AN OPERATOR INEQUALITY RELATED TO JENSEN’S INEQUALITY

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Abstract. For bounded non-negative operators $A$ and $B$, Furuta showed

$$0 \leq A \leq B \implies A^s B^t A^r \leq (A^{r+} B^{s+} A^{r+})^{\frac{r}{s+t}} (0 \leq r, 0 \leq s \leq t).$$

We will extend this as follows: $0 \leq A \leq B^\lambda C^{\lambda} (0 < \lambda < 1)$ implies

$$A^{r+} (\lambda B^t + (1 - \lambda)C^s) A^{r+} \leq \{A^{r+} (\lambda B^t + (1 - \lambda)C^s) A^{r+}\}^{\frac{r}{s+t}},$$

where $B^\lambda C^{\lambda}$ is a harmonic mean of $B$ and $C$. The idea of the proof comes from Jensen’s inequality for an operator convex function by Hansen-Pedersen.

1. Introduction

Throughout this article, an operator means a bounded linear operator on a Hilbert space. For selfadjoint operators $A, B$ we write $A \leq B$ as usual if $B - A$ is positive semidefinite. A real continuous function $f$ defined on an interval $I$ is said to be operator monotone if $f$ preserves this order, that is, for bounded selfadjoint operators $A, B$ with spectra in $I$,

$$A \leq B \implies f(A) \leq f(B);$$

and it is said to be operator convex if for all selfadjoint operators $A, B$ with spectra in $I$ and for all $\lambda$ in $[0, 1]$

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B).$$

An operator concave function is similarly defined. In [2] Hansen and Pedersen showed that for a non-negative continuous function $f$ on $[0, \infty)$ the following conditions are equivalent:

(i) $f$ is operator monotone,

(ii) $f$ is operator concave,

(iii) $T^* f(A) T \leq f(T^* A T)$ for every contraction $T$ (i.e., $\|T\| \leq 1$) and for every non-negative operator $A$,

(iv) $S^* f(A) S + T^* f(B) T \leq f(S^* A S + T^* B T)$ for every pair of $S, T$ with $S^* S + T^* T \leq 1$ and for all non-negative operators $A, B$.

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It is well-known that \( f(x) = x^a \) \((0 < a \leq 1)\) is operator monotone on \([0, \infty)\), that is,
\[
0 \leq A \leq B \Rightarrow A^a \leq B^a,
\]
which is called the L"owner-Heinz inequality. Therefore, (iv) yields
\[
(1) \quad S^* A^a S + T^* B^a T \leq (S^* A S + T^* B T)^a \quad (0 < a \leq 1).
\]

Related to the L"owner-Heinz inequality, Furuta showed (cf. [5]): for non-negative real numbers \( r, s \) and \( t \) such that \( t \geq s \) and \((r, t) \neq (0, 0)\),
\[
(2) \quad 0 \leq A \leq B \Rightarrow A^\frac{r}{t} B^\frac{s}{t} \leq (A^\frac{r}{s} B^\frac{s}{s})^\frac{r}{s}, \quad (B^\frac{r}{s} A^\frac{s}{s} B^\frac{s}{s})^\frac{s}{r} \leq B^\frac{r}{s} A^\frac{s}{s} B^\frac{s}{s}.
\]

For non-negative operators \( A \) and \( B \) and for a real number \( \lambda \) with \( 0 < \lambda < 1 \), the harmonic mean is defined by
\[
A^\lambda B := (\lambda A^{-1} + (1 - \lambda) B^{-1})^{-1}
\]
if \( A \) and \( B \) are invertible, and defined by the weak limit of \((A + \epsilon)^\lambda(B + \epsilon)^{-1}\) as \( \epsilon \to +0 \) if not. If non-negative operators \( A, B \) are invertible, then we have
\[
\lambda A^{-1} + \mu B^{-1} = (\lambda A + \mu B)^{-1} = (A^{-1} - B^{-1})(\lambda A^{-1} + (\mu B)^{-1})^{-1}(A^{-1} - B^{-1}),
\]
where \( 0 \leq \lambda, \mu \leq 1 \), \( \lambda + \mu = 1 \) (see [10], p. 117 of [2]). This shows that a function \( f(x) = 1/x \) is operator convex on \((0, \infty)\) and that
\[
A^\lambda B \leq \lambda A + (1 - \lambda) B.
\]

We need the following properties of the harmonic mean (cf. [1], [7]):
\[
(\alpha A)^\lambda (\alpha B) = \alpha (A^\lambda B),
\]
\[
A^\lambda B \leq C^\lambda B \text{ if } A \leq C, \text{ and}
\]
\[
A^\lambda B + C^\lambda D \leq (A + C)^\lambda (B + D).
\]

2. Main theorem

From now on, \( \lambda \) and \( \mu \) represent real numbers such that
\[
0 \leq \lambda, \mu \leq 1 \quad \text{and} \quad \lambda + \mu = 1.
\]

We start this section with a simple inequality.

**Lemma 2.1.** Let \( S, T \) be contractions such that \( S^* S + T^* T \leq 1 \), and let \( A, B \) be non-negative operators. Then for \( 0 \leq r \leq s \leq t \) and \( s \neq 0 \),
\[
(S^* A^s S + T^* B^s T)^\frac{r}{s} \leq (S^* A^s S + T^* B^s T)^\frac{r}{s},
\]
\[
(S^* A^s S + T^* B^s T)^\frac{s}{r} \leq (S^* A^s S + T^* B^s T)^\frac{s}{r}.
\]

*Proof.* The first inequality follows from (1). By using the L"owner-Heinz inequality we get the second inequality.

We remark that the above implies that for fixed \( r > 0 \) the operator valued function \((S^* A^s S + T^* B^s T)^\frac{r}{s}\) is increasing on \( r \leq t < \infty \).

**Lemma 2.2.** Let \( H \) and \( K \) be bounded selfadjoint operators such that \( 0 \leq H \leq K \). Then for real numbers \( p, q \) such that \( 0 \leq p \leq q \)
\[
H^\frac{p}{q} K^p H^\frac{q}{q} \leq (H^\frac{p}{q} K^q H^\frac{q}{q})^\frac{p}{q}, \quad K^\frac{p}{q} H^p K^\frac{q}{q} \geq (K^\frac{p}{q} H^q K^\frac{q}{q})^\frac{p}{q}.
\]
Proof. To show the first inequality we may assume that $K$ is invertible. Since $\|K^{-\frac{1}{2}}H\| \leq 1$, by Lemma 2.1, we get
\[
\left(H^{\frac{1}{2}}K^{-\frac{1}{2}}K^{p+1}K^{-\frac{1}{2}}H^{\frac{1}{2}}\right) \leq \left(H^{\frac{1}{2}}K^{-\frac{1}{2}}K^{q+1}K^{-\frac{1}{2}}H^{\frac{1}{2}}\right)^{\frac{p+q}{2}}.
\]
This gives the first inequality. By considering the inverse of $K$ and $H$, the second inequality follows from the first one.

Now we give the main theorem that is an extension of (2).

**Theorem 2.3.** Let $A, B$ and $C$ be non-negative operators. Then for non-negative real numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0),$

\[
A \leq B^\lambda C \implies A^\frac{1}{r}(\lambda B^s + \mu C^s)A^\frac{1}{t} \leq \left\{A^\frac{1}{r}(\lambda B^t + \mu C^t)A^\frac{1}{t}\right\}^{\frac{r+s}{2}}.
\]

Proof. Suppose $r = 0$. Then, since $f(x) = x^{s/t}$ is operator concave, we obtain (3). Thus, we need to show (3) in the case of $r > 0$. Since $A \leq (B + \epsilon)^\lambda (C + \epsilon)$, we may assume that $B$ and $C$ are invertible. We first suppose $0 < r \leq 1$. Since $f(x) = x^r$ is operator monotone and operator concave,

\[
A \leq (\lambda B^{-1} + \mu C^{-1})^{-1}
\]
yields

\[
A^r \leq (\lambda B^{-1} + \mu C^{-1})^{-r} \leq (\lambda B^{-r} + \mu C^{-r})^{-1}.
\]

This implies

\[
A^\frac{1}{r}(\lambda B^{-r} + \mu C^{-r})A^\frac{1}{t} \leq 1.
\]

By Lemma 2.1, for $0 \leq s \leq t$ we get

\[
A^\frac{1}{r}(\lambda B^s + \mu C^s)A^\frac{1}{t} = (\lambda A^\frac{1}{r} B^s A^\frac{1}{t} + \mu A^\frac{1}{r} C^s A^\frac{1}{t}),
\]

\[
\leq (\lambda A^\frac{1}{r} B^t A^\frac{1}{t} + \mu A^\frac{1}{r} C^t A^\frac{1}{t})^{\frac{r+s}{2}}.
\]

This means (3) holds for $0 < r \leq 1$ and for $0 \leq s \leq t$.

Assume (3) holds for $0 < r \leq 2^n$ and for $0 \leq s \leq t$. Take an arbitrary $r$ in $(2^n, 2^{n+1})$. Since $r/2 \leq 2^n$, the assumption says

\[
A^\frac{1}{r}(\lambda B^s + \mu C^s)A^\frac{1}{t} \leq \left\{A^\frac{1}{r}(\lambda B^t + \mu C^t)A^\frac{1}{t}\right\}^{\frac{r+s}{2}}.
\]

in particular,

\[
A^\frac{1}{r} \leq \left\{A^\frac{1}{r}(\lambda B^t + (1 - \lambda) C^t)A^\frac{1}{t}\right\}^{\frac{r+s}{2}}.
\]

Let us apply this to the first inequality of Lemma 2.2 with $p = (2s + r)/r$, $q = (2t + r)/r$. Then we get

\[
A^\frac{1}{r} \leq \left\{A^\frac{1}{r}(\lambda B^t + (1 - \lambda) C^t)A^\frac{1}{t}\right\}^{\frac{r+s}{2}}.
\]

This in conjunction with (4) gives

\[
A^\frac{1}{r}(\lambda B^s + (1 - \lambda) C^s)A^\frac{1}{t} \leq \left\{A^\frac{1}{r}(\lambda B^t + (1 - \lambda) C^t)A^\frac{1}{t}\right\}^{\frac{r+s}{2}}.
\]

\[
\Box
\]

**Corollary 2.4.** Let $A, B$ and $C$ be non-negative operators. Then

\[
A \leq B^\lambda C \implies A^{1+r} \leq \left\{A^\frac{1}{r}(\lambda B^t + (1 - \lambda) C^t)A^\frac{1}{t}\right\}^{\frac{r+s}{2}}.
\]
Theorem 2.3 and

By the continuity of harmonic mean in the norm sense, we may assume that $B; C$ and $(5)$

This has been found by Furuta [4] and called the Furuta inequality. Furuta showed (2) by using (5). Our proof seems to help us clear up the significance of the exponent $(s + r)/(t + r)$ in (2) and hence the exponent $(1 + r)/(t + r)$ in (5) (cf. [3]).

If $A \leq B$ and $A \leq C$, then $A \leq B^1C$ for every $\lambda$; therefore, the inequality in

We proved that Theorem 2.3 and Corollary 2.4 hold for just the harmonic mean of two operators $B$ and $C$; however, it is easy to see that they do even for the harmonic mean of a finite number of operators, too.

Proposition 2.5. Let $B, C$ and $D$ be non-negative operators. Then for non-negative real numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0)$,

$$\lambda B + \mu C \leq D \implies \{D^\frac{s}{t}(B^t \downarrow C^t)D^\frac{s}{t}\}^{\frac{r}{t}} \leq D^\frac{s}{t}(B^t \downarrow C^t)^{\frac{r}{t}}.$$

Proof. By the continuity of harmonic mean in the norm sense, we may assume that $B, C$ and $D$ are all invertible. The condition $\lambda B + \mu C \leq D$ implies

$$D^{-1} \leq B^{-1} \downarrow C^{-1}.$$

By Theorem 2.3, this yields

$$D^{-\frac{s}{t}}(\lambda B^{-s} + \mu C^{-s})D^{-\frac{s}{t}} \leq \{D^{-\frac{s}{t}}(\lambda B^{-t} + \mu C^{-t})D^{-\frac{s}{t}}\}^{\frac{r}{t}}.$$

Take the inverse of both sides of this inequality to get the required inequality. □

Put $s = 1$ in Proposition 2.5. Since

$$B^\downarrow C \leq \lambda B + (1 - \lambda)C \leq D,$$

we get

Corollary 2.6. Let $B, C$ and $D$ be non-negative operators. Then for $0 \leq r$ and $1 \leq t$,

$$\lambda B + \mu C \leq D \implies \{D^\frac{s}{t}(B^t \downarrow C^t)D^\frac{s}{t}\}^{\frac{r}{t}} \leq D^{1+r}.$$

By setting $B = C$ in the above, we get an alternate Furuta inequality:

$$(5)' \quad 0 \leq C \leq D \implies (D^\frac{s}{t}C^tD^\frac{s}{t})^{\frac{r}{t}} \leq D^{1+r}.$$

By combining (5) and $(5)'$, we get, for $0 \leq r$, $1 \leq t$,

$$0 \leq A \leq B \leq C \implies (B^\frac{s}{t}A^tB^\frac{s}{t})^{\frac{r}{t}} \leq B^{1+r} \leq (B^\frac{s}{t}C^tB^\frac{s}{t})^{\frac{r}{t}}.$$

Suppose that $B$ and $C$ are non-negative operators. Replace $A$ with $B^\downarrow C$ in Theorem 2.3 and $D$ with $\lambda B + \mu C$ in Proposition 2.5. Then we get
Corollary 2.7. Let $B, C$ be non-negative operators. Then for non-negative numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0)$,

\[ (B \frac{1}{\lambda} C) \frac{r}{t} (\lambda B^* + \mu C^*) (B \frac{1}{\lambda} C) \frac{r}{t} \leq \{(B \frac{1}{\lambda} C) \frac{r}{t} (\lambda B^t + \mu C^t) (B \frac{1}{\lambda} C) \frac{r}{t}\} \frac{r}{t} \frac{r}{r}, \]

\[ \{(\lambda B + \mu C) \frac{s}{t} (B^t \frac{1}{\lambda} C^t) (\lambda B + \mu C) \frac{s}{t}\} \frac{s}{s} \leq (\lambda B + \mu C) \frac{s}{s} (B^* \frac{1}{\lambda} C^*) (\lambda B + \mu C) \frac{s}{s}. \]

3. Exponential inequality

Theorem 3.1. Let $A, B$ and $C$ be non-negative operators. Then for non-negative real numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0)$,

\[ A \leq B \frac{1}{\lambda} C \implies e^{\frac{r}{t} A} (\lambda e^{sB} + \mu e^{tC}) e^{\frac{r}{t} A} \leq (e^{\frac{r}{t} A} e^{\frac{s}{t} B} e^{\frac{s}{t} A}) \frac{r}{t}. \]

Proof. For every natural number $n, 1 + A/n \leq (1 + B/n)! (1 + C/n)$ because of the properties of harmonic mean given in Section 1.

Therefore, (3) gives

\[ (1 + A/n) \frac{r}{t} \{\lambda(1 + B/n)^{ns} + \mu(1 + C/n)^{nt}\} (1 + A/n) \frac{r}{t} \]

\[ \leq [(1 + A/n) \frac{r}{t} \{\lambda(1 + B/n)^{ns} + \mu(1 + C/n)^{nt}\} (1 + A/n) \frac{r}{t}] \frac{r}{t}. \]

Letting $n \to \infty$, we get the required inequality. □

Remark. Harmonic mean is just defined for non-negative operators. However, it is clear that the above inequality holds for general selfadjoint operators $A, B, C$ if there is a real number $a$ so that $A + a, B + a, C + a \geq 0$ and $A + a \leq (B + a)! (C + a)$.

In particular, putting $B = C$ we can obtain:

Let $A, B$ be (not necessarily non-negative) selfadjoint operators. Then

\[ A \leq B \implies e^{\frac{r}{t} A} e^{sB} e^{\frac{r}{t} A} \leq (e^{\frac{r}{t} A} e^{\frac{s}{t} B} e^{\frac{s}{t} A}) \frac{r}{t}. \]

Proposition 3.2. Let $B, C$ and $D$ be (not necessarily non-negative) selfadjoint operators. Then for non-negative real numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0)$,

\[ \lambda B + \mu C \leq D \implies \{e^{\frac{r}{t} D} (e^{sB} \frac{1}{s} e^{sA}) e^{\frac{r}{t} D}\} \frac{r}{r} \leq e^{\frac{r}{t} D} (e^{\frac{s}{t} B} \frac{1}{s} e^{sC} e^{\frac{s}{t} D}). \]

Proof. For sufficiently large $n, (1 + B/n), (1 + C/n)$ and $(1 + D/n)$ are all non-negative. Since $\lambda (1 + B/n) + \mu (1 + C/n) \leq (1 + D/n)$, taking notice that

\[ (1 + B/n)^{nt} (1 + C/n)^{nt} \to e^{\frac{s}{t} B} \frac{1}{s} e^{sC} \] as $n \to \infty,$

we can derive the desired inequality from Proposition 2.5. □

By putting $s = 0$ in the above remark, we get

\[ A \leq B \implies e^{rA} \leq (e^{\frac{r}{t} A} e^{\frac{s}{t} B} e^{\frac{s}{t} A}) \frac{r}{r}. \]

Similarly, by putting $B = C$ and $s = 0$ in Proposition 3.2 we get

\[ C \leq D \implies (e^{\frac{r}{t} D} e^{\frac{s}{t} C} e^{\frac{s}{t} D}) \frac{s}{s} \leq e^{rD}. \]

These are known inequalities (cf. [3, 9]). Combining them, we see:

For $r, t \geq 0$ with $(r, t) \neq (0, 0)$,

\[ A \leq B \leq C \implies (e^{\frac{r}{t} D} e^{\frac{s}{t} A} e^{\frac{s}{t} B}) \frac{r}{r} \leq e^{rB} \leq (e^{\frac{r}{t} B} e^{\frac{s}{t} C} e^{\frac{s}{t} B}) \frac{r}{r}. \]
REFERENCES

4. T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1 + 2r)q \geq p + 2r$, Proc. Amer. Math. Soc. 101(1987), 85–88. MR 89b:47028

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