VARIATIONALLY COMPLETE REPRESENTATIONS ARE POLAR

ANTONIO J. DI SCALA AND CARLOS OLMOS

(Communicated by Wolfgang Ziller)

Abstract. A recent result of C. Gorodski and G. Thorbergsson, involving classification, asserts that a variationally complete representation is polar. The aim of this paper is to give a conceptual and very short proof of this fact, which is the converse of a result of Conlon.

The concept of a variationally complete action was introduced by R. Bott [B] in 1956. Two years later, Bott and Samelson [BS] proved that $s$-representations (i.e. isotropy representation of semisimple symmetric spaces) are variationally complete. This class of representations contains examples of what L. Conlon [C] called polar representations, or more generally hyperpolar actions. He proved that a hyperpolar action of a compact Lie group on a complete Riemannian manifold is variationally complete. Polar representations were classified by J. Dadok [D] who proved that they are orbit equivalent to $s$-representations (see also [EH]). Recently, C. Gorodski and G. Thorbergsson classified variationally complete representations of compact Lie groups [GT]. From this classification they obtained that a variationally complete representation is also orbit equivalent to an $s$-representation (from this they obtained, with different methods, Dadok’s list). So, they obtained the following equivalent theorem, some of whose history can be found in [TT, p. 196].

Theorem (GT). A variationally complete orthogonal representation of a compact Lie group is polar.

The object of this short note is to give a direct and geometric proof of the above theorem.

Recall that a compact connected Lie subgroup $G$ of $SO(n)$ acts polarly on $\mathbb{R}^n$ if there exists an affine subspace which meets orthogonally all $G$-orbits. This is equivalent to the fact that the tangent space $T_v(G.v)$ of a principal orbit $G.v$ contains the tangent spaces of all orbits through points in the normal space $\nu_v(G.v)$. The $G$-action is called variationally complete if any $G$-transversal Jacobi field (i.e. produced by variations of transversal geodesics) that is tangent to orbits at two points is the restriction of some Killing field on $\mathbb{R}^n$ induced by the action. Recall that a geodesic $\gamma(t)$ in $\mathbb{R}^n$ is $G$-transversal if it is orthogonal to the $G$-orbit through...
\(\gamma(t)\) for every \(t\) (or equivalently, for some \(t_0\) since a Killing field projects constantly to any geodesic).

**Proof.** Let \(G\) be a compact connected Lie subgroup of \(SO(n)\) such that the \(G\)-action is variationally complete, and let \(v \in \mathbb{R}^n\) be a principal vector. Let \(\xi\) be a normal vector to \(G.v\) at \(v\) whose shape operator \(A_\xi\) has all eigenvalues different from zero.

(Such normal vectors define an open and dense subset of the normal space. This is because the determinant of the shape operator is a nonzero polynomial on the normal space at a given point \(v\), since \(A_v = -\text{Id}\).)

Let \(c(s)\) be a curve in \(G.v\) with \(c(0) = v\) and such that \(w := c'(0) \neq 0\) is an eigenvector of \(A_\xi\) with associated eigenvalue \(\lambda\). Extend \(\xi_v\) to a parallel normal field \(\xi(s)\) to \(G.v\) along \(c(s)\). Let us consider the variation by \(G\)-transversal geodesics given by \(\gamma_s(t) = c(s) + t\xi(s)\) and set \(J(t) = \frac{\partial}{\partial s}|_{s=0}\gamma_s(t) = (1 - t\lambda)w\). Then \(J(t)\) is a \((G\text{-transversal})\) Jacobi field along the geodesic \(\gamma_0(t) = v + t\xi_v\) of \(\mathbb{R}^n\). Observe that \(J(0) = w \in T_v(G.v)\) and \(J(1/\lambda) = 0 \in T_{\gamma_0(1/\lambda)}(G.\gamma_0(1/\lambda))\). By the variational completeness of the \(G\)-action, \(J\) is the restriction to \(\gamma_0\) of a Killing field induced by \(G\). Thus, for all \(t\), \(J(t) = (1 - t\lambda)w\), and so \(w\) belongs to \(T_{\gamma_0(t)}(G.\gamma_0(t))\). Since the eigenvectors of \(A_\xi\) generate \(T_v(G.v)\), we obtain that \(T_{\gamma_0(t)}(G.\gamma_0(t)) = T_v(G.v)\) for \(t\) small. This easily implies that \(G\) acts polarly.

**Remark.** It is also true that a variationally complete action of a noncompact Lie subgroup \(G\) of \(Iso(\mathbb{R}^n)\) is also polar. In fact, from [DI] (see also [O]) there always exists a principal orbit \(G.p\) with a normal vector \(\xi_p\) whose shape operator \(A_\xi\) is positive definite (otherwise all \(G\)-orbits are parallel and totally geodesic), and so invertible. Then, the same proof applies.

**References**


Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba,
5000 Córdoba, Argentina

E-mail address: discala@mate.uncor.edu

Department of Mathematics, Ciudad Universitaria, 5000 Córdoba, Argentina

E-mail address: olmos@mate.uncor.edu