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A REMARK ON THE DEBS–SAINT-RAYMOND THEOREM

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ABSTRACT. A theorem of Debs and Saint-Raymond gives sufficient conditions for a σ-ideal of compact sets to have the covering property. We discuss the necessity of these conditions. Namely, we show that there exists a $\mathbf{\Pi}^1_1$ σ-ideal that is locally non-Borel, has no Borel basis and has the covering property. This partially answers a question posed by Kechris.

We start with basic definitions. Let E be a compact metric space, and let $\mathcal{K}(E)$ be the space of all closed subsets of E with the Hausdorff metric

$$\begin{aligned}\delta(K,L) &= \sup\{\max(\operatorname{dist}(x,K),\operatorname{dist}(y,L)); x\in L,\, y\in K\},\ \text{if } K,\, L\neq\emptyset,\\ &= \operatorname{diam}(E)+1,\ \text{if } K=\emptyset,\, L\neq\emptyset,\ \text{or } K\neq\emptyset,\, L=\emptyset,\\ &= 0,\ \text{if } K=L=\emptyset.\end{aligned}$$

If $A\subset E$, then $\mathcal{K}(A)$ stands for the set of all compact subsets of A. A set $I\subset\mathcal{K}(E)$ is called a *σ-ideal*

(i) if I is *hereditary,* i.e. if $K,\, L\in\mathcal{K}(E),\, K\in I,\, L\subset K$, then $L\in I$,
(ii) if $K,\, K_1,\, K_2,\dots\in\mathcal{K}(E), K_n\in I$ for all $n\in\mathbb{N}$, and $K=\bigcup_{n=1}^{+\infty}K_n$, then $K\in I$.

We say that a σ-ideal I is *calibrated* if, for every $F\in\mathcal{K}(E),\ F_n\in I,\ n\in\mathbb{N}$, with $\mathcal{K}(F\setminus\bigcup_{n=1}^{+\infty}F_n)\subset I$, we have $F\in I$. A σ-ideal I is called *locally non-Borel* if for every compact set $F\in\mathcal{K}(E)\setminus I$ the set $I\cap\mathcal{K}(F)$ is not Borel. We say that a σ-ideal $I\subset\mathcal{K}(E)$ has the *covering property* if every analytic subset of E which cannot be covered by countably many elements of I contains a closed set which is not in I. *A basis* for a σ-ideal $I\subset\mathcal{K}(E)$ is a subset $B\subset I$ such that

$$I=B_\sigma\overset{\text{def}}{=}\{K\in\mathcal{K}(E);\ K=\bigcup_{n=1}^{\infty}K_n,\ K_n\in B\}$$

and B is hereditary. Let $I\subset\mathcal{K}(E)$ be a σ-ideal. Then we denote

$$I^{\text{loc}}=\{K\in\mathcal{K}(E);\ \text{there exists an open set } G\subset E \text{ with } K\cap G\neq\emptyset \text{ and } K\cap\overline{G}\in I\}.$$

The theory of σ-ideals of compact sets has been motivated by problems in harmonic analysis, namely by problems concerning U-sets and U_0-sets. Results on the

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structure of σ-ideals of compact sets due to Kechris, Louveau and Woodin ([KLW]) were used by Debs and Saint-Raymond ([DSR]) to give a positive answer to the old question of whether every Borel U-set is meager. The following theorem plays the key role in their proof.

Theorem A ([DSR])**.** *Let E be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a $\mathbf{\Pi}^1_1$ σ-ideal satisfying*

(i) *I is locally non-Borel,*
(ii) *I is calibrated,*
(iii) *I has a Borel basis.*

Then I has the covering property.

Kechris posed in [K1] the question of whether all assumptions in Theorem A are necessary. If I has the covering property, then clearly I is calibrated. The necessity of condition (i) is discussed in [K1] but this question remains open. We prove the following theorem which shows that condition (iii) is not necessary.

Theorem. *There exists a compact metric space E and a $\mathbf{\Pi}^1_1$ σ-ideal $I \subset \mathcal{K}(E)$ such that*

(i) *I is locally non-Borel,*
(ii) *I has the covering property,*
(iii) *I has no Borel basis.*

We need the following results.

Theorem B ([KLW, p. 271])**.** *Let E be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a $\mathbf{\Pi}^1_1$ σ-ideal. Then I has a Borel basis if and only if I^{loc} is Borel.*

Theorem C ([L, Chapter 3, pp. 48-54], see also [KLW, p. 271])**.** *Let E be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a σ-ideal. Then I is $\mathbf{\Pi}^1_1$ if and only if I has a $\mathbf{\Pi}^1_1$ basis.*

Theorem D ([H])**.** *Let E be an uncountable compact metric space. Denote*

$$\mathcal{K}_\omega(E) = \{K \in \mathcal{K}(E);\ K \text{ is at most countable}\}.$$

Then $\mathcal{K}_\omega(E)$ is $\mathbf{\Pi}^1_1$ and non-Borel.

Lemma. *Let X and Y be Polish spaces, and let $A \subset X \times Y$ be in $\mathbf{\Sigma}^1_1$. If the projection of A to Y is uncountable, then there exists an uncountable closed set $K \subset X \times Y$ such that K^y ($\overset{\text{def}}{=} \{x \in X;\ [x, y] \in K\}$) contains at most one point for every $y \in Y$.*

Proof. Denote the projection of $X \times Y$ to Y by π and let $T \subset \pi(A)$ be a nonempty perfect set. According to the Jankov–von Neumann uniformization theorem (see [K2, 18.1]) there exists a $\sigma(\mathbf{\Sigma}^1_1)$-measurable function $\varphi : T \to X$ such that $(\varphi(y), y) \in A$ for every $y \in T$. The function φ is Baire measurable and therefore φ is continuous on some G_δ set H that is dense in T (see [K2, 8.38]). Find an uncountable compact set $L \subset H$ and put $K = \{[\varphi(y), y];\ y \in L\}$. □

Proof of the Theorem. Put $X = Y = 2^{\mathbb{N}}$ and $E = X \times Y$. The spaces X, Y, E are equipped with the usual topologies. Take a non-Borel set $C \subset Y$ in $\mathbf{\Pi}^1_1$ (for the existence of such a set see [K2, 14.2]). Put

$$B_0 = \{\{[x,y]\};\ x \in X,\ y \in Y\} \cup \{X \times \{y\};\ y \in C\},$$
$$B = \{K \in \mathcal{K}(E);\ \text{ there exists } L \in B_0 \text{ with } K \subset L\},$$
$$I = B_\sigma.$$

The projection of E to Y is denoted by π. The set B can be written as follows:

$$B = \{\{[x,y]\};\ x \in X, y \in Y\}$$
$$\cup\, (\mathcal{K}(X \times C) \cap \{K \in \mathcal{K}(E); \pi(K) \text{ is a singleton}\}) \cup \{\emptyset\}.$$

The set $X \times C$ is $\mathbf{\Pi^1_1}$ and therefore $\mathcal{K}(X \times C)$ is $\mathbf{\Pi^1_1}$ (cf. [K1]). The sets $\{[x,y];\ x \in X, y \in Y\}$ and $\{K \in \mathcal{K}(E); \pi(K) \text{ is a singleton}\}$ are closed. Thus we see that B is $\mathbf{\Pi^1_1}$ in $\mathcal{K}(E)$ and therefore I is a $\mathbf{\Pi^1_1}$ σ-ideal by Theorem C.

Define a mapping $\varphi : Y \to \mathcal{K}(E)$ by $\varphi(y) = X \times \{y\}$. The mapping φ is clearly continuous. If $y \in C$, then $\varphi(y) \in B_0 \subset I^{\text{loc}}$. If $y \in Y \setminus C$ and an open set $G \subset E$ intersects $\varphi(y)$, then $\varphi(y) \cap \overline{G}$ is an uncountable closed set not intersecting $X \times C$. Therefore such a set cannot be covered by countably many elements of B and thus does not belong to I. This implies $\varphi(y) \notin I^{\text{loc}}$. Thus we have proved $\varphi^{-1}(I^{\text{loc}}) = C$. This gives that I^{loc} is non-Borel and I has no Borel basis by Theorem B.

Take $K \in \mathcal{K}(E) \setminus I$. Denote $H = K \setminus (X \times C)$. We distinguish two possibilities.

1) The set H is uncountable. The set H is analytic and therefore there exists a non-empty perfect set $K^\star \subset H$. We have $\mathcal{K}(K^\star) \cap I = \mathcal{K}_\omega(K^\star)$. This and Theorem D show that $\mathcal{K}(K^\star) \cap I$ is not Borel and consequently $\mathcal{K}(K) \cap I$ is not Borel.

2) The set H is countable. Then the set $P = K \cap (X \times C)$ is analytic. The set P cannot be covered by countably many elements of I, otherwise $K \in I$. This implies that the projection of P to Y is uncountable and, according to the Lemma, P contains an uncountable closed set $K^\star$ such that $(K^\star)^y$ contains at most one point for every $y \in Y$. We have $\mathcal{K}(K^\star) \cap I = \mathcal{K}_\omega(K^\star)$ and $\mathcal{K}(K) \cap I$ is not Borel again.

It remains to prove that I has the covering property. Let $A \subset E$ be a $\mathbf{\Sigma^1_1}$ set which cannot be covered by countably many elements of I. If $A \setminus (X \times C)$ is uncountable, then there exists an uncountable closed set $K \subset A \setminus (X \times C)$ and we have $K \notin I$. If $A \setminus (X \times C)$ is countable, then $\tilde{A} = A \cap (X \times C)$ is $\mathbf{\Sigma^1_1}$. The set $\tilde{A}$ cannot be covered by countably many elements of I and therefore the projection of $\tilde{A}$ to Y is uncountable. Using the Lemma we can find an uncountable closed set $K \subset \tilde{A}$ such that K^y contains at most one point for every $y \in Y$. This implies that $K \notin I$. ☐

Remark. A similar construction of a σ-ideal used for another purpose is due to Dougherty (see [K1, p. 133]).

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