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A GENERALIZATION OF BENDIXSON'S CRITERION

MICHAL FEČKAN

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ABSTRACT. Bendixson's condition on the nonexistence of periodic solutions for planar ordinary differential equations is extended to higher dimensional ordinary differential equations with first integrals to preclude the existence of certain invariant Lipschitz compact submanifolds for those equations.

1. Introduction

Criteria of Bendixson and Dulac are well-known on the nonexistence of periodic solutions for planar ordinary differential equations [1], [6]. Extensions to higher dimensional cases are given by W.B. Demidowitsch [5], R.A. Smith [15], [16], S.N. Busenberg and P. van den Driessche [2], G. Butler, R. Schmid and P. Waltman [3], J.S. Muldowney [14], Y. Li and J.S. Muldowney [9], [10], and M.Y. Li and J.S. Muldowney [11], [12]. Conditions precluding the existence of nontrivial periodic orbits for mappings in $\mathbb{R}^n$ are given by C.C. McCluskey and J.S. Muldowney [13].

Stimulated by the papers [8]–[14], the author of this note derived in [7] criteria on the nonexistence of certain invariant objects for finite dimensional dynamical systems. There are also studied dynamical systems on invariant submanifolds which naturally arise in differential equations possessing first integrals. It was already shown by W.B. Demidowitsch [5] that for a differential equation given by

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

and possessing a first integral with $n = 3$, the Bendixson condition $\operatorname{div} f \neq 0$ in a simply connected region of a nondegenerate level set of the first integral precludes periodic solutions of (1.1) in the region. Unfortunately the Demidowitsch proof is incorrect and we present a correction here. This result was extended by M.Y. Li in his Ph.D. dissertation [8] in the context of former generalizations of Bendixson's condition proved by J.S. Muldowney [14].

We show in this note that if (1.1) has p independent first integrals, the Bendixson criterion $\operatorname{div} f \neq 0$ implies the nonexistence of certain invariant $n-p-1$-dimensional objects of (1.1) on each nondegenerate level set of the first integrals. Similar results are proved in [12].

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2. Preliminaries

We suppose that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. We need the following definition and result of [7].

Let $M \subset \mathbb{R}^l$ be an m-dimensional compact smooth orientable submanifold with a nonempty border ∂M [17]. Hence ∂M is an $m-1$-dimensional compact smooth orientable submanifold. Of course, we always assume that $m \geq 2$.

Let $V \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold of $\mathbb{R}^n$ with empty border $\partial V = \emptyset$.

Definition 2.1 ([7])**.** Let $\beta \in \text{Lip } (M, \mathbb{R}^n)$ be such that $\beta(M) \subset V$ and $\tau = \beta/\partial M$ satisfy:

(I) τ is injective on ∂M.
(II) The inverse $\tau^{-1}\colon \tau(\partial M) \to \mathbb{R}^l$ is Lipschitz on the set $\tau(\partial M) \subset \mathbb{R}^n$.

We shall call the set $\mathcal{S} = \tau(\partial M)$ an $m-1$**-V-L-boundary** of V. It is a generalization of smooth submanifolds of V.

Let $T_vV^{\perp}$, $v \in V$, be the orthogonal vector bundle to the tangent vector bundle T_vV and let Q_v be the orthogonal projection onto $T_vV^{\perp}$ in $\mathbb{R}^n$ with respect to the usual scalar product on $\mathbb{R}^n$. Let $N(v)$ be given by

$$N(v) = Q_v Df(v)/T_vV^{\perp}\colon T_vV^{\perp} \to T_vV^{\perp}\,.$$

We note that $\text{div} f(v) = \text{tr} Df(v) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $Df(v)$.

Theorem 2.2 ([7])**.** *If manifold V is invariant for* (1.1) *and the inequality*

$$\text{div} f(v) - \text{tr}\, N(v) \neq 0 \tag{2.1}$$

holds for any $v \in V$, then there is no $k-1$-V-L-boundary $\mathcal{S}$ of V which is invariant for (1.1).

3. Systems with first integrals

We assume the existence of p first integrals $g_1, g_2, \cdots, g_p \in C^2(\mathbb{R}^n, \mathbb{R})$ of (1.1). Let $0 \in \mathbb{R}^p$ be a regular value of the mapping $G = (g_1, g_2, \cdots, g_p)$. Hence $V = G^{-1}(0)$ is a smooth $n-p$-dimensional submanifold of $\mathbb{R}^n$ without border.

We now show that condition (2.1) for this case can be simplified. Of course we have $\text{tr}N(v) = \text{tr}Q_vS(v)/T_vV^{\perp}$, where $S(v) = \big(Df(v) + Df(v)^*\big)/2$.

Lemma 3.1. *If $\lambda \neq 0$ is an eigenvalue of $Q_vS(v)$, then it is also an eigenvalue of $Q_vS(v)/T_vV^{\perp}$.*

Proof. Let $Q_vS(v)w = \lambda w$ with $\lambda = \lambda_1 + \imath\lambda_2$, $w = w_1 + \imath w_2$. Then

$$Q_vS(v)w_1 = \lambda_1 w_1 - \lambda_2 w_2, \quad Q_vS(v)w_2 = \lambda_1 w_2 + \lambda_2 w_1\,.$$

If $w_3 \in T_vV$, then we get

$$\begin{aligned} \lambda_1\langle w_1, w_3\rangle - \lambda_2\langle w_2, w_3\rangle = 0\,, \\ \lambda_1\langle w_2, w_3\rangle + \lambda_2\langle w_1, w_3\rangle = 0\,. \end{aligned}$$

Since $\lambda \neq 0$ we get $\langle w_1, w_3\rangle = \langle w_2, w_3\rangle = 0$, i.e. $w_1, w_2 \in T_vV^{\perp}$. □

Lemma 3.1 implies that $\mathrm{tr}N(v) = \mathrm{tr}Q_v S(v)$. Now we compute $\mathrm{tr}Q_v S(v)$. Since $0 \in \mathbb{R}^p$ is a regular value of G, the vectors $\mathrm{grad}g_i(v)$, $i = 1, 2, \cdots, p$, form a basis of $T_v V^\perp$. The vector $Q_v x = \sum_{i=1}^{p} c_i \mathrm{grad}g_i$ for an $x \in \mathbb{R}^n$ is determined by the equations

$$\left\langle x - \sum_{i=1}^{p} c_i \mathrm{grad}g_i, \mathrm{grad}g_j \right\rangle = 0\,, \quad j = 1, 2, \cdots, p\,.$$

Let us put $e_k = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$, where 1 is on the k-th position. Then

$$S(v)e_k = \frac{1}{2}\Big(\frac{\partial f_1}{\partial x_k} + \frac{\partial f_k}{\partial x_1}, \frac{\partial f_2}{\partial x_k} + \frac{\partial f_k}{\partial x_2}, \cdots, \frac{\partial f_n}{\partial x_k} + \frac{\partial f_k}{\partial x_n}\Big)\,,$$

and $Q_v S(v) e_k = \sum_{i=1}^{p} c_{ki} \mathrm{grad}g_i$ is determined by the equations

$$\frac{1}{2}\sum_{s=1}^{n}\Big(\frac{\partial f_s}{\partial x_k} + \frac{\partial f_k}{\partial x_s}\Big)\frac{\partial g_j}{\partial x_s} = \sum_{i=1}^{p} c_{ki}\langle \mathrm{grad}g_i, \mathrm{grad}g_j\rangle, \qquad j = 1, 2, \cdots, p\,. \tag{3.1}$$

Consequently we get

$$\mathrm{tr}Q_v S(v) = \sum_{k=1}^{p}\sum_{i=1}^{p} c_{ki}\frac{\partial g_i}{\partial x_k}\,. \tag{3.2}$$

The right-hand side of (3.2) has, according to (3.1), the form

$$\frac{1}{2}\sum_{k=1}^{p}\sum_{i=1}^{p} A_{ki}\frac{\partial g_i}{\partial x_k}/B\,, \tag{3.3}$$

where

$$A_{ki} =$$

$$\det\begin{pmatrix} \langle \mathrm{grad}g_1, \mathrm{grad}g_1\rangle & \cdots & \sum_{s=1}^{n}\Big(\frac{\partial f_s}{\partial x_k} + \frac{\partial f_k}{\partial x_s}\Big)\frac{\partial g_1}{\partial x_s} & \cdots & \langle \mathrm{grad}g_p, \mathrm{grad}g_1\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \mathrm{grad}g_1, \mathrm{grad}g_p\rangle & \cdots & \sum_{s=1}^{n}\Big(\frac{\partial f_s}{\partial x_k} + \frac{\partial f_k}{\partial x_s}\Big)\frac{\partial g_p}{\partial x_s} & \cdots & \langle \mathrm{grad}g_p, \mathrm{grad}g_p\rangle \end{pmatrix}$$

and

$$B = \det\begin{pmatrix} \langle \mathrm{grad}g_1, \mathrm{grad}g_1\rangle & \cdots & \langle \mathrm{grad}g_p, \mathrm{grad}g_1\rangle \\ \vdots & \vdots & \vdots \\ \langle \mathrm{grad}g_1, \mathrm{grad}g_p\rangle & \cdots & \langle \mathrm{grad}g_p, \mathrm{grad}g_p\rangle \end{pmatrix}.$$

By using the identities

$$\sum_{s=1}^{n}\frac{\partial g_j}{\partial x_s} f_s = 0, \quad j = 1, 2, \cdots, p\,,$$

we get the new ones

$$\sum_{k=1}^{n}\sum_{s=1}^{n}\frac{\partial g_i}{\partial x_k}\frac{\partial^2 g_j}{\partial x_k \partial x_s} f_s + \frac{\partial g_i}{\partial x_k}\frac{\partial g_j}{\partial x_s}\frac{\partial f_s}{\partial x_k} = 0\,, \qquad i, j = 1, 2, \cdots, p\,.$$

These identities imply

$$-\sum_{s=1}^{n} f_s \frac{\partial}{\partial x_s}\langle \mathrm{grad} g_i, \mathrm{grad} g_j\rangle = \sum_{s=1}^{n}\sum_{k=1}^{n}\Big(\frac{\partial f_s}{\partial x_k}+\frac{\partial f_k}{\partial x_s}\Big)\frac{\partial g_i}{\partial x_k}\frac{\partial g_j}{\partial x_s}, \quad i,j=1,2,\cdots,p\,.$$

By using these equalities in the formulas for A_{ki}, (3.3) has the form

$$\sum_{s=1}^{p}\sum_{i=1}^{p} -f_s B_{si}/B\,, \tag{3.4}$$

where

$$2B_{si} = \det\begin{pmatrix} \langle \mathrm{grad} g_1, \mathrm{grad} g_1\rangle & \cdots & \frac{\partial}{\partial x_s}\langle \mathrm{grad} g_i, \mathrm{grad} g_1\rangle & \cdots & \langle \mathrm{grad} g_p, \mathrm{grad} g_1\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \mathrm{grad} g_1, \mathrm{grad} g_p\rangle & \cdots & \frac{\partial}{\partial x_s}\langle \mathrm{grad} g_i, \mathrm{grad} g_p\rangle & \cdots & \langle \mathrm{grad} g_p, \mathrm{grad} g_p\rangle \end{pmatrix}.$$

Consequently (3.4) implies

$$\mathrm{tr} N = -\langle f, \mathrm{grad} B\rangle/(2B)\,. \tag{3.5}$$

Furthermore, clearly the dynamics of (1.1) on V are the same as for $\dot{x}=\alpha(x)f(x)$ for a positive C^1-smooth function α defined near V. Since

$$\mathrm{div}\big(\alpha(x)f(x)\big) = \alpha(x)\mathrm{div} f(x) + \langle \mathrm{grad}\alpha(x), f(x)\rangle\,,$$

according to (3.5) condition (2.1) can be replaced by

$$\alpha \mathrm{div} f + \langle \mathrm{grad}\alpha + \alpha W, f\rangle \neq 0$$

on V, where $W = \mathrm{grad} B/(2B)$.

Lemma 3.2. *There is a C^1-smooth positive function α defined near V such that $\langle \mathrm{grad}\alpha + \alpha W, f\rangle = 0$ on V.*

Proof. We have $\langle f, W\rangle = \frac{1}{2B}\langle f, \mathrm{grad} B\rangle$ on V. Now by taking $\alpha = B^{-1/2}$ we get the result. □

Summarizing, we have the following main result of this note.

Theorem 3.3. *Let $g_1, g_2, \cdots, g_p \in C^2(\mathbb{R}^n, \mathbb{R})$ be first integrals of* (1.1). *If $V = G^{-1}(0)$ is a nondegenerate level set of the mapping $G = (g_1, g_2, \cdots, g_p)$ and in addition* $\mathrm{div} f \neq 0$ *on V, then there is no $n-p-1$-V-L-boundary $\mathcal{S}$ of V which is invariant for* (1.1).

Theorem 3.3 gives for $n = 3$ the Demidowitsch result [5], but the proof of [5] is incorrect: formula (13) is not right. Furthermore, similar results are proved by M.Y. Li and J.S. Muldowney [8], [12], and the author was strongly motivated by [5], [8], [12].

Finally we note that clearly an open bounded subset of V, for V from Theorem 3.3, with a Lipschitz boundary is an $n-p-1$-V-L-boundary of V. Furthermore, any minimally invariant closed set of (1.1) is lying on a level set of G. The minimality means that the invariant set does not contain a smaller nonempty invariant subset of (1.1). Consequently, Theorem 3.3 together with Sard's theorem [17] imply that (1.1) possessing p first integrals of C^{n-p+1}-smoothness and satisfying $\mathrm{div} f \neq 0$ on $\mathbb{R}^n$ generically may have a minimally invariant object, which is an r-V-L-boundary of the nondegenerate level set V, with the dimension r less than $n-p-1$.

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Department of Mathematical Analysis, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia

E-mail address: Michal.Feckan@fmph.uniba.sk