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EXTENSION OF BILINEAR FORMS ON BANACH SPACES

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ABSTRACT. We study the extension of bilinear and multilinear forms from a given subspace of a Banach space to the whole space. Precisely, an isomorphic embedding $j : E \to X$ is said to be (linearly) N-exact if N-linear forms on E can be (linear and continuously) extended to X through j. We present some necessary and sufficient conditions for j to be 2-exact, as well as several examples of 2-exact embeddings. We answer a problem of Zalduendo: in a cotype 2 space bilinear extendable and integral forms coincide.

1. INTRODUCTION AND PRELIMINARIES

Bilinear continuous forms on Banach spaces, contrary to what happens with linear continuous forms, do not extend from subspaces to the whole space. The typical example exhibited to prove that is the inner product on the real Hilbert space. This bilinear form cannot be extended to l_∞ since bilinear forms on l_∞ are weakly sequentially continuous (see [22]), something that the inner product is not. However, the preceding argument is misleading: it suggests that the property of making all polynomials weakly sequentially continuous matters, and suggests some unspecified essential property of either l_2 or l_∞ to be uncovered. However, the only thing that matters, as we shall see, is the inclusion map $l_2 \to l_\infty$.

A number of papers have considered the problem of the extension of bilinear and multilinear forms from a subspace E to a larger space X: Carando [6]; Carando and Zalduendo [7]; Galindo, García, Maestre and Mujica [13]; Kirwan and Ryan [18] and Zalduendo [23]. However, all those papers dodge to consider the role of the embedding $E \to X$ which, in our opinion, should be the central object to be studied. We justify this assertion with a simple observation: Let $j : E \to X$ be an into isomorhism; a bilinear form b on E can be extended to a bilinear form B on X through j if and only if there exists an operator $T \in L(X, X^*)$ making commutative the diagram

$$\begin{array}{ccc} E & \xrightarrow{j} & X \\ \tau_b \downarrow & & \downarrow T \\ E^* & \xleftarrow{j^*} & X^* \end{array}$$

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in which $\tau_b \in L(E, E^*)$ is the operator associated to b ($\tau_b(x)(y) = b(x, y)$). We shall say (see section 3 for a precise statement) that the embedding $j : E \to X$ is 2-exact if all bilinear forms on E can be extended to X through j.

In section 2 we shall consider the case in which j is an embedding into some $\mathcal{L}_\infty$-space. We shall show that if some embedding $j : E \to \mathcal{L}_\infty$ is 2-exact, then every embedding of E into any $\mathcal{L}_\infty$-space is again 2-exact (something that does not happen for embedding into $\mathcal{L}_p$-spaces, $p < +\infty$); moreover, that happens if and only if $\pi = \omega_2$ on $E \otimes E$ (see Theorem 2.2). This, and the classical interpretation of bilinear forms on a Banach space E as the elements of $(E \otimes_\pi E)^*$, opens the door to the possibility of characterizing different types of bilinear forms via different topologies on the tensor space. For instance, recall that a bilinear form B on a Banach space E is said to be *integral* if there exists a measure μ on the compact space $K = (Ball(E^*), weak^*) \times (Ball(E^*), weak^*)$ such that $B(x, y) = \int_K x^*(x)y^*(y)d\mu(x^*, y^*)$. In the language of tensor products, integral bilinear forms on E correspond to the elements of $(E \otimes_\varepsilon E)^*$ (see [23]). Hence, corollaries of Theorem 2.2 are: 1) All bilinear forms on E are extendable if and only if $E \otimes_\pi E = E \otimes_{w_2} E$; 2) Bilinear extendable and integral forms on E coincide if and only if $E \otimes_\varepsilon E = E \otimes_{w_2} E$. In this way we obtain in Proposition 2.5 that in cotype 2 spaces extendable and integral bilinear forms coincide, thus answering a question of Zalduendo.

We enter into details. In [9] the interested reader can find the most important topologies on a tensor space. Let us recall here the definition of the tensornorm ω_2. A sequence $(x_n)_n$ in a Banach space E is said to be weakly p-summable if

$$l_p^w[(x_n)_n] = \sup\{\|(f(x_n))_n\|_{l_p} : \ \|f\|_{E^*} \le 1\} < +\infty.$$

Now, let $z \in E \otimes E$; the ω_2-norm is defined as

$$\omega_2(z) = \inf\{l_2^w[(x_n)_n] l_2^w[(y_n)_n] : \ z = \sum_n x_n \otimes y_n\}.$$

An operator acting between Banach spaces is said to be p-summing, $1 \le p < +\infty$, if it transforms weakly p-summable sequences into absolutely p-summable sequences. The spaces of p-summing (resp. compact, weakly compact, all) operators acting between the spaces X and Y is denoted by $\Pi_p(X, Y)$ (resp. by $\mathcal{K}(X, Y)$, $\mathcal{W}(X, Y), L(X, Y)$). We shall adopt the slightly incorrect custom of writing $\mathcal{L}_p$ to denote an unspecified $\mathcal{L}_p$-space. Two facts that shall be needed through the paper are (see [12]): $L(\mathcal{L}_\infty, \mathcal{L}_1) = \Pi_2(\mathcal{L}_\infty, \mathcal{L}_1)$ and that 2-summing operators factorize through the canonical inclusion $L_\infty(\mu) \to L_2(\mu)$ for some measure μ.

2. Extendable bilinear forms

In the aforementioned papers a multilinear form on E is defined to be extendable if it admits an extension to every superspace X containing E. Let us clarify this point.

Proposition 2.1. *Let E be a Banach space and b a bilinear form on E. The following are equivalent:*

1. *b is extendable;*
2. *b can be extended through the canonical embedding $E \to C(B_{E^*}, w^*)$;*
3. *b can be extended through the canonical embedding $E \to l_\infty(B_{E^*})$;*
4. *b can be extended thrugh some embedding $E \to \mathcal{L}_\infty$ into some $\mathcal{L}_\infty$-space.*

Proof. It is clear that $1 \implies 2, 3$ and 4. We show that 4 implies 1 (and thus that 3 and 2 also imply 1): let B be an extension of b through some embedding $j : E \to \mathcal{L}_\infty$. Since τ_B is 2-summing, it factorizes $\tau_B = \gamma\beta\alpha$ as in the diagram

$$\begin{array}{ccccc}
E & & \xrightarrow{u} & & X \\
\| & & & & \downarrow v \\
E & \xrightarrow{j} & \mathcal{L}_\infty & \xrightarrow{\alpha} & L_\infty(\mu) \\
\tau_b \downarrow & & \downarrow \tau_B & & \downarrow \beta \\
E^* & \xleftarrow{j^*} & \mathcal{L}_\infty^* & \xleftarrow{\gamma} & L_2(\mu) \\
\| & & \downarrow J & & \\
E^* & \xleftarrow{u^*} & X^* & &
\end{array}$$

Now, the injectivity of $L_\infty(\mu)$ yields an extension v of αj through the embedding $u : E \to X$: i.e., $vu = \alpha j$. Moreover, j^* can be lifted to an operator $J : \mathcal{L}_\infty^* \to X^*$ through the quotient map u^* (i.e., $u^* J = j^*$) using Lindenstrauss lifting technique (see [19]). The operator $J\gamma\beta v : X \to X^*$ is the extension of τ_b through u we were looking for. ☐

We pass to characterize extendable bilinear forms.

Theorem 2.2. *Let E be a Banach space and b a bilinear form on E. The following are equivalent:*

1. *b is extendable.*
2. *There exist an $\mathcal{L}_\infty$-space, two operators $u, v : E \to \mathcal{L}_\infty$ and a bilinear form A on $\mathcal{L}_\infty$ such that $b(x, y) = A(u(x), v(y))$.*
3. *There exist a Hilbert space H and two 2-summing operators $u, v : E \to H$ such that $b(x, y) = \langle u(x), v(y)\rangle$.*
4. *The associated operator $\tau_b : E \to E^*$ factorizes as $\tau_b = v^* u$ where u and v are 2-summing operators (i.e., τ_b is 2-dominated; see* [9, 21]*; or else, $\tau_b \in \Pi_2^{dual}(H, E^*)\Pi_2(E, H)$).*
5. *$b \in (E \otimes_{w_2} E)^*$.*

Proof. $1 \Rightarrow 2$: Certainly, some $\mathcal{L}_\infty$-space containing E as a subspace with inclusion i and the choices $u = v = i$ and A the extension of b that exists by hypothesis verify 2.

$2 \Rightarrow 3$**:** By the proof of Grothendieck's theorem (as can be seen in [9, sec. 14 and 20.17]) and the fact that $L(\mathcal{L}_\infty, \mathcal{L}_1) = \Pi_2(\mathcal{L}_\infty, \mathcal{L}_1)$ (see, e.g., [12]), the bilinear form A can be written as $A(x, y) = \langle U(x), V(y)\rangle$ for two 2-summing operators $U, V : \mathcal{L}_\infty \to l_2$. Therefore, $b(x, y) = \langle U(u(x)), V(v(y))\rangle$.

$3 \Leftrightarrow 4$**:** It is clear that $b(x, y) = \langle u(x), v(y)\rangle$ if and only if $T_b = v^* u$.

$3 \Rightarrow 5$**:** If $b(x, y) = \langle u(x), v(y)\rangle$ for two 2-summing operators $u, v : E \to H$ explicitly state how it acts as an element of $(E \otimes_{w_2} E)^*$. If $z = \sum_i x_i \otimes y_i \in E \otimes_{w_2} E$, then $b(z) = \sum \langle u(x_i), v(y_i)\rangle$ is well defined since

$$\begin{aligned}
|\sum \langle u(x_i), v(y_i)\rangle| &\leq \left(\sum \|u(x_i)\|^2\right)^{1/2} \left(\sum \|v(y_i)\|^2\right)^{1/2} \\
&\leq \pi_2(u) w_2\left((x_i)\right) \pi_2(v) w_2\left((y_i)\right)
\end{aligned}$$

from whence $|b(z)| \leq \pi_2(u)\pi_2(v)w_2(z)$. Thus, b is w_2-continuous with $\|b\|_{w_2}^* \leq \inf\{\pi_2(u)\pi_2(v)\}$, where the infimum is taken over all factorizations as in 3 (the inequality is actually an equality; see [9, 20.15]).

$5 \Rightarrow 1$: Since functionals on $E \otimes_{w_2} E$ extend to functionals on some $l_\infty(I) \otimes_{\omega_2} l_\infty(I) = l_\infty(I) \otimes_\pi l_\infty(I)$, by Grothendieck's inequality, the sufficiency is proved. □

Our approach to the extendable bilinear forms has considered only the isomorphic (not the isometric) nature of the problem, as we now explain. In [18], the space of extendable bilinear forms on E is endowed with a natural norm $\|b\|_e$ which is the infimum of those constants $C > 0$ such that b admits an extension b_X to every superspace X of E with norm $\|b_X\| \leq C\|b\|$. Our Proposition 2.1 shows that $\|\cdot\|_e$ is well defined without appealing to the amalgamations of Kirwan and Ryan (see [18]). They prove in [18] that $\|\cdot\|_e$ is isometric to the dual norm of a certain tensornorm η_2 on $E \otimes E$. If $/\pi\backslash$ denotes the injective hull of the projective norm (see [9, sec. 20]), then it is immediate that $\eta_2 = /\pi\backslash$. One has $\omega_2 \leq \eta_2 = /\pi\backslash \leq K_G\omega_2$, where K_G is the Grothendieck's constant, is Grothendieck's inequality in the tensorial form (see [9, 20.17]).

Let us present some relevant examples.

Example 1. Bilinear forms on the disk algebra, on the space H^∞, and on Pisier's spaces are extendable.

Proof. It was proved by Bourgain [3] that Grothendieck's theorem holds for the disk algebra and for H^∞, and therefore if E denotes any of these spaces, $E \otimes_\pi E = E \otimes_{\omega_2} E$. Let us recall that the disk algebra and H^∞ are not $\mathcal{L}_\infty$-spaces.

By a Pisier's space we understand a Banach space verifying $P \otimes_\pi P = P \otimes_\epsilon P$ such as the examples exhibited in [21]. Thus, the characterization 5 can be applied. Also, Pisier's spaces are not $\mathcal{L}_\infty$-spaces. □

As for necessary conditions, let us recall that a Banach space is called regular if $L(X, X^*) = \mathcal{W}(X, X^*)$. Since $\Pi_2(E, E^*) \subset \mathcal{W}(E, E^*)$ the corollary below follows from 4:

Corollary 2.3. *If all bilinear forms on E are extendable, then E is regular.*

The spaces E such that $L(E, l_2) = \Pi_2(E, l_2)$ have been called Hilbert-Schmidt spaces by Jarchow [15].

Lemma 2.4. *If all bilinear forms on E are extendable, then E is Hilbert-Schmidt.*

Proof. Let $u : E \to H$ be an operator. The bilinear form $b(x, y) = \langle u(x), u(y)\rangle$ can, by hypothesis, be extended to some B having the form $B(x, y) = \langle U(x), V(y)\rangle$, where U and V are 2-summing operators. Now let (x_n) be a weakly 2-summable sequence of E. Since both U and V are 2-summing, by Theorem 2.2, one gets:

$$\begin{aligned} \sum \| u(x_n) \|^2 &= \sum \langle u(x_n), u(x_n)\rangle \\ &= \sum | \langle U(x_n), V(x_n)\rangle | \\ &\leq \left(\sum \|Ux_n\|^2\right)^{1/2} \left(\sum \|Vx_n\|^2\right)^{1/2} < +\infty, \end{aligned}$$

and thus u is 2-summing. □

Since cotype 2 spaces verify $E \otimes_\varepsilon E = E \otimes_{w_2} E$ (see [9, Ex. 31.2]) we have the following proposition:

Proposition 2.5. *On a cotype 2 space the extendable and integral bilinear forms coincide.*

This applies, in particular, to $\mathcal{L}_p$-spaces ($1 \leq p \leq 2$), thus answering a question of Zalduendo. That not all bilinear forms are extendable is clear since for $1 < p < \infty$, $L(\mathcal{L}_p, l_2) \neq \Pi_2(\mathcal{L}_p, l_2)$. The $\mathcal{L}_1$ and $\mathcal{L}_\infty$ spaces are Hilbert-Schmidt spaces. Nevertheless, there exist non-extendable bilinear forms on $\mathcal{L}_1$-spaces since they are not regular: a non-weakly compact operator $l_1 \to l_\infty$ seems very easy to find. Bilinear forms on $\mathcal{L}_\infty$-spaces are obviously extendable since all operators from an $\mathcal{L}_\infty$ to an $\mathcal{L}_1$-space are 2-summing (see also Proposition 2.1). Nevertheless, since this case conceals some surprise, we shall consider it again in section 4. We cannot resist the temptation of giving another proof of Proposition 2.5: recall that Pisier's construction in [21] shows that for every cotype 2 space E there exists a superspace $X(E)$ with the property that $X(E) \otimes_\varepsilon X(E) = X(E) \otimes_\pi X(E)$. If a bilinear form extends to $X(E)$ it is ε-continuous, hence integral.

One of the main results of Kirwan and Ryan [18] can be now easily deduced and extended: that a 2-homogeneous polynomial on an $\mathcal{L}_1$-space is extendable if and only if its associated operator is 2-summing. Recall that it has already been proved that bilinear extendable forms have 2-summing associated operators.

Corollary 2.6. *On a Hilbert-Schmidt space all bilinear forms with 2-summing associated operator are extendable.*

Proof. Let b be a bilinear form on a Hilbert-Schmidt space E, with 2-summing asociated operator $\tau_b : E \to E^*$; it factorizes as

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & L_\infty(\mu) \\ \tau_b \downarrow & & \downarrow \beta \\ E^* & \xleftarrow{\gamma} & L_2(\mu) \end{array}$$

The hypothesis yields that both $\beta\alpha$ and $\gamma^*|E$ are 2-summing. Now apply Theorem 2.2. ☐

Well-known examples of Hilbert-Schmidt spaces (see [15]) are quotients of $\mathcal{L}_1$-spaces by reflexive subspaces, and subspaces of $\mathcal{L}_\infty$-spaces yielding reflexive quotients.

Question 1. Characterize extendable bilinear forms on $\mathcal{L}_p$ spaces, $2 < p < \infty$.

3. 2-exact inclusions

Let us justify the starting assertion that only the into isomorphism j matters. We shall use a notation, borrowed from homological algebra, that stresses the role of j. An exact sequence of Banach spaces and operators is a diagram

$$0 \to E \xrightarrow{j} X \xrightarrow{q} Z \to 0$$

with the property that the kernel of each arrow coincides with the image of the preceding one. The open mapping theorem guarantees that E is a subspace of X through the into isomorphism j and Z is the corresponding quotient X/E through the quotient map q. From now on we shall maintain the notation j for the embedding $E \to X$ and q for the quotient map $X \to X/E$. The sequence is said to split (also called a trivial exact sequence) if j admits a linear and continuous left-inverse; i.e., if $j(E)$ is complemented in X. We suggest [8] for everything we shall use about exact sequences and functors in the category of Banach spaces; a sounder, more general, background can be found in [14].

Let $\mathcal{B}$ denote the (contravariant) functor that associates to a Banach space X the Banach space $\mathcal{B}(X)$ of all bilinear scalar forms on X and to an operator $T : X \to Y$ the composition operator $\mathcal{B}(T) : \mathcal{B}(Y) \to \mathcal{B}(X)$ defined as $\mathcal{B}(T)(M)(\cdot) = M(T(\cdot), T(\cdot))$. Clearly, the functor $\mathcal{B}(\cdot)$ transforms an exact sequence $0 \to E \stackrel{j}{\to} X \stackrel{q}{\to} Z \to 0$ into a nonnecessarily exact complex

$$0 \to \mathcal{B}(Z) \stackrel{\mathcal{B}(q)}{\to} \mathcal{B}(X) \stackrel{\mathcal{B}(j)}{\to} \mathcal{B}(E) \to 0.$$

We shall say that the into isomorphism $j : E \to X$ is 2-exact if the restriction map $\mathcal{B}(j) : \mathcal{B}(X) \to \mathcal{B}(E)$ is surjective (i.e., if every bilinear form on E admits an extension through j to a bilinear form on X).

Proposition 3.1. *Let E be a Banach space isomorphic to its dual. An into isomorphism $j : E \to X$ is 2-exact if and only if $j(E)$ is complemented in X.*

Proof. If $I : E \to E^*$ is an isomorphism and for some operator $T : X \to X^*$ one has $j^*Tj = I$, then $jI^{-1}j^*T : X \to X$ is a projection onto $j(E)$. □

Thus, Hilbert spaces, James quasi-reflexive space or Kalton-Peck Z_2 space ([17]) do not admit other 2-exact embeddings apart from trivial ones. To show, as we stated, that the weak sequential continuity of polynomials is only a secondary matter, consider the Kalton and Peck nontrivial exact sequence [17]: $0 \to l_2 \to Z_2 \to l_2 \to 0$. The existence of such a sequence shows that the inclusion $l_2 \to Z_2$ is not 2-exact although both spaces are reflexive and none of them has all polynomials weakly sequentially continuous. On the other hand an embedding $l_1 \to l_\infty$ is not 2-exact although the two spaces have all polynomials weakly sequentially continuous.

Moreover, if one considers the space $l_\infty(l_2^n)$, then an inclusion $l_2 \to l_\infty \to l_\infty(l_2^n)$ is not $\mathcal{B}$-exact, while it is possible to construct a complemented copy of l_2 inside $l_\infty(l_2^n)$ (see [8]), and thus it is possible to construct a 2-exact inclusion $l_2 \to l_\infty(l_2^n)$.

We now show that the property of an embedding being 2-exact has a local nature. The following result is the "localization" of the observation made at the introduction.

Proposition 3.2. *A bilinear form b on E is extendable to X through $j : E \to X$ if and only if there exists a constant C such that for every finite dimensional subspace $A \subset E$ and $F \subset X$ there exists an extension b_{AF} of $b_{|A}$ through $j : A \to A + F$ having norm no greater than $C\|b\|$.*

Proof. The necessity is clear. Let us prove the sufficiency. Given finite dimensional subspaces $A \subset E$ and $F \subset X$, observe the commutative diagram

$$\begin{array}{ccccc} A & \stackrel{j}{\to} & A+F & \to & X \\ {\scriptstyle b|_A}\downarrow & & \downarrow{\scriptstyle b_{AF}} & & \\ A^* & \stackrel{j^*}{\leftarrow} & (E+F)^* & & \end{array}$$

where $\|b_F\| \le C\|b\|$. Consider the set of pairs (A, F) of finite dimensional subspaces of E and X partially ordered by inclusion, and let $\mathcal{U}$ be a free ultrafilter refining the Fréchet filter. Since every point in X eventually belongs to some $A+F$ the formula $B(x, y) = \lim_{\mathcal{U}(F)} b_{AF}(x, y)$ defines a bilinear extension of b through $j : E \to X$ with norm $\|B\| \le C\|b\|$. □

Observe that the extension defined via the ultrafilter is symmetric when the original form is symmetric. Therefore, it works to extend polynomials. Also, observe that we could have considered without effort the corresponding problem of extension of N-linear forms through the notion of N-exact embedding. The following question, however, remained elusive (although we conjecture a negative answer).

Problem. Does there exist an N-exact embedding that is not $(N+1)$-exact?

4. Linear extension means local splitting

We consider now the case in which E is itself an $\mathcal{L}_\infty$-space. Recall from [16] that given an embedding $j : E \to X$ the subspace E is said to be *locally complemented* in X through j if there exists a constant λ such that given a finite dimensional subspace F of X there exists an operator $T : F \to E$ having norm no greater than λ and such that $Tj = id_{E\cap F}$. In [16], Kalton proved that E is locally complemented in X if and only if the dual sequence

$$0 \to (X/E)^* \xrightarrow{q^*} X^* \xrightarrow{j^*} E^* \to 0$$

splits, i.e., if and only if there exists a linear continuous section for the natural restriction map $j^* : X^* \to E^*$. We say that the embedding $j : E \to X$ is *linearly 2-exact* if there exists an operator $S : \mathcal{B}(E) \to \mathcal{B}(X)$ such that $\mathcal{B}(j)S = id$. We denote by $\mathcal{L}^N(X)$ (resp. $\mathcal{P}^N(X)$) the space of continuous N-linear forms (resp. N-homogenous polynomials) in X. We denote by $H_b(X)$ the Fréchet algebra of holomorphic maps of bounded type in X (see [10]).

The next result is organized as follows: the statements labelled (n) give an assertion in terms of either the spaces or their tensor products, while the statements (n') are, in a sense, the corresponding dual assertions. Observe that in the statements (n') we have omitted mentioning that the quotient maps are the natural restriction maps, i.e., the transpose of those appearing in (n). In the case of $(5')$, where no "predual" statement is given, the description of the involved map is given during the proof.

Proposition 4.1. *Let $j : E \to X$ be an embedding. The following are equivalent:*

(1) *$j : E \to X$ is linearly 2-exact.*
(2) *E is locally complemented in X through j.*
(2′) *E^* is complemented in X^*.*
(3) *$\otimes_\pi^N E$ is locally complemented in $\otimes_\pi^N X$ through $\otimes_\pi^N j$.*
(3′) *$\mathcal{L}^N(E)$ is complemented in $\mathcal{L}^N(X)$.*
(4) *$\otimes_{\pi,s}^N E$ is locally complemented in $\otimes_{\pi,s}^N X$ through $\otimes_{\pi,s}^N j$.*
(4′) *$\mathcal{P}^N(E)$ is complemented in $\mathcal{P}^N(X)$.*
(5′) *$H_b(E)$ is complemented in $H_b(X)$.*

Proof. That assertions (n) and (n') are equivalent follows from the very definition.

To show that (2) implies (1), (3′) and (4′), let us first recall the natural isomorphism $A \to A^t$ that identifies the spaces $L(E, L(X,R)) = L(X, L(E,R))$. Hence, if the natural restriction operator j^* admits a linear and continuous section ϕ, then $\mathcal{B}(j)$ also admits a linear and continuous section ϕ_2 as follows: if $b \in L(E, E^*)$, then $\phi_2(b) = \phi(\phi b)^t \in L(X, X^*)$; it is clear that $\mathcal{B}(j)\phi_2(b) = j^*\phi(\phi b)^t j = (\phi b)^t j = b$, which yields (1).

The same method can be used for N-linear forms and even for polynomials. If $\phi_N : \mathcal{L}^N(E) \to \mathcal{L}^N(X)$ denotes the N^{th} Nicodemi operator associated with ϕ (see [4, 5]), then ϕ_N is a linear continuous section for $(\otimes_\pi^N j)^*$, which implies statement $(3')$. The symmetrization of ϕ_N yields $\phi_{N,s}$: $\mathcal{P}^N(E) \to \mathcal{P}^N(X)$, a linear continuous section for $(\otimes_{\pi,s}^N j)^*$, which means statement $(4')$ (see also [1, 4, 5, 13, 23]).

The implication $(3') \Rightarrow (2)$ can be obtained in different ways. One of them begins by considering $E = E \oplus_\pi \oplus_\pi^{N-1}\mathbb{R}$ as a complemented subspace of $\otimes_\pi^N E$: fix a norm one $e_0 \in E$ and define isometric embeddings $u : E \to \otimes_\pi^N E$ by $u(e) = e \otimes e_0 \otimes \cdots \otimes e_0$ and $v : X \to \otimes_\pi^N X$ by $v(x) = x \otimes e_0 \otimes \cdots \otimes e_0$ (see [2]). By dualization, one has the commutative diagrams

$$\begin{array}{ccc} \otimes_\pi^N E & \stackrel{\otimes_\pi^N j}{\to} & \otimes_\pi^N X \\ u \uparrow & & \uparrow v \\ E & \to & X \end{array} \qquad \begin{array}{ccc} \mathcal{L}^N(E) & \stackrel{(\otimes_\pi^N j)^*}{\leftarrow} & \mathcal{L}^N(X) \\ u^* \downarrow & & \downarrow v^* \\ E^* & \to & X^* \end{array}$$

It is clear that the map $s : E^* \to \mathcal{L}^N(E)$ defined as $s(f)(x_1 \otimes \cdots \otimes x_N) = f(x_1)$ is a linear continuous section for u^*. The hypothesis $(3')$ yields a linear continuous section S for $(\otimes_\pi^N j)^*$, and thus v^*Ss is a linear continuous section for j^*, as can easily be shown:

$$j^* v^* S s = u^* (\otimes_\pi^N j)^* S s = u^* s = id_{E^*}.$$

That $(2')$ implies $(5')$ can be seen as follows: if $\phi : E^* \to X^*$ is a linear continuous section for j^*, then a linear continuous section $\psi : H_b(E) \to H_b(X)$ for the natural restriction map can be defined as

$$\psi(f) = \psi\left(\sum_n \frac{d^n f}{n!}(0)\right) = \sum \phi_{n,s}\left(\frac{d^n f}{n!}(0)\right).$$

That ψ is a morphism of Fréchet algebras is a consequence from the fact that Nicodemi operators have the following property: if $A \in \mathcal{L}^k(E)$, $B \in \mathcal{L}^m(E)$ and $k + m = n$, then $\phi^n(A \otimes B) = \phi^k(A) \otimes \phi^m(B)$ (see [13]).

The other implication, that $(5')$ implies $(2')$, is clear: since $\psi : H_b(E) \to H_b(X)$ is a linear continuous section for the natural restriction map, then the operator $d\psi(\cdot)(0) : E^* \to X^*$ given by $d\psi(\cdot)(0)(f) = (d\psi(f))(0)$ is a linear continuous section for j^*. The equivalence between $(2')$ and $(5')$ can also be seen in [11, Prop. 6.18] (equivalence between 2 and 4). Neither Nicodemi operators nor the notion of locally complemented subspace appear there; instead, the author uses that $E**$ is complemented in X^{**} (which can be shown to be equivalent to the local complementation of E in X) and then the Aron-Berner (see [1]) extension of multilinear forms to the bidual space. $\square$

In particular, this result implies that linearly N-exact embeddings are automatically linearly $(N+1)$-exact. As an application we have

Proposition 4.2. *Let $0 \to E \xrightarrow{j} X \to X/E \to 0$ be an exact sequence in which X/E is an $\mathcal{L}_1$-space or E is an $\mathcal{L}_\infty$-space. Then all multilinear forms on E can be linearly extended to X.*

Proof. Since the dual of an $\mathcal{L}_1$-space is injective the dual sequence splits. $\square$

We close this section with the general result about linear extendability.

Proposition 4.3. *A Banach space E is an $\mathcal{L}_\infty$-space if and only if all multilinear forms on E can be linear and continuously extended to any superspace.*

Proof. Since $\mathcal{L}_\infty$-spaces are locally complemented in every larger superspace, one implication is clear. As for the converse, if E is locally complemented in some $\mathcal{L}_\infty$ space, then E^{**} is complemented in some $\mathcal{L}_\infty$-space, thus being itself an $\mathcal{L}_\infty$-space. □

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