AN OPERATOR INEQUALITY RELATED TO JENSEN’S INEQUALITY

MITSURU UCHIYAMA

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Abstract. For bounded non-negative operators $A$ and $B$, Furuta showed

$$0 \leq A \leq B \quad \text{implies} \quad A^{s}B^{t} \leq (A^{r}B^{s})^{\frac{r}{s+t}} \quad (0 \leq r, \ 0 \leq s \leq t).$$

We will extend this as follows: $0 \leq A \leq B' \ C (0 < \lambda < 1)$ implies

$$A^{\frac{1}{\lambda}}(\lambda B^{s} + (1 - \lambda)C^{s}) A^{\frac{1}{\lambda}} \leq \{A^{\frac{1}{\lambda}}(\lambda B^{t} + (1 - \lambda)C^{t}) A^{\frac{1}{\lambda}}\}^{\frac{r}{s+t}},$$

where $B' \ C$ is a harmonic mean of $B$ and $C$. The idea of the proof comes from Jensen’s inequality for an operator convex function by Hansen-Pedersen.

1. Introduction

Throughout this article, an operator means a bounded linear operator on a Hilbert space. For selfadjoint operators $A, B$ we write $A \leq B$ as usual if $B - A$ is positive semidefinite. A real continuous function $f$ defined on an interval $I$ is said to be operator monotone if $f$ preserves this order, that is, for bounded selfadjoint operators $A, B$ with spectra in $I$,

$$A \leq B \quad \Rightarrow \quad f(A) \leq f(B);$$

and it is said to be operator convex if for all selfadjoint operators $A, B$ with spectra in $I$ and for all $\lambda$ in $[0, 1]$

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B).$$

An operator concave function is similarly defined. In [13] Hansen and Pedersen showed that for a non-negative continuous function $f$ on $[0, \infty)$ the following conditions are equivalent:

(i) $f$ is operator monotone,

(ii) $f$ is operator concave,

(iii) $T^{*}f(A)T \leq f(T^{*}AT)$ for every contraction $T$ (i.e., $||T|| \leq 1$) and for every non-negative operator $A$,

(iv) $S^{*}f(A)S + T^{*}f(B)T \leq f(S^{*}AS + T^{*}BT)$ for every pair of $S, T$ with $S^{*}S + T^{*}T \leq 1$ and for all non-negative operators $A, B$. 

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It is well-known that \( f(x) = x^a \) \((0 < a \leq 1)\) is operator monotone on \([0, \infty)\), that is,
\[
0 \leq A \leq B \Rightarrow A^a \leq B^a,
\]
which is called the \(\text{Löwner-Heinz}\) inequality. Therefore, (iv) yields
\[
S^*A^aS + T^*B^aT \leq (S^*AS + T^*BT)^a \quad (0 < a \leq 1).
\]

Related to the Lowner-Heinz inequality, Furuta showed (cf. [5]): for non-negative real numbers \(r, s\) and \(t\) such that \(t \geq s\) and \((r, t) \neq (0, 0)\),
\[
0 \leq A \leq B \Rightarrow A^{\frac{r}{s}}B^{\frac{t}{s}} \leq (A^{\frac{r}{s}}B^{\frac{t}{s}})^{\frac{s}{r+t}}; \quad (B^{\frac{r}{s}}A^{\frac{t}{s}})^{\frac{s}{r+t}} \leq B^{\frac{r}{s}}A^{\frac{t}{s}}B^{\frac{r}{s}}.
\]

For non-negative operators \(A\) and \(B\) and for a real number \(\lambda\) with \(0 < \lambda < 1\), the \textit{harmonic mean} is defined by
\[
A!B := (\lambda A^{-1} + (1 - \lambda)B^{-1})^{-1}
\]
if \(A\) and \(B\) are invertible, and defined by the weak limit of \((A + \epsilon)!B\) as \(\epsilon \to +0\) if not. If non-negative operators \(A, B\) are invertible, then we have
\[
\lambda A^{-1} + \mu B^{-1} - (\lambda A + \mu B)^{-1} = (A^{-1} - B^{-1})((\lambda A)^{-1} + (\mu B)^{-1})^{-1}(A^{-1} - B^{-1}),
\]
where \(0 \leq \lambda, \mu \leq 1\), \(\lambda + \mu = 1\) (see [10], p. 117 of [2]). This shows that a function \(f(x) = 1/x\) is operator convex on \((0, \infty)\) and that
\[
A!_{\lambda}B \leq \lambda A + (1 - \lambda)B.
\]

We need the following properties of the harmonic mean (cf. [1], [7]):
\[
(\alpha A)!_{\lambda}(\alpha B) = \alpha(A!_{\lambda}B),
\]
\[
A!_{\lambda}B \leq C!_{\lambda}B \quad \text{if} \quad A \leq C, \quad \text{and}
\]
\[
A!_{\lambda}B + C!_{\lambda}D \leq (A + C)!_{\lambda}(B + D).
\]

2. Main theorem

From now on, \(\lambda\) and \(\mu\) represent real numbers such that
\[
0 \leq \lambda, \mu \leq 1 \quad \text{and} \quad \lambda + \mu = 1.
\]

We start this section with a simple inequality.

\textbf{Lemma 2.1.} Let \(S, T\) be contractions such that \(S^*S + T^*T \leq 1\), and let \(A, B\) be non-negative operators. Then for \(0 \leq r \leq s \leq t\) and \(s 
eq 0\),
\[
(S^*A^rS + T^*B^rT)^{\frac{s}{r}} \leq (S^*A^rS + T^*B^rT)^{\frac{s}{s}}; \quad (S^*A^rS + T^*B^rT)^{\frac{s}{t}} \leq (S^*A^rS + T^*B^rT)^{\frac{s}{t}}.
\]

\textit{Proof.} The first inequality follows from (1). By using the Lowner-Heinz inequality we get the second inequality. \(\Box\)

We remark that the above implies that for fixed \(r > 0\) the operator valued function \((S^*A^rS + T^*B^rT)^{\frac{s}{t}}\) is increasing on \(r \leq s < \infty\).

\textbf{Lemma 2.2.} Let \(H\) and \(K\) be bounded selfadjoint operators such that \(0 \leq H \leq K\). Then for real numbers \(p, q\) such that \(0 \leq p \leq q\)
\[
H^{\frac{p}{q}}K^{\frac{q}{p}} \leq (H^{\frac{p}{q}}K^{\frac{q}{p}})^{\frac{q}{p+q}}; \quad K^{\frac{p}{q}}H^{\frac{q}{p}} \geq (K^{\frac{p}{q}}H^{\frac{q}{p}})^{\frac{q}{p+q}}.
\]
Proof. To show the first inequality we may assume that \( K \) is invertible. Since 
\[ \|K^{-\frac{1}{2}}H^\frac{1}{2}\| \leq 1, \] 
by Lemma 2.1, we get 
\[ (H^\frac{1}{2}K^{-\frac{1}{2}}K^{r+1}K^{-\frac{1}{2}}H^\frac{1}{2}) \leq (H^\frac{1}{2}K^{-\frac{1}{2}}K^{r+1}K^{-\frac{1}{2}}H^\frac{1}{2})^\frac{r+1}{r}. \]
This gives the first inequality. By considering the inverse of \( K \) and \( H \), the second 
inequality follows from the first one.

Now we give the main theorem that is an extension of (2).

**Theorem 2.3.** Let \( A, B \) and \( C \) be non-negative operators. Then for non-negative 
real numbers \( r, s, t \) such that \( s \leq t \) and \( (r, t) \neq (0, 0) \),
\[ A \leq B^\lambda C \implies A^\frac{r}{s} (\lambda B^s + \mu C^s) A^\frac{r}{s} \leq \{ A^\frac{r}{s} (\lambda B^s + \mu C^s) A^\frac{r}{s} \}^{\frac{r+s}{s}}. \]

Proof. Suppose \( r = 0 \). Then, since \( f(x) = x^{s/t} \) is operator concave, we obtain (3).
Thus, we need to show (3) in the case of \( r > 0 \). Since \( A \leq (B + \epsilon)^\lambda (C + \epsilon) \), we may 
assume that \( B \) and \( C \) are invertible. We first suppose \( 0 < r \leq 1 \). Since \( f(x) = x^r \) 
is operator monotone and operator concave,
\[ A \leq (\lambda B^{-1} + \mu C^{-1})^{-1} \]
yields
\[ A^r \leq (\lambda B^{-1} + \mu C^{-1})^{-r} \leq (\lambda B^{-r} + \mu C^{-r})^{-1}. \]
This implies
\[ A^\frac{r}{s} (\lambda B^{-r} + \mu C^{-r}) A^\frac{r}{s} \leq 1. \]
By Lemma 2.1, for \( 0 \leq s \leq t \) we get
\[ A^\frac{r}{s} (\lambda B^s + \mu C^s) A^\frac{r}{s} = \lambda A^\frac{r}{s} B^{-\frac{r}{s}} B^{s + s} B^{-\frac{r}{s}} A^\frac{r}{s} + \mu A^\frac{r}{s} C^{-\frac{r}{s}} C^{s + s} C^{-\frac{r}{s}} A^\frac{r}{s} \leq (\lambda A^\frac{r}{s} B^{-\frac{r}{s}} B^{s + s} B^{-\frac{r}{s}} A^\frac{r}{s} + \mu A^\frac{r}{s} C^{-\frac{r}{s}} C^{s + s} C^{-\frac{r}{s}} A^\frac{r}{s})^{\frac{r+s}{s}} = (\lambda A^\frac{r}{s} B^{s} A^\frac{r}{s} + \mu A^\frac{r}{s} C^{s} A^\frac{r}{s})^{\frac{r+s}{s}}. \]
This means (3) holds for \( 0 < r \leq 1 \) and for \( 0 \leq s \leq t \).
Assume (3) holds for \( 0 < r \leq 2^n \) and for \( 0 \leq s \leq t \). Take an arbitrary \( r \) in \( (2^n, 2^{n+1}) \). Since \( r/2 \leq 2^n \), the assumption says
\[ A^\frac{r}{s} (\lambda B^s + \mu C^s) A^\frac{r}{s} \leq \{ A^\frac{r}{s} (\lambda B^s + \mu C^s) A^\frac{r}{s} \}^{\frac{r+s}{s}} \quad (0 \leq s \leq t); \]
in particular,
\[ A^\frac{r}{s} \leq \{ A^\frac{r}{s} (\lambda B^s + (1 - \lambda) C^s) A^\frac{r}{s} \}^{\frac{r+s}{s}}. \]
Let us apply this to the first inequality of Lemma 2.2 with \( p = (2s + r)/r, \ q = (2t + r)/r \). Then we get
\[ A^\frac{r}{s} \{ A^\frac{r}{s} (\lambda B^s + (1 - \lambda) C^s) A^\frac{r}{s} \}^{\frac{r+s}{s}} A^\frac{r}{s} \leq [A^\frac{r}{s} \{ A^\frac{r}{s} (\lambda B^s + (1 - \lambda) C^s) A^\frac{r}{s} \} A^\frac{r}{s}]^{\frac{r+s}{s}}. \]
This in conjunction with (4) gives
\[ A^\frac{r}{s} (\lambda B^s + (1 - \lambda) C^s) A^\frac{r}{s} \leq \{ A^\frac{r}{s} (\lambda B^s + (1 - \lambda) C^s) A^\frac{r}{s} \}^{\frac{r+s}{s}}. \]

\[ \Box \]

**Corollary 2.4.** Let \( A, B \) and \( C \) be non-negative operators. Then
\[ A \leq B^\lambda C \quad \Rightarrow \quad A^{1+r} \leq \{ A^\frac{r}{s} (\lambda B^s + (1 - \lambda) C^s) A^\frac{r}{s} \}^{\frac{r+s}{s}} \quad (1 \leq t). \]
Proof. Putting \( s = 1 \) in (3), in virtue of \( A \leq B ! C \leq \lambda B + \mu C \), we get the above. \( \square \)

Considering the previous inequality with \( B = C \), we get

\[
0 \leq A \leq B \implies A^{1+r} \leq (A^\lambda B^t A^\lambda)^{1+r}.
\]

(5)

This has been found by Furuta and called the Furuta inequality. Furuta showed (2) by using (5). Our proof seems to help us clear up the significance of the exponent \( (s + r)/(t + r) \) in (2) and hence the exponent \( (1 + r)/(t + r) \) in (5) (cf. [3]).

If \( A \leq B \) and \( A \leq C \), then \( A \leq B ! C \) for every \( \lambda \); therefore, the inequality in (3) holds. We remark that in this case we can show it by (2) and the concavity of \( f(x) = x^{(s+r)/(t+r)} \).

We proved that Theorem 2.3 and Corollary 2.4 hold for just the harmonic mean of two operators \( B \) and \( C \); however, it is easy to see that they do even for the harmonic mean of a finite number of operators, too.

**Proposition 2.5.** Let \( B, C \) and \( D \) be non-negative operators. Then for non-negative real numbers \( r, s, t \) such that \( s \leq t \) and \( (r, t) \neq (0, 0) \),

\[
\lambda B + \mu C \leq D \implies \{ D^\lambda (B^t C^s) D^\lambda \}^{1+r} \leq D^\lambda (B^s C^r) D^\lambda.
\]

**Proof.** By the continuity of harmonic mean in the norm sense, we may assume that \( B, C \) and \( D \) are all invertible. The condition \( \lambda B + \mu C \leq D \) implies

\[
D^{-1} \leq B^{-1} ! C^{-1}.
\]

By Theorem 2.3, this yields

\[
D^{-\frac{r}{s}} (\lambda B^{-s} + \mu C^{-s}) D^{-\frac{t}{t}} \leq \{ D^{-\frac{r}{s}} (\lambda B^{-t} + \mu C^{-t}) D^{-\frac{t}{t}} \}^{1+r}.
\]

Take the inverse of both sides of this inequality to get the required inequality. \( \square \)

Put \( s = 1 \) in Proposition 2.5. Since

\[
B ! C \leq \lambda B + (1 - \lambda) C \leq D,
\]

we get

**Corollary 2.6.** Let \( B, C \) and \( D \) be non-negative operators. Then for \( 0 \leq r \) and \( 1 \leq t \),

\[
\lambda B + \mu C \leq D \implies \{ D^\lambda (B^t C^s) D^\lambda \}^{1+r} \leq D^{1+r}.
\]

By setting \( B = C \) in the above, we get an alternate Furuta inequality:

(5)'

\[
0 \leq C \leq D \implies (D^\lambda C^t D^\lambda)^{1+r} \leq D^{1+r}.
\]

By combining (5) and (5)', we get, for \( 0 \leq r \), \( 1 \leq t \),

\[
0 \leq A \leq B \leq C \implies (B^\lambda A^t B^\lambda)^{1+r} \leq B^{1+r} \leq (B^\lambda C^t B^\lambda)^{1+r}.
\]

Suppose that \( B \) and \( C \) are non-negative operators. Replace \( A \) with \( B ! C \) in Theorem 2.3 and \( D \) with \( \lambda B + \mu C \) in Proposition 2.5. Then we get
Corollary 2.7. Let $B, C$ be non-negative operators. Then for non-negative numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0)$,
\[
(B \lambda C)^{\frac{r}{2}}(\lambda B^* + \mu C^*)(B \lambda C)^{\frac{t}{2}} \leq \{(B \lambda C)^{\frac{r}{2}}(\lambda B^* + \mu C^*)(B \lambda C)^{\frac{t}{2}}\}^{\frac{s}{r+t}},
\]
\[
\{(\lambda B + \mu C)^{\frac{r}{2}}(B^* \lambda C)^{\frac{t}{2}}(\lambda B + \mu C)^{\frac{t}{2}}\}^{\frac{s}{r+t}} \leq (\lambda B + \mu C)^{\frac{s}{r+t}}(B^* \lambda C)^{\frac{s}{r+t}}.(\lambda B + \mu C)^{\frac{s}{r+t}}.
\]

3. Exponential Inequality

Theorem 3.1. Let $A, B$ and $C$ be non-negative operators. Then for non-negative real numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0),$
\[
A \leq B \lambda C \implies e^{\frac{r}{2}}(\lambda e^{rB} + \mu e^{rC})e^{\frac{t}{2}} \leq \{e^{\frac{r}{2}}(\lambda e^{rB} + \mu e^{rC})e^{\frac{t}{2}}\}^{\frac{s}{r+t}}.
\]

Proof. For every natural number $n$, $1 + A/n \leq (1 + B/n)^{(n/A)}(1 + C/n)$ because of the properties of harmonic mean given in Section 1.

Therefore, (3) gives
\[
(1 + A/n)^{\frac{s}{n}}\{\lambda(1 + B/n)^{nt} + \mu(1 + C/n)^{nt}\}(1 + A/n)^{\frac{t}{n}} \leq \{(1 + A/n)^{\frac{s}{n}}\{\lambda(1 + B/n)^{nt} + \mu(1 + C/n)^{nt}\}(1 + A/n)^{\frac{t}{n}}\}^{\frac{s}{r+t}}.
\]

Letting $n \to \infty$, we get the required inequality. \qed

Remark. Harmonic mean is just defined for non-negative operators. However, it is clear that the above inequality holds for general selfadjoint operators $A, B, C$ if there is a real number $a$ so that $A+a, B+a, C+a \geq 0$ and $A+a \leq (B+a)^{(n/A)}(C+a)$.

In particular, putting $B = C$ we can obtain:

Let $A, B$ be (not necessarily non-negative) selfadjoint operators. Then
\[
A \leq B \implies e^{\frac{r}{2}}e^{rB}e^{\frac{t}{2}} \leq \{e^{\frac{r}{2}}e^{rB}e^{\frac{t}{2}}\}^{\frac{s}{r+t}}.
\]

Proposition 3.2. Let $B, C$ and $D$ be (not necessarily non-negative) selfadjoint operators. Then for non-negative real numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0),$
\[
\lambda B + \mu C \leq D \implies \{e^{\frac{r}{2}}(e^{rB} \lambda e^{rC})e^{\frac{t}{2}}\}^{\frac{s}{r+t}} \leq e^{\frac{r}{2}}(e^{rB} \lambda e^{rC})e^{\frac{t}{2}}.
\]

Proof. For sufficiently large $n$, $(1 + B/n), (1 + C/n)$ and $(1 + D/n)$ are all non-negative. Since $\lambda(1 + B/n)^{nt} + \mu(1 + C/n)^{nt} \leq (1 + D/n)$, taking notice that
\[
(1 + A/n)^{\frac{s}{n}}(1 + C/n)^{nt} \to e^{rB} \lambda e^{rC} \quad \text{as} \quad n \to \infty,
\]
we can derive the desired inequality from Proposition 2.5. \qed

By putting $s = 0$ in the above remark, we get
\[
A \leq B \implies e^{rA} \leq (e^{rA}e^{rB}e^{rA})^{\frac{s}{s+t}}.
\]

Similarly, by putting $B = C$ and $s = 0$ in Proposition 3.2 we get
\[
C \leq D \implies (e^{rD}e^{rC}e^{rD})^{\frac{s}{s+t}} \leq e^{rD}.
\]

These are known inequalities (cf. [3], [9]). Combining them, we see:

For $r, t \geq 0$ with $(r, t) \neq (0, 0),$
\[
A \leq B \leq C \implies (e^{rB}e^{rA}e^{rB})^{\frac{s}{s+t}} \leq e^{rB} \leq (e^{rB}e^{rC}e^{rB})^{\frac{s}{s+t}}.
\]
REFERENCES

4. T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc. 101(1987), 85–88. MR 89b:47028

DEPARTMENT OF MATHEMATICS, FUKUOKA UNIVERSITY OF EDUCATION, MUNAKATA, FUKUOKA, 811-4192, JAPAN
E-mail address: uchiyama@fukuoka-edu.ac.jp