A WEAK ASPUND SPACE WHOSE DUAL
 IS NOT WEAK* FRAGMENTABLE

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Abstract. Under the assumption that there exists in the unit interval \([0, 1]\)
an uncountable set \(A\) with the property that every continuous mapping from
a Baire metric space \(B\) into \(A\) is constant on some non-empty open subset of
\(B\), we construct a Banach space \(X\) such that \((X^*, \text{weak}^*)\) belongs to Stegall’s
class but \((X^*, \text{weak}^*)\) is not fragmentable.

1. Introduction

We say that a Banach space \(X\) is weak Asplund if every continuous convex function
defined on a non-empty open convex subset \(A\) of \(X\) is Gâteaux differentiable
at the points of a residual subset of \(A\). In the study of weak Asplund spaces Stegall
introduced the following class of topological spaces, which are defined in terms
of minimal uscos. Recall that a set-valued mapping \(\varphi : X \to 2^Y\) acting between
topological spaces \(X\) and \(Y\) is called an usco mapping if for each \(x \in X\), \(\varphi(x)\) is a
non-empty compact subset of \(Y\) and for each open set \(W\) in \(Y\), \(\{x \in X : \varphi(x) \subseteq W\}\)
is open in \(X\). An usco mapping \(\varphi : X \to 2^Y\) is called minimal if its graph does
not properly contain the graph of any other usco defined on \(X\). We say that a
topological space \(Y\) belongs to Stegall’s class \((\mathcal{S})\) if for every Baire space \(B\) and
minimal usco \(\varphi : B \to 2^X\), \(\varphi\) is single-valued at the points of a residual subset of
\(B\). In [3] Stegall showed that a Banach space \(X\) is weak Asplund if \((X^*, \text{weak}^*)\) lies
in class \((\mathcal{S})\). In fact, Stegall proved that if the dual unit ball \(B_{X^*}\) of \(X\) equipped
with the weak* topology belongs to class \((\mathcal{S})\), then \(X\) is weak Asplund. Another
class of topological spaces that have played a significant role in the study of weak
Asplund spaces is the class of fragmentable spaces. We say that a topological space
\(Y\) is fragmented by a pseudo metric \(\rho\) if every non-empty subset of \(Y\) contains
a non-empty relatively open set of arbitrarily small \(\rho\)-diameter. A space that is
fragmented by some metric is called fragmentable. An easy argument shows that
fragmentable spaces belong to Stegall’s class \((\mathcal{S})\) (see Theorem 5.1.11 in [2]). The
converse question was considered in [4]. Indeed, in that paper the author shows

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that under some additional set-theoretic assumptions there are compact spaces in Stegall’s class \(\mathcal{S}\) that are not fragmentable. We show in this paper that under similar set-theoretic assumptions there are Banach spaces \(X\) such that \((X^*, \text{weak}^*)\) lies in Stegall’s class \(\mathcal{S}\) but \((X^*, \text{weak}^*)\) is not fragmentable.

2. Construction of a Banach space

Given a subset \(A\) of \((0, 1)\) we shall consider the Banach space \(D_A\) of all real-valued functions on \((0, 1]\) that have finite right-hand limits at the points of \([0, 1)\), are left-continuous at the points of \([0, 1]\) and are continuous at the points of \((0, 1]\setminus A\), endowed with sup-norm. Then we shall characterise the duals of these spaces in terms of functions of bounded variation. Given bounded functions \(f\) and \(\alpha\) defined on \((0, 1]\) and \([0, 1]\) respectively and a partition \(P := \{t_k : 0 \leq k \leq n\}\) of \([0, 1]\) where
\[
0 = t_0 < t_1 < t_2 < \cdots < t_n = 1,
\]
the Riemann-Stieltjes sum of \(f\) with respect to \(\alpha\), determined by \(P\), is the real number
\[
S(P, f, \alpha) := \sum_{k=1}^{n} f(t_k) \cdot [\alpha(t_k) - \alpha(t_{k-1})].
\]

We say that \(f\) is Riemann-Stieltjes integrable with respect to \(\alpha\) if there exists a real number \(I\) such that for every \(\varepsilon > 0\) there exists a partition \(P_\varepsilon\) of \([0, 1]\) such that \(|S(P, f, \alpha) - I| < \varepsilon\) for all partitions \(P\) that refine \(P_\varepsilon\). In this case \(I\) is denoted by \(I := \int_{[0,1]} f(t) d\alpha(t)\) and is called the Riemann-Stieltjes integral of \(f\) with respect to \(\alpha\).

For any subset \(A\) of \((0, 1)\) we shall denote by \(BV_A[0, 1]\) the space of all real-valued functions of bounded variation on \([0, 1]\) that are right-continuous at the points of \((0, 1]\setminus A\) and map 0 to 0. We will consider this space endowed with the total variation norm, i.e. for each \(\alpha \in BV_A[0, 1]\)
\[
\|\alpha\| := \text{Var}(\alpha) = \sup\left\{\sum_{k=1}^{n} |\alpha(t_k) - \alpha(t_{k-1})| : \{t_k : 0 \leq k \leq n\} \text{ is a partition of } [0, 1]\right\}.
\]

The proof of the following lemma is straightforward.

**Lemma 1** (Uniform approximation lemma). Let \(A\) be any dense subset of \((0, 1)\), \(f \in D_A\) and \(\varepsilon > 0\). Then there exists a partition \(P_\varepsilon := \{t_k : 0 \leq k \leq n\}\) of \([0, 1]\) with \(t_k \in A\) for all \(1 \leq k < n\) such that \(\|f - f_{P_\varepsilon}\|_\infty < \varepsilon\), where \(f_{P_\varepsilon} : (0, 1) \to \mathbb{R}\) is defined by \(f_{P_\varepsilon}(t) := \sum_{k=1}^{n} f(t_k) \cdot \chi_{(t_{k-1}, t_k]}(t)\).

One can now use the previous lemma to prove the following theorem.

**Theorem 1.** Suppose that \(\alpha : [0, 1] \to \mathbb{R}\) has bounded variation and \(f \in D_{(0,1)}\). Then \(f\) is Riemann-Stieltjes integrable with respect to \(\alpha\).

**Proof.** First note that to show \(f\) is Riemann-Stieltjes integrable with respect to \(\alpha\) we need only show that for every \(\varepsilon > 0\) there exists a partition \(P_\varepsilon\) of \([0, 1]\) such that \(|S(P_\varepsilon, f, \alpha) - S(P', f, \alpha)| < \varepsilon\) for all partitions \(P'\) that refine \(P_\varepsilon\). Further, an elementary calculation shows that for any \(g, g' \in D_{(0,1)}\) and partition \(P\) we have
that $|S(P, g, \alpha) - S(P, g', \alpha)| \leq \|g - g'\| \cdot \text{Var}(\alpha)$. Therefore, if we fix $\varepsilon > 0$ and choose a partition $P$ of $[0, 1]$ such that $\|f - f_P\| < \varepsilon/(\text{Var}(\alpha) + 1)$, then

$$|S(P, f, \alpha) - S(P', f, \alpha)| \leq |S(P, f, \alpha) - S(P, f_P, \alpha)| + |S(P, f_P, \alpha) - S(P', f_P, \alpha)| + |S(P', f_P, \alpha) - S(P', f, \alpha)| < 0 + 0 + \varepsilon = \varepsilon$$

for all partitions $P'$ that refine $P$. \qed

By a slight adaption of the proof of Riesz’s representation theorem for the dual of $(C[0, 1], \|\cdot\|_\infty)$ we can obtain the following representation theorem. Note: it is easiest to make the adaption to the proof of Riesz’s representation theorem that relies upon the Hahn-Banach extension theorem. In fact the standard proof only uses extensions to the space $D_{(0,1)}$ and not to all of $B[0,1]$ - the space of bounded functions on $[0, 1]$; see [1]. Further details may also be found in the paper [6].

**Theorem 2.** Let $A$ be any subset of $(0, 1)$. Then the dual of $D_A$ is isometrically isomorphic to $BV_A[0,1]$. In particular the mapping $T : BV_A[0,1] \to D_A^*$ defined by $T(\alpha)(x) := \int_{[0,1]} x(t)\alpha(t)$ for each $x \in D_A$ is an isometry from $BV_A[0,1]$ onto $D_A^*$.

For a non-empty subset $A$ of $[0,1]$ we shall denote by $\tau_A$ the topology (on $BV_A[0,1]$) of pointwise convergence on $A \cup \{1\}$. If $A$ is dense in $[0,1]$, then $\tau_A$ is a Hausdorff topology. Moreover, the closed unit ball in $BV_A[0,1]$ (with respect to the total variation norm) is $\tau_A$-compact.

**Corollary 1.** For a non-empty subset $A$ of $(0,1)$, $(BV_A[0,1], \tau_A)$ is homeomorphic to $D_A^*$ endowed with the weak topology generated by the functions $\chi_{(0,a]}$ with $a \in A \cup \{1\}$. If $A$ is dense in $(0,1)$, then $\tau_A$ is Hausdorff and the closed unit ball $B_{BV_A[0,1]}$ in $BV_A[0,1]$ with the $\tau_A$-topology is homeomorphic to $(B_{D_A^*}, \text{weak}^*)$. In fact the mapping $T$ defined in the previous theorem, restricted to the ball $B_{BV_A[0,1]}$, realizes such a homeomorphism.

**Proof.** The proof of the first assertion is based upon the simple fact that for each $\alpha \in BV_A[0,1]$ and $t \in A \cup \{1\}$, $T(\alpha)(\chi_{(0,t]}) = \alpha(t)$. The fact that $T$ restricted to $B_{BV_A[0,1]}$ realizes a homeomorphism onto $(B_{D_A^*}, \text{weak}^*)$ follows from the fact that on $B_{D_A^*}$ the relative weak$^*$ topology and the relative topology generated by the functions $\chi_{(0,t]$, $t \in A \cup \{1\}$ coincide (see Lemma 1). \qed

3. $(BV_A[0,1], \tau_A)$ belongs to class($S$)

We begin this section with the following preliminary theorem.

**Theorem 3.** Let $Y$ be a compact topological space and $\rho$ a metric on it. Then $Y$ belongs to class($S$) if (and only if) for $\varepsilon > 0$, each Baire metric space $B$ and each minimal usco $\varphi : B \to 2^Y$ there exists a point $x \in B$ such that $\rho$-diam $\varphi(x) \leq \varepsilon$.

**Proof.** By the “factorization theorem” in [5] we need only show that for every complete metric space $M$ and minimal usco $\varphi : M \to 2^Y$ there exists a residual set $R$ of $M$ such that $\varphi$ is single-valued at the points of $R$. If we now apply the proof of Theorem 3.2.6 in [2] to our current situation we obtain the desired result. \qed
Lemma 2. Let \( \varphi : X \to 2^Y \) be a minimal usco acting between topological spaces \( X \) and \( Y \) and let \( f : Y \to \mathbb{R} \) be a continuous function. Then there is a residual set \( R \) in \( X \) such that the composition mapping \( f \circ \varphi : X \to 2^\mathbb{R} \) defined by \( (f \circ \varphi)(x) := \{ f(y) : y \in \varphi(x) \} \) is single-valued at the points of \( R \).

Proof. By Lemma 3.1.2(iv) in [2], \( f \circ \varphi \) is a minimal usco on \( X \) and so the result follows from Theorem 5.1.11 in [2].

In the remainder of this section \( A \) will always denote a dense subset of \( (0,1) \) that satisfies the property: \( (*) \) Every continuous function from a Baire metric space \( B \) into \( A \) is constant on some non-empty open subset of \( B \).

Of course every countable dense subset of \( (0,1) \) has this property; however we shall be particularly interested in the case when \( A \) is uncountable, if indeed such a set exists.

Theorem 4. Let \( A \) be a dense subset of \( (0,1) \) that satisfies property \( (*) \). Then \( (BV_A[0,1], \tau_A) \) belongs to \( \text{class}(S) \).

Proof. First, let us note that by Theorem 3.1.5, part(iv) in [2], we need only show that the closed unit ball \( B_{BV_A[0,1]} \) of \( BV_A[0,1] \) belongs to \( \text{class}(S) \). In fact, we need only show that the \( (\tau_A\text{-compact}) \) set \( M_A[0,1] \) of all non-decreasing functions in \( B_{BV_A[0,1]} \), endowed with the \( \tau_A \)-topology lies in Stegall’s \( \text{class}(S) \). Since if \( M_A[0,1] \in \text{class}(S) \), then by Theorem 3.1.5, part(iii) in [2], \( M_A[0,1] \times M_A[0,1] \in \text{class}(S) \). However, by the Jordan decomposition theorem \( B_{BV_A[0,1]} \subseteq \Delta(M_A[0,1] \times M_A[0,1]), \) where \( \Delta : M_A[0,1] \times M_A[0,1] \to BV_A[0,1] \) is defined by \( \Delta(f,g) := f - g \). Hence the result follows from Theorem 3.1.5, part(i) in [2], since \( \Delta \) is a perfect mapping. For any \( \alpha, \beta \in M_A[0,1] \) we define

\[
\rho_1(\alpha, \beta) := |(\alpha - \beta)(1)|, \quad \rho_1(\alpha, \beta) := \int_0^1 |(\alpha - \beta)(t)|dt, \\
\rho_f(\alpha, \beta) := \sum_{t \in A} |(\alpha - \beta)(t^+) - (\alpha - \beta)(t)|.
\]

Note: \( \{ t \in A : |(\alpha - \beta)(t^+) - (\alpha - \beta)(t)| > 0 \} \) is at most countable. Then we define \( \rho(\alpha, \beta) := \rho_1(\alpha, \beta) + \rho_f(\alpha, \beta) \). With a little thought it should be clear that \( \rho \) defines a metric on the set \( M_A[0,1] \). We now proceed via Theorem 3. To this end, let \( \varepsilon > 0 \), \( B \) be a Baire metric space and \( \varphi : B \to 2^{M_A[0,1]} \) be a minimal usco.

Step 1. It is not too difficult to check that \( \rho_f \) is a continuous pseudo-metric on \( M_A[0,1] \), i.e. for each \( \alpha \in M_A[0,1] \) and \( r > 0 \) the set \( \{ \beta \in M_A[0,1] : \rho_f(\alpha, \beta) < r \} \) is \( \tau_A \)-open in \( M_A[0,1] \). Hence it follows that \( \rho_f \) “fragments” \( M_A[0,1] \). It is also very easy to see that \( \rho_1 \) is a continuous pseudo-metric on \( M_A[0,1] \) and so \( \rho_f \) also “fragments” \( M_A[0,1] \). In particular this means that there is a residual set \( R \subseteq B \) such that both \( \rho_1 \)-diam \( \varphi(x) = 0 \) and \( \rho_f \)-diam \( \varphi(x) = 0 \) at each point \( x \in R \) (see the proof of Theorem 5.1.11 in [2]). Therefore by restricting \( \varphi \) to \( R \) and re-labeling we may assume, without loss of generality, that both \( \rho_1 \)-diam \( \varphi(x) = 0 \) and \( \rho_f \)-diam \( \varphi(x) = 0 \) for all \( x \in B \). One immediate consequence of this is that for each \( x \in B \) we may unambiguously refer to the left-hand and right-hand limits of \( \varphi(x) \), since if \( \alpha, \beta \in \varphi(x) \), then both the left-hand and right-hand limits of \( \alpha \) and \( \beta \) coincide on \([0,1]\).
Step 2. In this step we decompose the space $M_A[0,1]$ into countably many parts, \( \{M_{m,n,(F,f)} : (m,n,(F,f)) \in \mathbb{N}^2 \times \mathcal{F} \} \), but first we introduce some notation. For each $\alpha \in M_A[0,1]$ and $m \in \mathbb{N}$,

\[
S(\alpha, m) := \{ t \in A : \alpha(t^+) - \alpha(t^-) > 1/m \} \quad \text{and} \quad L^1(\alpha, m) := \sum_{t \in S(\alpha, m)} [\alpha(t^+) - \alpha(t^-)].
\]

The notation $S(\alpha, \infty)$ and $L^1(\alpha, \infty)$ will have the expected meaning. For each $m \in \mathbb{N}$ we define, $M_m := \{ \alpha \in M_A[0,1] : L^1(\alpha, m) > L^1(\alpha, \infty) - \varepsilon/2 \}$ and for each partition $P := \{ t_k : 0 \leq k \leq n \}$ of $[0,1]$ we let $I_k(P) := [t_{k-1}, t_k]$, $1 \leq k \leq n$. Then for each $n \in \mathbb{N}$ we let $P_n$ denote the uniform $1/n$-partition of $[0,1]$ and we define

\[
M_{m,n} := \{ \alpha \in M_m : P_n \cap S(\alpha, m) = \emptyset \} \quad \text{and} \quad \text{card}[S(\alpha, m) \cap I_k(P_n)] \leq 1 \quad \text{for } k \in \{1,2,\ldots,n\}.
\]

One can check that $\bigcup\{M_{m,n} : (m,n) \in \mathbb{N}^2 \} = M_A[0,1]$. Now, with $m$ and $n$ fixed we further decompose $M_A[0,1]$ as follows: For each fixed non-empty subset $F \subseteq \{1,2,\ldots,n\}$ and function $f : F \to \mathbb{Q}^2$, i.e. $f(k) := (f_1(k), f_2(k)) \in \mathbb{Q}^2$, we consider the set

\[
M_{m,n,(F,f)} := \{ \alpha \in M_{m,n} : \text{card}[I_k(P_n) \cap S(\alpha, m)] = 1 \} \quad \text{and only if, } k \in F,
\]

\[
\text{and } \max\{|\alpha(t^-) - f_1(k)|,|\alpha(t^+) - f_2(k)|\} < 1/(4m)
\]

for each $t \in I_k(P_n) \cap S(\alpha, m)$ and $k \in F$.

If we let $\mathcal{F}$ denote the family of all such pairs $(F,f)$, then $\mathcal{F}$ is at most countable. Hence $\{M_{m,n,(F,f)} : (m,n,(F,f)) \in \mathbb{N}^2 \times \mathcal{F} \}$ is a countable decomposition of $M_A[0,1]$.

Step 3. For any subset $X \subseteq M_A[0,1]$ we define $\varphi^{-1}(X) := \{ x \in B : \varphi(x) \cap X \neq \emptyset \}$. Now since $M_A[0,1] = \bigcup\{M_{m,n,(F,f)} : (m,n,(F,f)) \in \mathbb{N}^2 \times \mathcal{F} \}$, it follows that $\bigcup\{\varphi^{-1}(M_{m,n,(F,f)}) : (m,n,(F,f)) \in \mathbb{N}^2 \times \mathcal{F} \} = B$. Therefore there must be some $(m',n',(F',f')) \in \mathbb{N}^2 \times \mathcal{F}$ such that $\varphi^{-1}(M_{m',n',(F',f')})$ is second (Baire) category in $B$. Moreover, since the set $M_{m',n',(F',f')}$ is defined solely in terms of the left-hand and right-hand limits of its members it follows, by the note at the end of Step 1, that

\[
\varphi(\varphi^{-1}(M_{m',n',(F',f')})) \subseteq M_{m',n',(F',f')}.
\]

Further, by Proposition 3.2.5 in [2] there exists a non-empty open set $U$ in $B$ such that $B' := U \cap \varphi^{-1}(M_{m',n',(F',f')})$ is dense in $U$ and a Baire space with the relative topology. Now by applying Lemma 2 in [4] twice we see that the restriction of $\varphi$ to $B'$ is a minimal usco. In this way, we see that there is no loss of generality in assuming that $\varphi(B) \subseteq M_{m',n',(F',f')}$. 

Step 4. For each $k \in F' \subseteq \{1,2,\ldots,n'\}$ we define the function $g_k : B \to A$ by $g_k(x) := S(\varphi(x), m') \cap I_k(P_{n'})$. Note: this definition is sensible since for each $x \in B$ and $\alpha, \beta \in \varphi(x)$, $S(\alpha, m') = S(\beta, m')$. It now follows from the $\tau_A$-upper semi-continuity of $\varphi$ and the definition of $M_{m',n',(F',f')}$ that each $g_k$ is continuous on $B$. Hence by property (*) there exists a non-empty open subset $U$ of $B$ such that each $g_k$, $k \in F'$, is constant on $U$. 

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Step 5. For each $k \in F'$ define $t_k := g_k(x)$, $x \in U$. Then by Lemma 2 there exists a residual set $R$ in $U$ such that each of the uscos $t_k \circ \varphi : U \to 2^R$ defined by $(t_k \circ \varphi)(x) := \{\alpha(t_k) : \alpha \in \varphi(x)\}$ are single-valued on $R$. We claim that $\rho$-diam $\varphi(x) \leq \varepsilon$ for each $x \in R$. To see this, first note that it is sufficient to show that $\rho_J$-diam $\varphi(x) \leq \varepsilon$ for each $x \in R$. Now fix $x_0 \in R$ and consider $\alpha, \beta \in \varphi(x_0)$; then

$$\rho_J(\alpha, \beta) = \sum_{t \in A} |(\alpha - \beta)(t^+) - (\alpha - \beta)(t)| = \sum_{t \in S(\alpha, \infty)} |(\alpha - \beta)(t^+) - (\alpha - \beta)(t)|.$$

However, if $t \in S(\alpha, m')$, then $|(\alpha - \beta)(t^+) - (\alpha - \beta)(t)| = 0$ since (by Step 1) $\alpha(t^+) = \beta(t^+)$ and (as just noted) $\alpha(t) = \beta(t)$. On the other hand, if we write $S_{tail} := S(\alpha, \infty) \setminus S(\alpha, m')$, then we have

$$\sum_{t \in S_{tail}} |(\alpha - \beta)(t^+) - (\alpha - \beta)(t)| \leq \sum_{t \in S_{tail}} \alpha(t^+) - \alpha(t) + \sum_{t \in S_{tail}} \beta(t^+) - \beta(t)$$

$$\leq \sum_{t \in S_{tail}} \alpha(t^+) - \alpha(t^-) + \sum_{t \in S_{tail}} \beta(t^+) - \beta(t^-)$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that $\rho(\alpha, \beta) \leq \varepsilon$ and so $\rho$-diam $\varphi(x_0) \leq \varepsilon$, which completes the proof.

Corollary 2. Let $A$ be any dense subset of $(0, 1)$ that satisfies property $(*)$. Then $(D_A^*, \text{weak}^*)$ belongs to class$(S)$.

We end this section of the paper by returning to the question of the existence of uncountable sets that have property $(*)$. The good news is that there is such a subset $A$ of $(0, 1)$ that satisfies property $(*)$ in Gödel’s universe $(V = L)$ and hence the set $A' := A \cup \{(0, 1) \cap \mathbb{Q}\}$ will serve our needs; see [7]. However, the set $A$ necessarily relies upon additional axioms, as it is known that if we assume the existence of a precipitous ideal over $\omega_1$, then for every uncountable separable metric space $A$ there exists a Baire metric space $B$ and a continuous function $f : B \to A$ such that $\text{int}(f^{-1}(a)) = \emptyset$ for each $a \in A$ (see [3]).

4. WHEN IS $(BV_A[0,1], \tau_A)$ FRAGMENTABLE?

We will show that for every set $A \subseteq (0, 1)$, $D_A$ is isometrically isomorphic to $C(K_A)$ for some compact Hausdorff space $K_A$. Indeed, if $\emptyset \neq A \subseteq (0, 1)$, then we may define $K_A$ in the following manner: $K_A := \{(0) \cup A \times \{1\}\} \cup \{[0, 1] \times \{0\}\}$. We endow $K_A$ with the order topology (on $K_A$) generated by the lexicographic (i.e. dictionary) ordering, i.e. $(s_1, s_2) \leq (t_1, t_2)$ if, and only if, either $s_1 < t_1$ or $s_1 = t_1$ and $s_2 \leq t_2$. It is shown in [1] (see Proposition 2) that $K_A$ is always Hausdorff and compact. It is also shown that $K_A$ is fragmentable if, and only if, $A$ is countable and this occurs if, and only if, $K_A$ is metrizable.

Theorem 5. Let $A$ be a non-empty subset of $(0, 1)$. Then $(D_A^*, \text{weak}^*)$ is fragmentable if, and only if, $A$ is countable.

Proof. We define an isometry $T$ from $D_A$ onto $C(K_A)$ in the following way: $T(f)((t, 0)) := f(t)$ for all $t \in (0, 1)$ and $T(f)((t, 1)) := \lim_{t \to t^+} f(t)$ for $t \in \{0\} \cup A$. One can check, as in ([2], p. 47), that $T$ is in fact an isometry from $D_A$ onto $C(K_A)$. Indeed, it is routine to verify that $T$ is a linear isometry into $C(K_A)$, so it suffices to check that $T$ is surjective. To this end, let $g \in C(K_A)$ and define $f : (0, 1] \to \mathbb{R}$ by $f(t) := g((t, 0))$ for all $t \in (0, 1]$. Then $f \in D_A$ and $T(f) = g$. 


Corollary 3. If $A$ is an uncountable dense subset of $(0,1)$ that satisfies property $(\ast)$, then $(D_A^*, \text{weak}^*)$ belongs to class $\mathcal{S}$ (and so $D_A$ is weak Asplund) but $(D_A^*, \text{weak}^*)$ is not fragmentable.

References