

## VARIATIONALLY COMPLETE REPRESENTATIONS ARE POLAR

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**ABSTRACT.** A recent result of C. Gorodski and G. Thorbergsson, involving classification, asserts that a variationally complete representation is polar. The aim of this paper is to give a conceptual and very short proof of this fact, which is the converse of a result of Conlon.

The concept of a variationally complete action was introduced by R. Bott [B] in 1956. Two years later, Bott and Samelson [BS] proved that  $s$ -representations (i.e. isotropy representation of semisimple symmetric spaces) are variationally complete. This class of representations contains examples of what L. Conlon [C] called polar representations, or more generally hyperpolar actions. He proved that a hyperpolar action of a compact Lie group on a complete Riemannian manifold is variationally complete. Polar representations were classified by J. Dadok [D] who proved that they are orbit equivalent to  $s$ -representations (see also [EH]). Recently, C. Gorodski and G. Thorbergsson classified variationally complete representations of compact Lie groups [GT]. From this classification they obtained that a variationally complete representation is also orbit equivalent to an  $s$ -representation (from this they obtained, with different methods, Dadok's list). So, they obtained the following equivalent theorem, some of whose history can be found in [TT, p. 196].

**Theorem** ([GT]). *A variationally complete orthogonal representation of a compact Lie group is polar.*

The object of this short note is to give a direct and geometric proof of the above theorem.

Recall that a compact connected Lie subgroup  $G$  of  $SO(n)$  acts polarly on  $\mathbb{R}^n$  if there exists an affine subspace which meets orthogonally all  $G$ -orbits. This is equivalent to the fact that the tangent space  $T_v(G.v)$  of a principal orbit  $G.v$  contains the tangent spaces of all orbits through points in the normal space  $\nu_v(G.v)$ . The  $G$ -action is called variationally complete if any  $G$ -transversal Jacobi field (i.e. produced by variations of transversal geodesics) that is tangent to orbits at two points is the restriction of some Killing field on  $\mathbb{R}^n$  induced by the action. Recall that a geodesic  $\gamma(t)$  in  $\mathbb{R}^n$  is  $G$ -transversal if it is orthogonal to the  $G$ -orbit through

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$\gamma(t)$  for every  $t$  (or equivalently, for some  $t_0$  since a Killing field projects constantly to any geodesic).

*Proof.* Let  $G$  be a compact connected Lie subgroup of  $SO(n)$  such that the  $G$ -action is variationally complete, and let  $v \in \mathbb{R}^n$  be a principal vector. Let  $\xi_v$  be a normal vector to  $G.v$  at  $v$  whose shape operator  $A_{\xi_v}$  has all eigenvalues different from zero. (Such normal vectors define an open and dense subset of the normal space. This is because the determinant of the shape operator is a nonzero polynomial on the normal space at a given point  $v$ , since  $A_v = -Id$ .) Let  $c(s)$  be a curve in  $G.v$  with  $c(0) = v$  and such that  $w := c'(0) \neq 0$  is an eigenvector of  $A_{\xi_v}$  with associated eigenvalue  $\lambda$ . Extend  $\xi_v$  to a parallel normal field  $\xi(s)$  to  $G.v$  along  $c(s)$ . Let us consider the variation by  $G$ -transversal geodesics given by  $\gamma_s(t) = c(s) + t\xi(s)$  and set  $J(t) = \frac{\partial}{\partial s}|_{s=0} \gamma_s(t) = (1 - t\lambda)w$ . Then  $J(t)$  is a ( $G$ -transversal) Jacobi field along the geodesic  $\gamma_0(t) = v + t\xi_v$  of  $\mathbb{R}^n$ . Observe that  $J(0) = w \in T_v(G.v)$  and  $J(1/\lambda) = 0 \in T_{\gamma_0(1/\lambda)}(G.\gamma_0(1/\lambda))$ . By the variational completeness of the  $G$ -action,  $J$  is the restriction to  $\gamma_0$  of a Killing field induced by  $G$ . Thus, for all  $t$ ,  $J(t) = (1 - t\lambda)w$ , and so  $w$ , belongs to  $T_{\gamma_0(t)}(G.\gamma_0(t))$ . Since the eigenvectors of  $A_{\xi_v}$  generate  $T_v(G.v)$ , we obtain that  $T_{\gamma_0(t)}(G.\gamma_0(t)) = T_v(G.v)$  for  $t$  small. This easily implies that  $G$  acts polarly.  $\square$

*Remark.* It is also true that a variationally complete action of a noncompact Lie subgroup  $G$  of  $Iso(\mathbb{R}^n)$  is also polar. In fact, from [Di] (see also [O]) there always exists a principal orbit  $G.p$  with a normal vector  $\xi_p$  whose shape operator  $A_{\xi}$  is positive definite (otherwise all  $G$ -orbits are parallel and totally geodesic), and so invertible. Then, the same proof applies.

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