AN OPERATOR INEQUALITY RELATED TO JENSEN’S INEQUALITY

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(Communicated by Joseph A. Ball)

Abstract. For bounded non-negative operators $A$ and $B$, Furuta showed

$$0 \leq A \leq B \implies A^r B^s A^t \leq (A^\frac{r}{r+s+t} B^\frac{s}{r+s+t} A^\frac{t}{r+s+t})^{r+s+t} \quad (0 \leq r, 0 \leq s \leq t).$$

We will extend this as follows: $0 \leq A \leq B \implies A^r B^s C^t \leq (A^r B^t C^s)^{\lambda}$ implies

$$A^\frac{r}{\lambda} (\lambda B^r + (1 - \lambda) C^s) A^\frac{t}{\lambda} \leq (A^\frac{t}{\lambda} (\lambda B^t + (1 - \lambda) C^r) A^\frac{t}{\lambda})^{\lambda},$$

where $B^t C$ is a harmonic mean of $B$ and $C$. The idea of the proof comes from
Jensen’s inequality for an operator convex function by Hansen-Pedersen.

1. Introduction

Throughout this article, an operator means a bounded linear operator on a
Hilbert space. For selfadjoint operators $A, B$ we write $A \leq B$ as usual if $B - A$ is
positive semidefinite. A real continuous function $f$ defined on an interval $I$ is said
to be operator monotone if $f$ preserves this order, that is, for bounded selfadjoint
operators $A, B$ with spectra in $I$,

$$A \leq B \implies f(A) \leq f(B);$$

and it is said to be operator convex if for all selfadjoint operators $A, B$ with spectra
in $I$ and for all $\lambda$ in $[0, 1]$

$$f(\lambda A + (1 - \lambda) B) \leq \lambda f(A) + (1 - \lambda) f(B).$$

An operator concave function is similarly defined. In $[8]$, Hansen and Pedersen
showed that for a non-negative continuous function $f$ on $[0, \infty)$ the following
conditions are equivalent:

(i) $f$ is operator monotone,
(ii) $f$ is operator concave,
(iii) $T^* f(A) T \leq f(T^* A T)$ for every contraction $T$ (i.e., $||T|| \leq 1$) and for every
non-negative operator $A$,
(iv) $S^* f(A) S + T^* f(B) T \leq f(S^* A S + T^* B T)$ for every pair of $S, T$ with $S^* S + T^* T \leq 1$ and for all non-negative operators $A, B$.

Received by the editors March 21, 2000.

2000 Mathematics Subject Classification. Primary 47A63, 15A48.

Key words and phrases. Order of selfadjoint operators, Jensen inequality, Furuta inequality.

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It is well-known that $f(x) = x^a$ $(0 < a \leq 1)$ is operator monotone on $[0, \infty)$, that is,

$$0 \leq A \leq B \Rightarrow A^a \leq B^a,$$

which is called the L"owner-Heinz inequality. Therefore, (iv) yields

$$(1) \quad S^* A^a S + T^* B^a T \leq (S^* A S + T^* B T)^a \quad (0 < a \leq 1).$$

Related to the L"owner-Heinz inequality, Furuta showed (cf. [5]): for non-negative real numbers $r, s$ and $t$ such that $t \geq s$ and $(r, t) \neq (0, 0)$,

$$(2) \quad 0 \leq A \leq B \Rightarrow A^{\frac{r}{s}} B^{\frac{s}{r}} A^{\frac{s}{t}} \leq (A^{\frac{r}{s}} B^{\frac{s}{r}} A^{\frac{s}{t}})^{\frac{t}{s}}, \quad (B^{\frac{s}{r}} A^{\frac{r}{s}} B^{\frac{s}{r}})^{\frac{t}{s}} \leq B^{\frac{s}{r}} A^{\frac{r}{s}} B^{\frac{s}{r}}.$$

We need the following properties of the harmonic mean (cf. [1], [7]):

Let

$$A^{\lambda} := (\lambda A^{-1} + (1 - \lambda) B^{-1})^{-1}$$

if $A$ and $B$ are invertible, and defined by the weak limit of $(A + \epsilon)^!(B + \epsilon)$ as $\epsilon \to +0$ if not. If non-negative operators $A, B$ are invertible, then we have

$$\lambda A^{-1} + \mu B^{-1} = (\lambda A + \mu B)^{-1} = (A^{-1} - B^{-1})((\lambda A)^{-1} + (\mu B)^{-1})^{-1}(A^{-1} - B^{-1}),$$

where $0 \leq \lambda, \mu \leq 1$, $\lambda + \mu = 1$ (see [10], p. 117 of [2]). This shows that a function $f(x) = 1/x$ is operator convex on $(0, \infty)$ and that

$$A^{\lambda} B \leq \lambda A + (1 - \lambda) B.$$

We start this section with a simple inequality.

**Lemma 2.1.** Let $S, T$ be contractions such that $S^* S + T^* T \leq 1$, and let $A, B$ be non-negative operators. Then for $0 \leq r \leq s \leq t$ and $s \neq 0$,

$$S^* A^r S + T^* B^r T \leq (S^* A^s S + T^* B^s T)^{\frac{t}{s}}, \quad (S^* A^s S + T^* B^s T)^{\frac{t}{s}} \leq (S^* A^r S + T^* B^r T)^{\frac{t}{r}}.$$

**Proof.** The first inequality follows from (1). By using the L"owner-Heinz inequality we get the second inequality.

We remark that the above implies that for fixed $r > 0$ the operator valued function $(S^* A^r S + T^* B^r T)^{\frac{t}{s}}$ is increasing on $r \leq t < \infty$.

**Lemma 2.2.** Let $H$ and $K$ be bounded selfadjoint operators such that $0 \leq H \leq K$.

Then for real numbers $p, q$ such that $0 \leq p \leq q$

$$H^{\frac{p}{q}} K^p H^{\frac{q}{p}} \leq (H^{\frac{p}{q}} K^q H^{\frac{q}{p}})^{\frac{p}{q}}, \quad K^{\frac{p}{q}} H^p K^{\frac{q}{p}} \geq (K^{\frac{p}{q}} H^q K^{\frac{q}{p}})^{\frac{p}{q}}.$$
Suppose inequality (3) follows from the first one. This gives the first inequality. By considering the inverse of \( K \) and \( H \), the second inequality follows from the first one.

Now we give the main theorem that is an extension of (2).

**Theorem 2.3.** Let \( A, B \) and \( C \) be non-negative operators. Then for non-negative real numbers \( r, s, t \) such that \( s \leq t \) and \( (r, t) \neq (0, 0) \),

\[
A \leq B^t C \implies A_r^\frac{t}{s} (\lambda B^s + \mu C^s) A_r^\frac{t}{s} \leq \left\{ A_r^\frac{t}{s} (\lambda B^t + \mu C^t) A_r^\frac{t}{s} \right\}^{\frac{t}{r}}.
\]

**Proof.** Suppose \( r = 0 \). Then, since \( f(x) = x^{s/t} \) is operator concave, we obtain (3). Thus, we need to show (3) in the case of \( r > 0 \). Since \( A \leq (B + \epsilon)^t (C + \epsilon) \), we may assume that \( B \) and \( C \) are invertible. We first suppose \( 0 < r \leq 1 \). Since \( f(x) = x^r \) is operator monotone and operator concave,

\[
A \leq (\lambda B^{-1} + \mu C^{-1})^{-1}
\]

yields

\[
A^r \leq (\lambda B^{-1} + \mu C^{-1})^{-r} \leq (\lambda B^{-r} + \mu C^{-r})^{-1}.
\]

This implies

\[
A_r^\frac{t}{s} (\lambda B^{-r} + \mu C^{-r}) A_r^\frac{t}{s} \leq 1.
\]

By Lemma 2.1, for \( 0 \leq s \leq t \) we get

\[
A_r^\frac{t}{s} (\lambda B^s + \mu C^s) A_r^\frac{t}{s} = \lambda A_r^\frac{t}{s} B^{-s} B^{s+r} B^{-\frac{t}{s}} A_r^\frac{t}{s} + \mu A_r^\frac{t}{s} C^{-s} C^{s+r} C^{-\frac{t}{s}} A_r^\frac{t}{s} \leq (\lambda A_r^\frac{t}{s} B^{-s} B^{s+r} B^{-\frac{t}{s}} A_r^\frac{t}{s} + \mu A_r^\frac{t}{s} C^{-s} C^{s+r} C^{-\frac{t}{s}} A_r^\frac{t}{s})^{\frac{t}{s}} = (\lambda A_r^\frac{t}{s} B^{t} A_r^\frac{t}{s} + \mu A_r^\frac{t}{s} C^{t} A_r^\frac{t}{s})^{\frac{t}{r}}.
\]

This means (3) holds for \( 0 < r \leq 1 \) and for \( 0 \leq s \leq t \).

Assume (3) holds for \( 0 < r \leq 2^n \) and for \( 0 \leq s \leq t \). Take an arbitrary \( r \) in \((2^n, 2^{n+1}]\). Since \( r/2 \leq 2^n \), the assumption says

\[
A_r^\frac{t}{s} (\lambda B^s + \mu C^s) A_r^\frac{t}{s} \leq \{ A_r^\frac{t}{s} (\lambda B^t + \mu C^t) A_r^\frac{t}{s} \}^{\frac{s+r}{s}} \quad (0 \leq s \leq t);
\]

in particular,

\[
A_r^\frac{t}{s} \leq \{ A_r^\frac{t}{s} (\lambda B^t + (1 - \lambda) C^t) A_r^\frac{t}{s} \}^{\frac{s+r}{s}}.
\]

Let us apply this to the first inequality of Lemma 2.2 with \( p = (2s + r)/r, q = (2t + r)/r \). Then we get

\[
A_r^\frac{t}{s} \{ A_r^\frac{t}{s} (\lambda B^t + (1 - \lambda) C^t) A_r^\frac{t}{s} \}^{\frac{s+r}{s}} A_r^\frac{t}{s} \leq [ A_r^\frac{t}{s} \{ A_r^\frac{t}{s} (\lambda B^t + (1 - \lambda) C^t) A_r^\frac{t}{s} \} A_r^\frac{t}{s} ]^{\frac{s+r}{s}}.
\]

This in conjunction with (4) gives

\[
A_r^\frac{t}{s} (\lambda B^s + (1 - \lambda) C^s) A_r^\frac{t}{s} \leq \{ A_r^\frac{t}{s} (\lambda B^t + (1 - \lambda) C^t) A_r^\frac{t}{s} \}^{\frac{s+r}{s}}.
\]

**Corollary 2.4.** Let \( A, B \) and \( C \) be non-negative operators. Then

\[
A \leq B^t C \quad \implies \quad A^{1+r} \leq \{ A_r^\frac{t}{s} (\lambda B^t + (1 - \lambda) C^t) A_r^\frac{t}{s} \}^{\frac{s+r}{s}} \quad (1 \leq t).
\]
Theorem 2.3 and

By the continuity of harmonic mean in the norm sense, we may assume that

Proof. Putting $s = 1$ in (3), in virtue of $A \leq B^!C \leq \lambda B + \mu C$, we get the above. \[
\]

Considering the previous inequality with $B = C$, we get

This has been found by Furuta and called the Furuta inequality. Furuta showed (2) by using (5). Our proof seems to help us clear up the significance of the exponent $(s + r)/(t + r)$ in (2) and hence the exponent $(1 + r)/(t + r)$ in (5) (cf. [8]).

If $A \leq B$ and $A \leq C$, then $A \leq B^!C$ for every $\lambda$; therefore, the inequality in (3) holds. We remark that in this case we can show it by (2) and the concavity of $f(x) = x^{(s+r)/(t+r)}$.

We proved that Theorem 2.3 and Corollary 2.4 hold for just the harmonic mean of two operators $B$ and $C$; however, it is easy to see that they do even for the harmonic mean of a finite number of operators, too.

Proposition 2.5. Let $B, C$ and $D$ be non-negative operators. Then for non-negative real numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0)$,

\[
\lambda B + \mu C \leq D \implies \{D^\frac{t}{\lambda}(B^! C^t)D^\frac{t}{\lambda}\} \leq D^\frac{t}{\lambda}(B^\lambda C^s)D^\frac{t}{\lambda}.
\]

Proof. By the continuity of harmonic mean in the norm sense, we may assume that $B, C$ and $D$ are all invertible. The condition $\lambda B + \mu C \leq D$ implies

\[
D^{-1} \leq B^{-1} C^{-1}.
\]

By Theorem 2.3, this yields

\[
D^{-\frac{t}{\lambda}}(\lambda B^{-s} + \mu C^{-s})D^{-\frac{t}{\lambda}} \leq \{D^{-\frac{t}{\lambda}}(\lambda B^{-t} + \mu C^{-t})D^{-\frac{t}{\lambda}}\}.
\]

Take the inverse of both sides of this inequality to get the required inequality. \[
\]

Put $s = 1$ in Proposition 2.5. Since

\[
B^! C \leq \lambda B + (1 - \lambda)C \leq D,
\]

we get

Corollary 2.6. Let $B, C$ and $D$ be non-negative operators. Then for $0 \leq r$ and $1 \leq t$,

\[
\lambda B + \mu C \leq D \implies \{D^\frac{t}{\lambda}(B^t C^t)D^\frac{t}{\lambda}\} \leq D^{1+r}.
\]

By setting $B = C$ in the above, we get an alternate Furuta inequality:

\[
(5') \quad 0 \leq C \leq D \implies \{D^\frac{t}{\lambda}C^t D^\frac{t}{\lambda}\} \leq D^{1+r}.
\]

By combining (5) and (5'), we get, for $0 \leq r$, $1 \leq t$,

\[
0 \leq A \leq B \leq C \implies (B^\frac{t}{\lambda}C^t) \leq B^{1+r} \leq (B^\frac{t}{\lambda}C^t)B^\frac{t}{\lambda}.
\]

Suppose that $B$ and $C$ are non-negative operators. Replace $A$ with $B^\lambda C$ in Theorem 2.3 and $D$ with $\lambda B + \mu C$ in Proposition 2.5. Then we get
Corollary 2.7. Let $B, C$ be non-negative operators. Then for non-negative numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0)$,
\[
(B^{\lambda} C^{\lambda})(\lambda B^* + \mu C^*) (B^{\lambda} C^{\lambda}) \leq (\{B^{\lambda} C^{\lambda}\}(\lambda B^* + \mu C^*) (B^{\lambda} C^{\lambda})^{\lambda} + \mu C^{\lambda})^{\frac{1}{\lambda}}.
\]
\[
(\{\lambda B + \mu C\}^{\lambda} (B^* C^{\lambda})(\lambda B + \mu C)^{\lambda} \leq (\lambda B + \mu C)^{\lambda} (B^* C^{\lambda})(\lambda B + \mu C)^{\lambda}.
\]

3. Exponential Inequality

Theorem 3.1. Let $A, B$ and $C$ be non-negative operators. Then for non-negative real numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0)$,
\[
A \leq B^{\lambda} C^{\lambda} \implies e^{\frac{r}{A}A} (\lambda e^{sB} + \mu e^{tC}) e^{\frac{r}{A}A} \leq (e^{\frac{r}{A}A} e^{sB} e^{\frac{r}{A}A})^{\frac{1}{\lambda}} + \mu C^{\lambda})^{\frac{1}{\lambda}}.
\]

Proof. For every natural number $n$, $1 + A/n \leq (1 + B/n)! (1 + C/n)$ because of the properties of harmonic mean given in Section 1.

Therefore, (3) gives
\[
(1 + A/n)^{\frac{1}{\lambda}} \{\lambda(1 + B/n)^{nt} + \mu(1 + C/n)^n\}(1 + A/n)^{\frac{1}{t}} \leq [(1 + A/n)^{\frac{1}{t}} \{\lambda(1 + B/n)^{nt} + \mu(1 + C/n)^n\}(1 + A/n)^{\frac{1}{t}}]^{\frac{1}{\lambda}}.
\]

Letting $n \to \infty$, we get the required inequality.

Remark. Harmonic mean is just defined for non-negative operators. However, it is clear that the above inequality holds for general selfadjoint operators $A, B, C$ if there is a real number $a$ so that $A + a, B + a, C + a \geq 0$ and $A + a \leq (B + a)! (C + a)$.

In particular, putting $B = C$ we can obtain:

Let $A, B$ be (not necessarily non-negative) selfadjoint operators. Then
\[
A \leq B \implies e^{\frac{r}{A}A} e^{sB} e^{\frac{r}{A}A} \leq (e^{\frac{r}{A}A} e^{sB} e^{\frac{r}{A}A})^{\frac{1}{\lambda}} + \mu C^{\lambda})^{\frac{1}{\lambda}}.
\]

Proposition 3.2. Let $B, C$ and $D$ be (not necessarily non-negative) selfadjoint operators. Then for non-negative real numbers $r, s, t$ such that $s \leq t$ and $(r, t) \neq (0, 0)$,
\[
\lambda B + \mu C \leq D \implies \{e^{\frac{r}{A}A} e^{sB} e^{\frac{r}{A}A}\}^{\frac{1}{\lambda}} \leq e^{\frac{r}{A}A} e^{sB} e^{\frac{r}{A}A} D^{\lambda}.
\]

Proof. For sufficiently large $n$, $(1 + B/n), (1 + C/n)$ and $(1 + D/n)$ are all non-negative. Since $\lambda(1 + B/n) + \mu(1 + C/n) \leq (1 + D/n)$, taking notice that
\[
(1 + B/n)^{nt} (1 + C/n)^n \to e^{tB} e^{sC} e^{rA} \text{ as } n \to \infty,
\]

we can derive the desired inequality from Proposition 2.5.

By putting $s = 0$ in the above remark, we get
\[
A \leq B \implies e^{rA} \leq (e^{rA} e^{sB} e^{rA})^{\frac{1}{\lambda}}.
\]

Similarly, by putting $B = C$ and $s = 0$ in Proposition 3.2 we get
\[
C \leq D \implies (e^{rA} e^{sB} e^{rA})^{\frac{1}{\lambda}} \leq e^{rD}.
\]

These are known inequalities (cf. [3], [9]). Combining them, we see:

For $r, t \geq 0$ with $(r, t) \neq (0, 0)$,
\[
A \leq B \leq C \implies (e^{rA} e^{sB} e^{rA})^{\frac{1}{\lambda}} \leq e^{rB} \leq (e^{rB} e^{sC} e^{rB})^{\frac{1}{\lambda}}.
\]
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