \mathfrak{S}_n |_{W \setminus W_n}$ and $\mathfrak{S}_n |_{W \setminus W_n} \cong \mathfrak{S}_n |_{W \setminus W_n}$. Furthermore

$$\mathfrak{S}_n |_{\mathbb{Z} \times \mathfrak{S}_n} = \{(k, x, w) \in \mathcal{F}^n \mid w \in \mathbb{Z}_{\geq 0}^n \} = \mathbb{Z} \times \mathcal{H}^n |_{\mathbb{Z}_{\geq 0}^n}.$$ 

and similarly $\mathfrak{S}_n |_{\mathbb{Z} \times \mathfrak{S}_n} = \mathcal{H}^n |_{\mathbb{Z}_{\geq 2}^n}$. So we get $\mathfrak{S}_n |_{W \setminus W_n} = \mathbb{Z} \times (\mathfrak{S}_n |_{W \setminus W_n})$, and the commutative diagram

$$\begin{array}{cccc}
0 & \to & C^* (\mathfrak{S}_n |_{W \setminus W_n}) & \overset{\sim}{\to} \ C^* (\mathfrak{S}_n) & \overset{\sim}{\to} \ C^* (\mathfrak{S}_n |_{W_n}) & \to & 0 \\
0 & \to & \mathcal{K} (\ell^2 (\mathbb{Z}_{\geq 0}^n)) & \to \mathcal{B} (\ell^2 (\mathbb{Z}_{\geq 0}^n))
\end{array}$$

via the faithful regular representation [R] [MR] of $C^* (\mathfrak{S}_n)$ on $\ell^2 (\mathbb{Z}_{\geq 0}^n)$.

On the other hand, $\mathfrak{S}_n |_{\mathbb{Z} \times \mathfrak{S}_n}$ consists of $(k, x, w) \in \mathfrak{S}_n$ with $w_i = \infty$ for some $i \leq n$ and hence $k = 0$. So $\mathfrak{S}_n |_{\mathbb{Z} \times \mathfrak{S}_n} = \{0\} \times \mathfrak{S}_n |_{\mathbb{Z} \times \mathfrak{S}_n}$ and

$$\mathfrak{S}_n |_{\mathbb{Z} \times \mathfrak{S}_n} = \{0\} \times \mathfrak{S}_n |_{W_n} \subset \mathbb{Z} \times \mathfrak{S}_n |_{W_n}.$$
Now it is clear that
\[
\mathfrak{G}_n = (\mathfrak{G}_n|_{W\setminus W'_n}) \cup (\mathfrak{G}_n|_{W'_n}) = (\mathbb{Z} \times (\mathfrak{F}_n|_{W\setminus W'_n})) \cup \{(0) \times \mathfrak{F}_n|_{W'_n}\}
\]
\[
= (\mathbb{Z} \times (\mathfrak{F}_n|_{W\setminus W'_n})) \cup \{(0) \times \mathfrak{F}_n\}
\]
is an open subgroupoid of \(\mathbb{Z} \times \mathfrak{F}_n\), and we have the commuting diagram
\[
\begin{array}{cccc}
0 & \rightarrow & C^* (\mathfrak{G}_n|_{W\setminus W'_n}) & \rightarrow & C^* (\mathfrak{G}_n) & \rightarrow & C^* ((0) \times \mathfrak{F}_n|_{W'_n}) & \rightarrow & 0 \\
0 & \rightarrow & C^* (\mathbb{Z} \times (\mathfrak{F}_n|_{W\setminus W'_n})) & \rightarrow & C^* (\mathbb{Z} \times \mathfrak{F}_n) & \rightarrow & C^* (\mathbb{Z} \times (\mathfrak{F}_n|_{W'_n})) & \rightarrow & 0 \\
0 & \rightarrow & C (\mathcal{T} \otimes \mathcal{K}) (\ell^2 (\mathbb{Z}^n_1)) & \xrightarrow{\text{id} \otimes \rho_*} & C (\mathcal{T} \otimes C^* (\mathfrak{F}_n)) & \xrightarrow{\text{id} \otimes \rho_*} & C (\mathcal{T} \otimes C^* (\mathfrak{F}_n|_{W'_n})) & \rightarrow & 0
\end{array}
\]
of exact rows, in which \(C^* (\mathfrak{G}_n)\) is embedded in \(C (\mathcal{T} \otimes C^* (\mathfrak{F}_n)) \cong C (\mathcal{T}, C^* (\mathfrak{F}_n))\) as an algebra containing \(C (\mathcal{T} \otimes \mathcal{K} (\ell^2 (\mathbb{Z}^n_1)))\) and \(C^* (\mathfrak{G}_n|_{W'_n})\) is embedded in \(C (\mathcal{T} \otimes C^* (\mathfrak{F}_n|_{W'_n}))\) as
\[
C^* (\{(0) \times (\mathfrak{F}_n|_{W'_n})\}) \cong C^* (\{(0)\}) \otimes C^* ((\mathfrak{F}_n|_{W'_n})) \cong \mathbb{C} \otimes C^* ((\mathfrak{F}_n|_{W'_n})).
\]
So
\[
C^* (\mathfrak{G}_n) \cong (\text{id} \otimes \rho_*)^{-1} (\mathbb{C} \otimes C^* ((\mathfrak{F}_n|_{W'_n}))).
\]

We claim that \(\mathfrak{G}_n|_{W'_n}\) is isomorphic to the groupoid \(\mathfrak{G}_{n-1}\). In fact, \(\mathfrak{G}_n|_{W'_n}\) consists of \((k, x, w) \in \mathfrak{G}_n\) with \(w_1 = \infty\) for some \(i \leq n\) and hence \(k = 0\). So by considering the smallest \(i\) with \(w_1 = \infty\), we get
\[
\mathfrak{G}_n|_{W'_n} = \{(0, x, w) \in \mathcal{F}_n|_n \mid \text{for some } i \leq n, \ w_i = \infty, \ x_{i+1} = \ldots = x_n = 0 \}
\]
but \(w_j < \infty\) for all \(j < i\).

Note that the map \(\tilde{\phi}\) sending \((0, x, w) \in \mathfrak{G}_n|_{W'_n}\) to \((k', x', w') \in \mathcal{F}^{n-1}\), where \(k' = x_n\), and \(x'_i = x_i\) and \(w'_i = w_i\) for all \(i \leq n-1\), takes values in \(\mathfrak{G}_{n-1}\), because if \(w'_i = \infty\) for some \(i \leq n-1\), then \(w_1 = \infty\) and hence \(k' = x_n = 0\) and \(x'_j = x_j = 0\) for all \(i < j \leq n-1\). It is not hard to verify that \(\tilde{\phi}\) is a surjective groupoid morphism from \(\mathfrak{G}_n|_{W'_n}\) to \(\mathfrak{G}_{n-1}\). Furthermore \(\tilde{\phi}\) preserves the equivalence relation \(\sim\) and hence induces a homeomorphic groupoid isomorphism \(\phi\) from the quotient groupoid \(\mathfrak{G}_n|_{W'_n} = \mathfrak{G}_n|_{W'_n}/\sim\) to the quotient groupoid \(\mathfrak{G}_{n-1} = \mathfrak{G}_{n-1}/\sim\). So we have
\[
C^* (\mathfrak{F}_n|_{W'_n}) \cong C^* (\mathfrak{G}_n|_{W'_n}) \cong C^* (\mathfrak{G}_{n-1}) \cong C (S_q^{2n-1}).
\]

We conclude the above discussion in the following theorem.

**Theorem 2.** There is a \([\mathcal{A}] \supset \mathcal{K} (\ell^2 (\mathbb{Z}^n_1))\) of \([\mathcal{B}] (\ell^2 (\mathbb{Z}^n_1))\) and a short exact sequence
\[
0 \rightarrow \mathcal{K} (\ell^2 (\mathbb{Z}^n_1)) \subset \mathcal{A} \xrightarrow{\rho_*} C (S_q^{2n-1}) \rightarrow 0
\]
such that
\[
C (S_q^{2n+1}) \cong (\text{id}_{\mathcal{T}} \otimes \rho_*)^{-1} (\mathbb{C} \otimes C (S_q^{2n-1}))
\]
\[
\cong \{f \in C (\mathcal{T}, \mathcal{A}) \mid \rho_* \circ f \text{ is a constant function on } \mathcal{T}\}
\]
where \(\text{id}_{\mathcal{T}} \otimes \rho_* : C (\mathcal{T}, \mathcal{A}) \rightarrow C (\mathcal{T} \otimes C (S_q^{2n-1}))\) and \(C (\mathcal{T}, \mathcal{A})\) is the algebra of continuous fields of operators in \(\mathcal{A}\) over the unit circle \(\mathcal{T}\).
The Structure of Quantum Spheres

References


