PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 12, Pages 3603–3610
S 0002-9939(01)05993-7
Article electronically published on June 28, 2001

SYMMETRY IN A FREE BOUNDARY PROBLEM FOR DEGENERATE PARABOLIC EQUATIONS ON UNBOUNDED DOMAINS

NICOLA GAROFALO AND ELENA SARTORI

(Communicated by David S. Tartakoff)

ABSTRACT. We use the method of Alexandroff-Serrin to establish the spherical symmetry of the ground domain and of the weak solution to a free boundary problem for a class of quasi-linear parabolic equations in an unbounded cylinder $\Omega\times(0,T)$, where $\Omega = (\mathbb{R}^n\setminus\overline{\Omega_1})$, with $\Omega_1 \subset \mathbb{R}^n$ a simply connected bounded domain. The equations considered are of the type $u_t - div(a(u,|Du|)Du) = c(u,|Du|)$, with a modeled on $|Du|^{p-2}$. We consider a solution satisfying the boundary conditions: $u(x,t) = f(t)$ for $(x,t) \in \partial\Omega_1 \times (O,T)$, and $u(x,0) = 0$, $u \to 0$ as $|x| \to \infty$. We show that the overdetermined co-normal condition $a(u,|Du|)\frac{\partial u}{\partial\nu} = g(t)$ for $(x,t) \in \partial\Omega_1 \times (O,T)$, with $g(\overline{T}) > 0$ for at least one value $\overline{T} \in (0,T)$, forces the spherical symmetry of the ground domain and of the solution.

1. Introduction

Let $\Omega_1 \subset \mathbb{R}^n$ be a simply connected, bounded open set. For a given $T > 0$ we consider the exterior cylindrical region $C_T = (\mathbb{R}^n\setminus\overline{\Omega_1}) \times (0,T) \subset \mathbb{R}^{n+1}$ and the following boundary value problem for the heat equation:

$$(1.1)\qquad \begin{cases} \Delta u - u_t = 0 & \text{in } C_T, \\ u(x,t) = f(t) & \text{for } (x,t) \in \partial\Omega_1 \times (0,T), \\ u(x,0) = 0, & \\ u(x,t) \to 0 & \text{uniformly in } t \in (0,T) \text{ as } |x| \to \infty, \\ 0 \le u(x,t) \le f(t) & \text{in } C_T. \end{cases}$$

When Ω_1 is an n-dimensional ball (e.g., centered at the origin), then the unique solution to the above problem is given by the Gauss-Weierstrass kernel

$$u(x,t) \;=\; (4\pi t)^{-\frac{n}{2}}\; exp(-\frac{|x|^2}{4t}), \qquad t > 0.$$

Obviously such u also satisfies the additional condition

$$(1.2)\qquad \frac{\partial u}{\partial \nu}(x,t) \;=\; g(t) \;>\; 0, \qquad (x,t) \in \partial\Omega_1 \times (0,T),$$

Received by the editors April 18, 2000.
1991 *Mathematics Subject Classification.* Primary 35K55.
The first author was supported by NSF Grant No. DMS-9706892.

where ν denotes the unit normal exterior to C_T. The question naturally arises as to whether problem (1.1) together with the overdetermined boundary condition (1.2) admits solution couples (C_T, u) different from the spherically symmetric ones. The answer is negative, as we will see from the following general result.

In an exterior cylindrical region C_T as above we consider the free boundary problem

$$\begin{cases} u_t - div(a(u,|Du|)Du) = c(u,|Du|) & \text{in } C_T, \\ u(x,t) = f(t), & (x,t) \in \partial\Omega_1 \times (0,T), \\ u(x,0) = 0, & x \in \mathbb{R}^n \backslash \overline{\Omega_1}, \\ u(x,t) \to 0, & \text{uniformly in } t \in (0,T) \text{ as } |x| \to \infty, \\ 0 \le u(x,t) \le f(t) & \text{in } C_T. \end{cases} \tag{1.3}$$

The simply connected bounded domain Ω_1 will be assumed to be of class $C^{2,\alpha}$. The functions $a(u,s)$, $c(u,s) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ are assumed to be respectively C^2 and C^1 in their arguments and to satisfy the following structural hypothesis:

$$a(u,s) \ge k_1 s^{p-2}, \tag{1.4}$$

$$a(u,s) + s a_s(u,s) \ge k_2 s^{p-2}, \tag{1.5}$$

$$|a(u,s)| + |a_u(u,s)| + s|a_s(u,s)| \le k_3(s^{p-2} + 1), \tag{1.6}$$

$$|c(u,s)| + |c_u(u,s)| + s|c_s(u,s)| \le k_3(s^{p-1} + 1), \tag{1.7}$$

for every $u \in \mathbb{R}$, $s \in \mathbb{R}^+$ and for some positive constants k_1, k_2, k_3. The exponent p ranges over the interval $(\frac{2n}{n+2}, \infty)$. The model equation that the reader should keep in mind is the parabolic p-Laplacian given by

$$u_t - div(|Du|^{p-2}Du) = 0. \tag{1.8}$$

On the boundary datum $f : [0,T) \to \mathbb{R}$ we make the following assumption. The function f is a Hölder continuous function on $[0,T)$ such that

$$f(0) \;=\; 0. \tag{1.9}$$

The Dirichlet boundary condition in (1.3) is supposed to hold in the sense of traces of $W^{1,p}(\Omega)$ functions for a.e. $t \in (0,T)$. The initial condition is intended in the sense that $lim_{t\to 0^+} \int_\Omega u(x,t)^2 dx = 0$. By the regularity theory developed in [4], [6], [7], [8], [5], [11], there exists $\beta > 0$ such that a solution to (1.3) is in fact $C^{\beta}_{loc}(\overline{C_T})$, so that boundary and initial values of u are taken in the strong sense.

The main result in this paper is the following.

Theorem 1.1. *Let u be a solution to* (1.3) *with f satisfying* (1.9). *If in addition u satisfies the overdetermined co-normal boundary condition*

$$a(u,|Du|)\, \frac{\partial u}{\partial \nu}(x,t) \;=\; g(t), \qquad (x,t) \in \partial\Omega_1 \times (0,T), \tag{1.10}$$

for a function $g \in C^\alpha([0,T))$, $0 < \alpha \le 1$, such that for some $\overline{T} \in (0,T)$

$$g(\overline{T}) \;>\; 0, \tag{1.11}$$

then the set Ω_1 must be a ball and for every $t \in (0,T)$, $u(.,t)$ is a non-increasing spherically symmetric function with respect to the center of the ball.

We emphasize that (1.11) is requested to hold only at one time level and not for all $t \in (0,T)$. By a solution to (1.3) satisfying (1.10) it is meant a function u in the space $C((0,T];L^2(\Omega)) \cap L^p((0,T);W^{1,p}(\Omega))$ such that for every $K \subset\subset \mathbb{R}^n$, and every $0 < t_1 < t_2 < T$, one has

$$\int_{K\cap\Omega} u\phi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{K\cap\Omega} -u\phi_t \ + \ \langle a(u,|Du|)Du, D\phi\rangle dxdt$$

$$= \int_{t_1}^{t_2}\int_{K\cap\Omega}\langle c(u,|Du|)Du, D\phi\rangle dxdt \ + \ \int_{t_1}^{t^2}\int_{K\cap\partial\Omega} g(t)\phi dxdt$$

for any $\phi \in W_0^{1,2}((0,T);L^2(K)) \cap L^p((0,T);W_0^{1,p}(K))$.

When $p > 2$, a remarkable family of self-similar solutions for (1.8) was discovered by Barenblatt [3]; see also Chap. 6 in [5]. Given $\rho > 0$, $k > 0$, the *Barenblatt fundamental solution* with pole at $x = 0$, $t = 0$ is given by

$$B_{k,\rho}(x,t) = \frac{k\rho^n}{S^{n/\lambda}(t)}\left\{1 - \left(\frac{|x|}{S^{n/\lambda}(t)}\right)^{\frac{p}{p-1}}\right\}_+^{\frac{p-1}{p-2}},$$

where $\lambda = n(p-2) + p$,

$$S(t) = bk^{p-2}\rho^{n(p-2)}t + \rho^\lambda, \qquad t \geq 0,$$

with $b = \lambda\left(\frac{p}{p-2}\right)^{p-1}$. For each $t > 0$ the function $B_{k,\rho}(.,t)$ is supported in the ball $\{x \in \mathbb{R}^n | |x| \leq S^{1/\lambda}(t)\}$ and vanishes on its boundary. One should also notice that $S(0)^{1/\lambda} = \rho$ and that $S(t)$ strictly increases. It is then clear that if we consider the infinite cylinder $C_T = (\mathbb{R}^n \backslash \overline{B(0,\rho)}) \times (0,T)$, then $u(x,t) = B_{k,\rho}(x,t)$ solves (1.8) in C_T, and satisfies all the conditions in (1.3). But it is not difficult to see that such a u also satisfies (1.10). We thus infer from Theorem 1.1 that for the degenerate parabolic equation (1.8) the Barenblatt function is the unique solution of the overdetermined problem (1.3), (1.10).

The proof of Theorem 1.1 is based on the method of the moving hyperplanes, due to A.D. Alexandroff [2] and J. Serrin [14]. For second order partial differential equations with uniform ellipticity, such a method was first used by Serrin in his seminal paper [14] to prove spherical symmetry for solutions to overdetermined boundary value problems in bounded domains. Many authors have since employed the technique of the moving hyperplanes to infer symmetry in various problems of interest in mathematical physics and geometry. Among the most prominent results we cite the celebrated works of Gidas, Ni and Nirenberg [10]. In 1989 Alessandrini and one of us [1] extended Serrin's ideas to quasilinear degenerate parabolic equations modeled on (1.8) on bounded domains. A key observation in [1] was the fact that the overdetermined Neumann condition such as that in (1.2) forces the equation to be uniformly parabolic in a region near the parabolic boundary. This fact allows us to implement a suitable modification of the moving hyperplanes technique. Recently, Reichel [12] has proved symmetry for solutions of uniformly elliptic equations in unbounded domains by introducing an appropriate form of reflection of external domains with respect to a hyperplane. He has also established symmetry for nonlinear equations such as

$$div \ \ (a(|Du|) \ Du) \ = \ c(u),$$

modeled on the p-Laplacian $div(|Du|^{p-2}Du) = 0$ [13]. It should be noted, however, that differently from (1.3), the function a does not depend on u and the function c is assumed to be decreasing with respect to u and independent of $|Du|$. For the p-Laplacian, and essentially simultaneously to [13], the authors of the present paper also established symmetry in unbounded domains with a completely different method based on a delicate modification of the method of the so-called P-functions. This result appeared subsequently in [9].

The plan of the paper is as follows. In section 2 we collect some basic results about the regularity of solutions of (1.3) which will be needed in the proof of Theorem 1.1. Section 3 is devoted to the proof of the latter.

2. Preliminary results

A crucial ingredient in the proof of Theorem 1.1 is the following version for an unbounded cylindrical region of the comparison theorem. Its proof is a standard modification of that of Lemma 1.1 in [1].

Lemma 2.1. *Let $G \subset \mathbb{R}^n$ be an unbounded connected open set and consider two solutions u, v to the equation $w_t - div(a(w|Dw|)Dw) = c(w, |Dw|)$ in $G \times (0,T)$. Assume that $|Du|$, $|Dv| \in L^\infty(G \times (0,T))$. If $v \geq u$ on $\partial G \times (0,T)$, $v = u = 0$ on $(\overline{G} \times \{0\})$ and u, $v \to 0$ uniformly in t as $|x| \to \infty$, then we have $v \geq u$ in $G \times (0,T)$. If $A \subset G$ is a bounded domain, then $v \geq u$ on $(\partial A \times (0,T)) \cup (\overline{A} \times \{0\})$ implies $v \geq u$ in $A \times (0,T)$. If in addition there exist T_1, T_2, $0 < T_1 < T_2 \leq T$ such that $|Dv| > 0$ in $A \times (T_1, T_2)$, then $v > u$ in $A \times (T_1, T_2)$.*

If u is a solution to the problem (1.3), then as a consequence of the Caccioppoli type inequality in Proposition 6.1 in [5], Chap. 5, and of the bound for $|Du|$ in Theorem 5.1 and 5.2 in [5], Chap. 8, one has

$$||Du||_{L^\infty(C_T)} \leq C, \tag{2.1}$$

with C depending on $||u||_{L^\infty(C_T)}$.

We now recall the following basic regularity result of solutions to the conormal problem ([11], Theorem 1; [5], Chap. 3, Theorem 1.3):

Theorem 2.2. *Consider the problem*

$$u_t \;-\; div\;\left(a(u, |Du|)Du\right) \;=\; c(u, |Du|) \quad in \quad C_T,$$

$$a(u, |Du|) < Du, \nu > + \;\Psi(t) \;=\; 0 \quad on \quad \partial\Omega \times (0,T),$$

$$u(x, 0) = 0,$$

where Ω is a bounded $C^{1,\beta}$ set in $\mathbb{R}^n$, $0 < \beta \leq 1$, ν is the outer unit normal to the lateral boundary. The function Ψ is such that for some positive constant Λ

$$|\Psi(t) - \Psi(s)| \;\leq\; \Lambda(|t-s|^{\beta/p})$$

for t, $s \in [0,T)$. Under these hypotheses u is Hölder continuous in $(\overline{C_T})$ and $|Du|$ is Hölder continuous up to $\partial\Omega \times [\epsilon, T)$, for any $\epsilon > 0$.

An analysis of the proof of Theorem 2.2 shows that it really is local in nature, so that, due to our hypothesis on a, f and g, we conclude that if u is as in Theorem 1.1, then u is in C^α up to the lateral boundary and to the bottom of C_T, and $C^{1,\alpha}$ up to any portion $\partial\Omega_1 \times [\epsilon, T)$ of the internal lateral boundary of C_T, for any fixed $\epsilon > 0$.

Estimates in Theorem 2.2 are based on the following theorem about interior regularity of $|Du|$ (see [6], [7], [8], see also [5]).

Theorem 2.3. *Let u be a solution in $C_T = \Omega \times (0,T)$ to the equation*

$$u_t - div\ (a(u,|Du|)Du) = c(u,|Du|), \tag{2.2}$$

with structural assumptions (1.4), (1.7)*. For every compact subset K of Ω and for every $\epsilon > 0$*

$$|Du(x,t) - Du(y,s)| \leq \gamma(|x-s| + |t-s|^{\frac{1}{2}})^{\alpha}, \tag{2.3}$$

for every pair of points (x,t) and $(y,s) \in K \times (\epsilon, T)$. Here, α and γ are positive constants depending only on p, n, K, ϵ, T, on the structural constants and on

$$\left(\sup_{0<t<T} \int_\Omega u(x,t)^2 dx\right)^{1/2} + \left(\int_{C_T} [|u|^p + |Du|^p] dxdt\right)^{1/p}.$$

Moreover, if the initial data are regular, (2.2) *can be extended to $K \times [0,T)$.*

3. The method of the moving hyperplanes

We finally come to the method of the moving hyperplanes. Following [1], we fix a direction ξ in $\mathbb{R}^n$ and consider the one-parameter family of n-dimensional hyperplanes $\{\Pi_\lambda\}$ in $\mathbb{R}^{n+1}$ which are orthogonal to $(\xi, 0)$. Here λ indicates the distance of the hyperplane Π_λ from the origin $(0,0)$, which for convenience we assume to belong to $\Omega_1 \times (0)$. Let Π_{λ_0} be a hyperplane in this family which does not intersect the cylinder $C_{1,T} = \Omega_1 \times (0,T)$. We move this hyperplane parallel to itself until it finally touches $C_{1,T}$. From that moment on there will be a portion of $C_{1,T}$ which lies on the same side of Π_λ as Π_{λ_0}, and a reflected portion which lies on the opposite side. We denote by $R''_T(\lambda)$ the former, and by $R'_T(\lambda)$ the latter. The basis of the cylindrical regions $R''_T(\lambda)$, $R'_T(\lambda)$ will be respectively denoted $A''(\lambda)$, $A'(\lambda)$. We call $H_T(\lambda)$ that portion of $\mathbb{R}^n \times (0,T)$ that lies on the same side as Π_{λ_0} with respect to Π_λ and set $\Sigma_T(\lambda) = H_T(\lambda) \setminus R''_T(\lambda)$. We denote by $A(\lambda) = \mathbb{R}^n \setminus A''(\lambda)$ the basis of $\Sigma_T(\lambda)$. If Ω_1 is not convex, $\Sigma_T(\lambda)$ may consist of more than one component, one of which will be unbounded.

As the hyperplane moves, $R'_T(\lambda)$ will lie inside $C_{1,T}$ until one of the following two situations occurs:

A) $R'_T(\lambda)$ meets $\partial\Omega_1$ tangentially and the set of tangency contains points not belonging to Π_λ.

B) Π_λ becomes orthogonal to the lateral boundary of $C_{1,T}$ and of $R'_T(\lambda)$ at some point of the intersection. Denote $\overline{\lambda}$ that value of λ for which either situation A) or B) occurs. Our goal is to show that, in both situations A) and B), Ω_1 is symmetric with respect to $\Pi_{\overline{\lambda}}$. Arguing by contradiction, we suppose that $R'_T(\overline{\lambda}) \subset C_{1,T}$. Following Reichel [12], for every $\lambda \geq \overline{\lambda}$ we define on $\Sigma_T(\lambda)$ the function $v_\lambda(x,t) = u(x',t)$, where (x',t) is reflected with respect to T_λ of the point $(x,t) \in \Sigma_T(\lambda)$. By our construction, $v_\lambda(x,t)$ is well defined in $\Sigma_T(\lambda)$ and solves equation (2.2). Furthermore, by the boundary conditions on $u(x,t)$ we have

$$\begin{aligned}
&v_\lambda(x,t) = u(x,t) && \text{on } (\partial A(\lambda) \times (0,T)) \cap T_\lambda,\\
&v_\lambda(x,t) = u(x,t) && \text{on } A(\lambda),\\
&v_\lambda(x,t) \geq u(x,t) && \text{on } (\partial A(\lambda) \times (0,T))/T_\lambda,\\
&v_\lambda(x,t) \to 0 && \text{as } |x| \to \infty \text{ uniformly in } t.
\end{aligned}$$

From (2.1) one obtains

$$|Du|, \quad |Dv_{\overline{\lambda}}| \in L^\infty(\Sigma_T(\overline{\lambda})), \tag{3.1}$$

so that the first part of Lemma 2.1 allows us to conclude $v_{\overline{\lambda}} \geq u$ on $\Sigma_T(\overline{\lambda})$. By Theorem 2.2, there exist T_1, T_2, $0 < T_1 < \overline{T} < T_2 \leq T$ such that $u \in C^{1,\alpha}_{loc}$ up to $\partial\Omega_1 \times (T_1, T_2)$ so that the overdetermined condition (1.10) can be interpreted in the classical sense. Consider now for $u \in \mathbb{R}$ and $s > 0$ the function

$$\Psi(u,s) \ = \ s \ a(u,s).$$

Since $\Psi(u,0) = 0$ for every $u \in \mathbb{R}$ and one has from (1.5)

$$\Psi_s(u,s) \ = \ a(u,s) \ + \ s \ a_s(u,s) \ \geq \ k_2 \ s^{p-2} \ > \ 0,$$

for $u \in \mathbb{R}$ and $s > 0$, we infer that the function $\Psi(u,\cdot)$ is strictly positive and invertible for every fixed $u \in \mathbb{R}$. Let $\Phi(u,\cdot)$ denote the inverse function. Re-writing (1.10) as $\Psi(u, |\partial u/\partial\nu|) = g(t)$ and using the second equation in (1.3) we obtain

$$\frac{\partial u}{\partial \nu} \ = \ h(t) \qquad \text{on} \quad \partial\Omega \times (0,T),$$

with $h(t) = \Phi(f(t), g(t))$. From (1.11) we have $h(\overline{T}) > 0$. By the regularity of h, $|Du| \neq 0$ on $\partial\Omega_1 \times (T_1, T_2)$ for some small interval (T_1, T_2) containing $\overline{T}$. It is then possible to find $\epsilon > 0$ such that, calling

$$S_\epsilon \ = \ \{x \in \Omega : dist(x, \partial\Omega_1) < \epsilon\},$$

one has $|Du| \neq 0$ in $S_\epsilon \times (T_1, T_2)$, $u(x,t) < f(t)$ on $S_\epsilon \times (T_1, T_2)$. If we consider $(S_\epsilon \cap A(\overline{\lambda})) \times (T_1, T_2)$, we can thus claim

$$\inf_{(S_\epsilon \cap A(\overline{\lambda})) \times (T_1,T_2)} |Dv_{\overline{\lambda}}| \ \geq \ \inf_{S_\epsilon \times (T_1,T_2)} |Du| \ > \ 0. \tag{3.2}$$

By the second part of Lemma 2.1 we infer

$$v_{\overline{\lambda}} > u \qquad on \quad (S_\epsilon \cap A(\overline{\lambda})) \times (T_1, T_2). \tag{3.3}$$

Now, $v_{\overline{\lambda}}$ and u are solutions in $(S_\epsilon \cap A(\overline{\lambda})) \times (T_1, T_2)$ of the equation

$$w_t - a(w, |Dw|) \ \Delta w \ + \ b(w, |Dw|) \ w_{x_i} w_{x_j} w_{x_i x_j} \ = \ f(w, |Dw|), \tag{3.4}$$

where we have adopted the summation convention over repeated indices. Here,

$$b(w,s) = \frac{a_s(w,s)}{s},$$

$$f(w,s) = c(w,s) + a_w(w,s)s^2.$$

Due to the smoothness assumptions on a and c, b and f are C^1. By (1.5) and (3.2) the equation (3.4) is uniformly parabolic, and u and v_λ are classical solutions of (3.4). In particular, they are $C^2((S_\epsilon \cap A(\overline{\lambda})) \times (T_1, T_2))$. By a standard linearization argument we obtain that $z = v_{\overline{\lambda}} - u$ is a solution in $(S_\epsilon \cap A(\overline{\lambda})) \times (T_1, T_2)$ of

$$z_t - A_{ij}(x,t) z_{x_i x_j} - B_i(x,t) z_{x_i} - C(x,t) z = 0, \tag{3.5}$$

where

$$A_{ij} = \int_0^1 [a(|Du_\sigma|)\delta_{ij} + b(|Du_\sigma|)u_{\sigma,x_i}u_{\sigma,x_j}]d\sigma,$$

$$B_i = \int_0^1 \frac{a_s(|Du_\sigma|)\Delta u_\sigma}{|Du_\sigma|}u_{\sigma,x_i}d\sigma + \int_0^1 \frac{\partial}{\partial \eta_i}(b(|\eta|)\eta_h\eta_k)_{\eta=Du_\sigma}u_{\sigma,x_hx_k}d\sigma + \int_0^1 \frac{f_s(u_\sigma,|Du_\sigma|)u_{\sigma,x_i}}{|Du_\sigma|}d\sigma,$$

$$C = \int_0^1 [a_w(u_\sigma,|Du_\sigma|) + b_w(u_\sigma,|Du_\sigma|)u_{\sigma,x_h}u_{\sigma,x_hx_k} + f_w(u_\sigma,|Du_\sigma|)]d\sigma.$$

Here, $u_\sigma = (1-\sigma)u + \sigma v_{\overline{\lambda}}$. Due to the structural conditions (1.4), (1.5), and to (3.2), $(A_{ij})_{i,j=1,\dots,n}$ is uniformly positive definite. In view of (1.6), (1.7), (3.1) and (3.2), and by the smoothness of u and $v_{\overline{\lambda}}$, the coefficients A_{ij} are Lipschitz continuous, B_i and C are bounded, so that equation (3.5) is uniformly parabolic and z is a classical solution of (3.5).

After these preparatory considerations we turn to the final step of the method and consider the two cases A) and B). In case A) we reach a contradiction with (3.3) by means of Hopf's boundary point lemma applied to equation (3.5). According to the latter, in fact, we should have $|Dv_{\overline{\lambda}}| > |Du|$ at each point of internal tangency, against the fact that at such points one has by construction $|Dv_{\overline{\lambda}}| = |Du|$. In case B), by differentiating initial and boundary conditions with respect to x, we find that on the set of tangency for every direction $\eta \in \mathbb{R}^n$

$$\frac{\partial u}{\partial \eta} = \frac{\partial v_{\overline{\lambda}}}{\partial \eta}, \qquad \frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 v_{\overline{\lambda}}}{\partial \eta^2}.$$

Therefore, $v_{\overline{\lambda}} > u$ in $(S_\epsilon \cap A(\overline{\lambda})) \times (T_1, T_2)$ would lead to a contradiction with a modified version of Hopf's boundary point lemma (see [1], lemma 2.1). This contradiction implies that Ω_1 is symmetric with respect to $\Pi_{\overline{\lambda}}$ and that u is symmetric with respect to ξ. Since ξ is arbitrary, Ω is a sphere and, if we fix the origin of $\mathbb{R}^n$ in the centre of the sphere, then $u = u(|x|, t)$. Since $v_\lambda \geq u$ for every $\lambda \leq \overline{\lambda}$, we conclude that $u(., t)$ is nonincreasing in $|x|$.

Acknowledgement

This paper was completed while the first author was at the Mittag-Leffler Institut, for the special year in *Potential Theory and Nonlinear Partial Differential Equations*. He gratefully acknowledges his indebtedness to the organizers of the program and to the Institut for the invitation and for the gracious hospitality.

References

1. G. Alessandrini and N. Garofalo, *Symmetry for degenerate parabolic equations*, Arch. Rat. Mech. Anal., **108** (1989), 161-174. MR **91a:**35086
2. A.D. Alexandroff, *A characteristic property of the spheres*, Ann. Mat. Pura Appl., **58** (1962), 303-315. MR **26:**722
3. G.I. Barenblatt, *On some unsteady motions of a liquid or a gas in a porous medium*, Prikl. Mat. Mech., **16** (1952), 67-78. MR **13:**700a
4. Y.Z. Chen and E. Di Benedetto, *Boundary estimates for solutions of nonlinear degenerate parabolic systems*, J. reine angew. Math., **395** (1989), 101-131. MR **90g:**35085
5. E. Di Benedetto, *Degenerate parabolic equations*, Springer Verlag, New York, Heidelberg, Paris, 1993. MR **94h:**35130

6. E. Di Benedetto and A. Friedman, *Regularity of solutions of nonlinear degenerate parabolic systems*, J. reine angew. Math., **349** (1984), 82-128. MR **85j:**35089
7. E. Di Benedetto and A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems*, J. reine angew. Math., **357** (1985), 1-22. MR **87f:**35134a
8. E. Di Benedetto and A. Friedman, *Addendum to Hölder estimates for nonlinear degenerate parabolic systems*, J. reine angew. Math., **363** (1985), 215-220. MR **87f:**35134b
9. N. Garofalo & E. Sartori, *Symmetry in exterior boundary value problems for quasilinear elliptic equations via blow-up and a priori estimates*, Adv. Diff. Equations, (2) **4** (1999), 137-161. MR **2000a:**35069
10. B. Gidas, W.M. Ni and L. Nierenberg, *Symmetry and related properties via the maximum principle*, Commun. Math. Phys., **68** (1979), 209-243. MR **80h:**35043
11. G.M. Lieberman, *Boundary regularity for solutions of degenerate parabolic equations*, Nonlinear Analysis, **14** (1990), 6, 501-524. MR **91h:**35183
12. W. Reichel, *Radial symmetry for elliptic boundary value problems on exterior domains*, Arch. Rat. Mech. Anal., (4) **137** (1997), 381-394. MR **98m:**35070
13. W. Reichel, *Radial symmetry for an elettrostatic, a capillarity and some fully nonlinear overdetermined problems on exterior domains*, Zeit. Anal. Anwerdungen, (3) **15** (1996), 619-635. MR **97g:**35055
14. J. Serrin, *A symmetry problem in potential theory*, Arch. Rat. Mech. Anal., **43** (1971), 304-318. MR **48:**11545

Institut Mittag-Leffler, Auravägen 17, S-182 62 Djursholm, Sweden
E-mail address: garofalo@ml.kva.se
Current address: Department of Mathematics, The Johns Hopkins University, 3400 N. Charles St., Baltimore, Maryland 21218

Dipartimento di Metodi e Modelli Matematici, Universitá di Padova, 35131 Padova, Italy
E-mail address: sartori@math.unipd.it